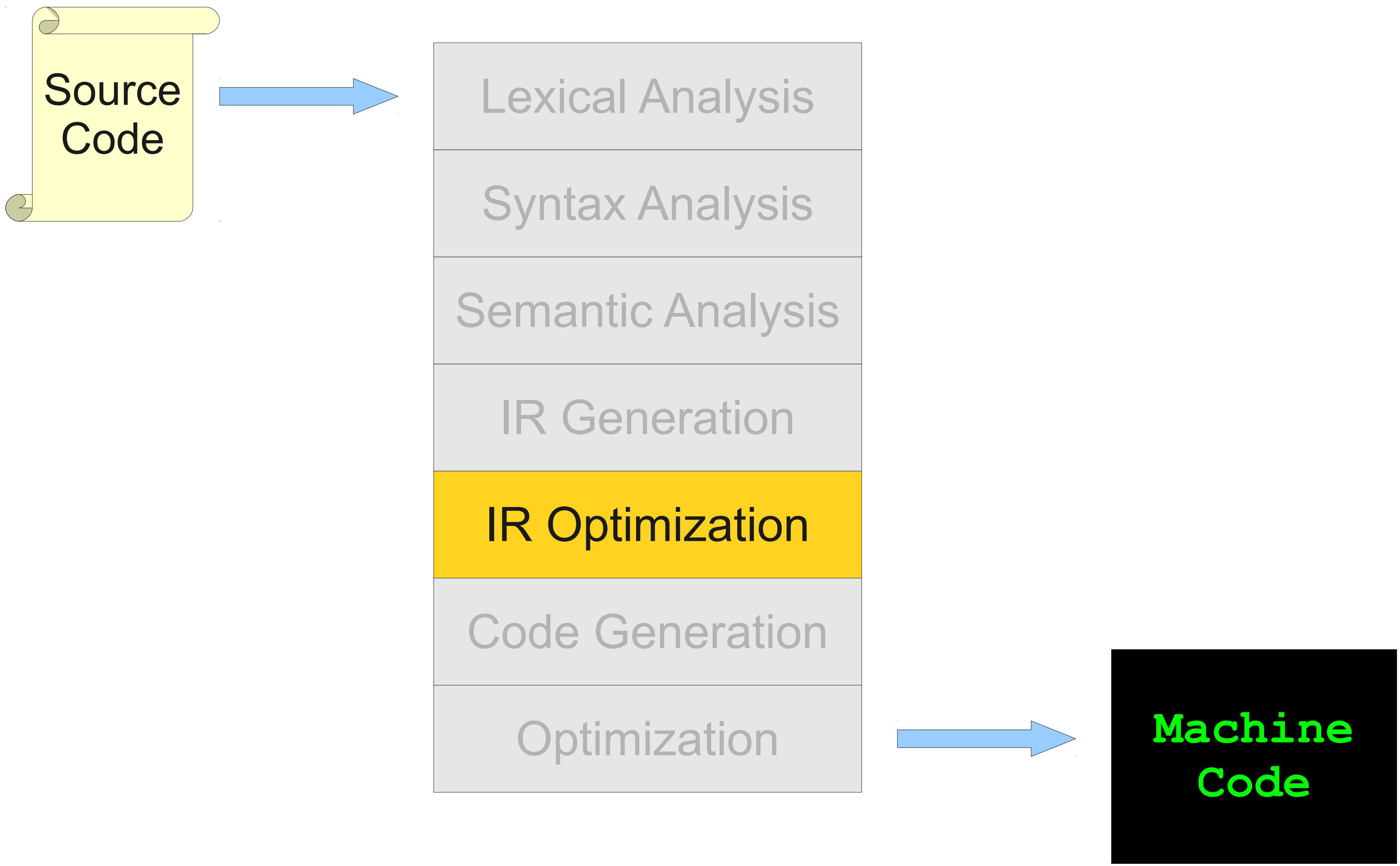
Global Optimization, Part II

Announcements

- Programming Project 4 due **Wednesday, August 10** at 11:59PM.
  - OH all this week and Sunday.
  - Ask questions via email!
  - Ask questions via Piazzza!
- Programming Assignment 2 grades/feedback available on Paperless.
  - Mean: **91/100**
  - Median: **96/100**
  - Stdev: **15**

Where We Are


Source Code
Lexical Analysis
Syntax Analysis
Semantic Analysis
IR Generation
IR Optimization
Code Generation
Optimization
Machine Code

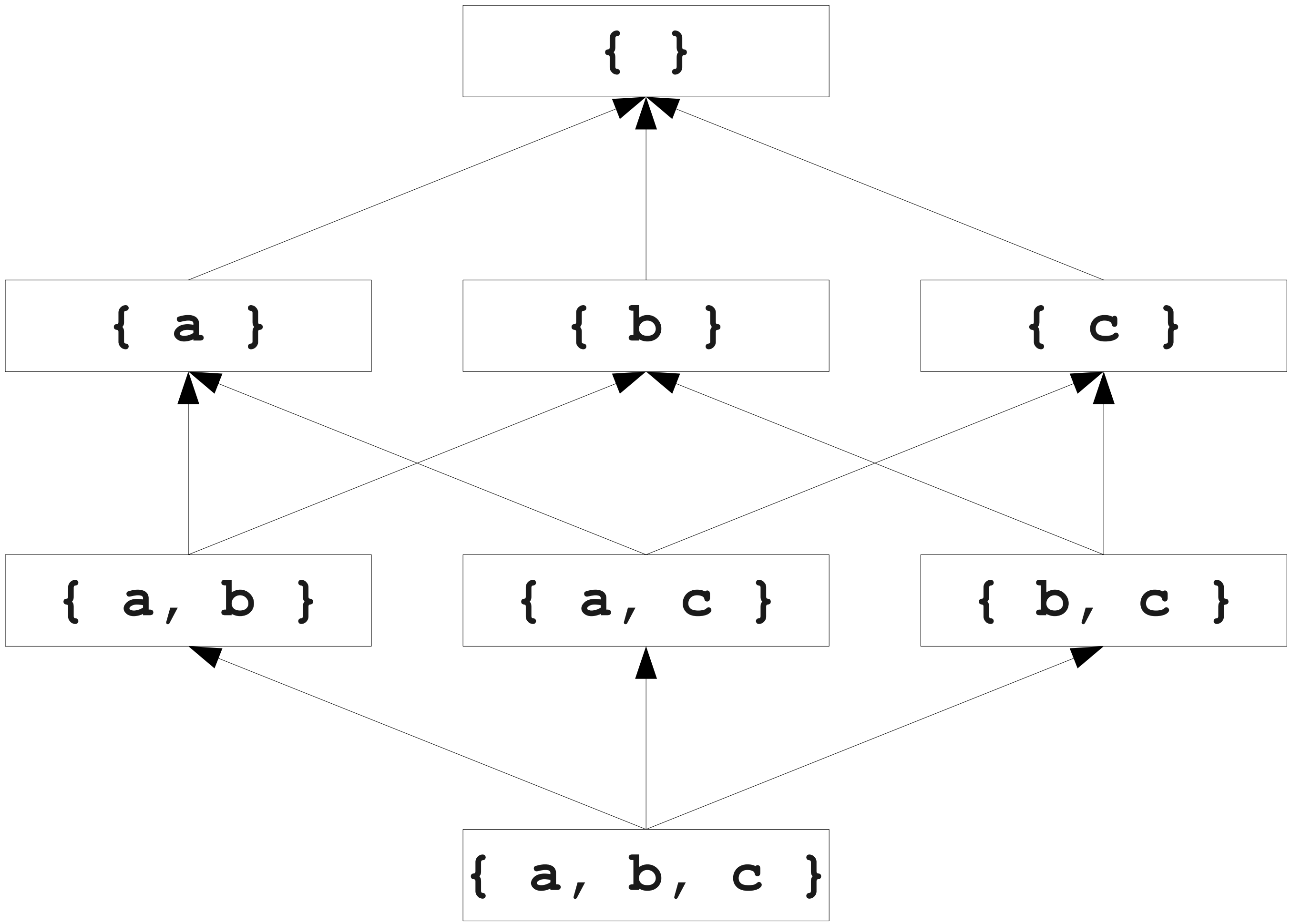
Review: Why Global Analysis is Hard

- Need to be able to handle multiple predecessors/successors for a basic block.
- Need to be able to handle multiple paths through the control-flow graph, and may need to iterate multiple times to compute the final value (but the analysis still needs to terminate!)
- Need to be able to assign each basic block a reasonable default value for before we've analyzed it.

Review: Meet Semilattices

- A **meet semilattice** is a ordering defined on a set of elements.
- Any two elements have some **meet** that is the largest element smaller than both elements.
- There is a unique **top element**, which is at least as large as any other element.
- Intuitively:
  - The meet of two elements represents combining information from two elements.
  - The top element element represents “no information yet.”

Meet Semilattices for Liveness



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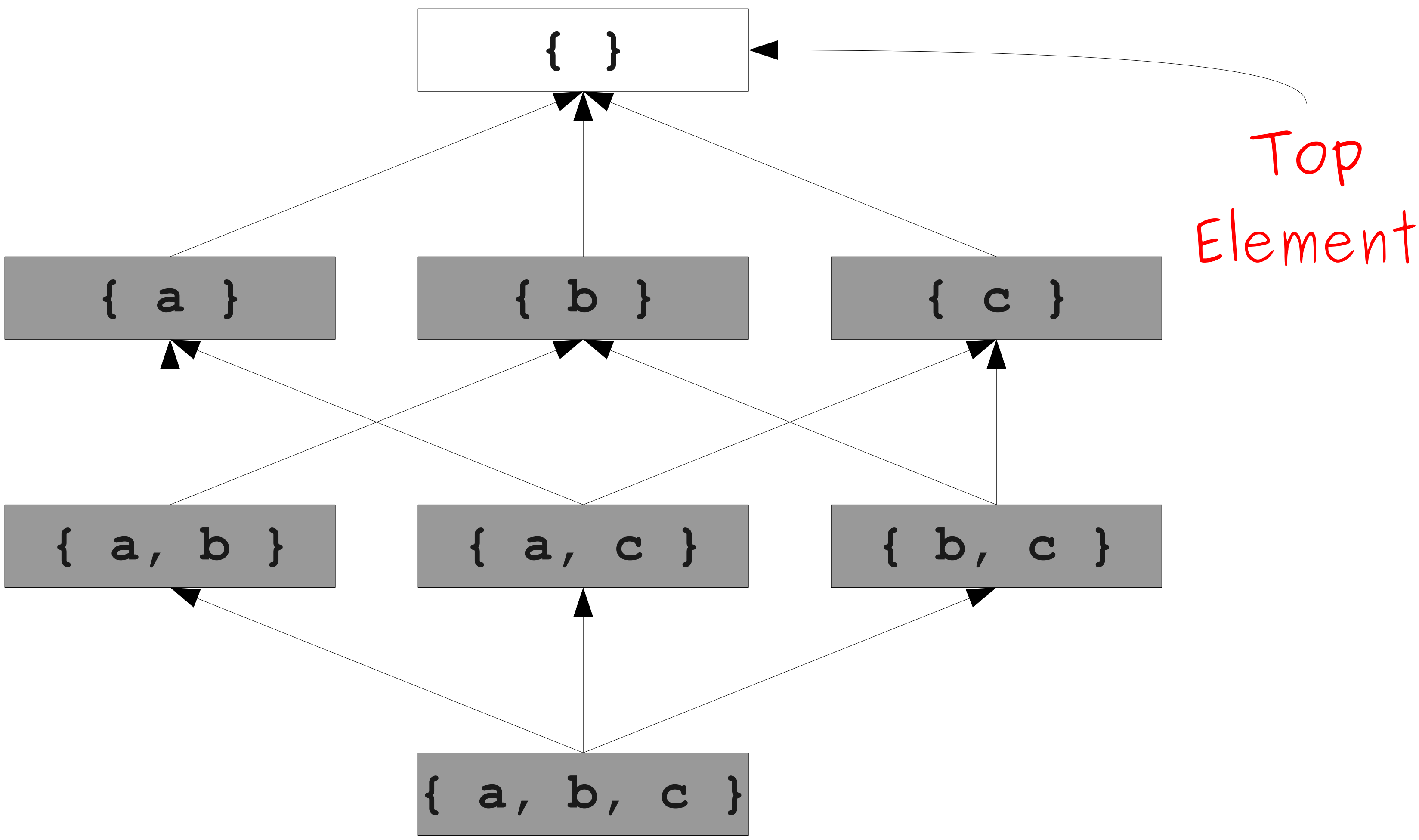

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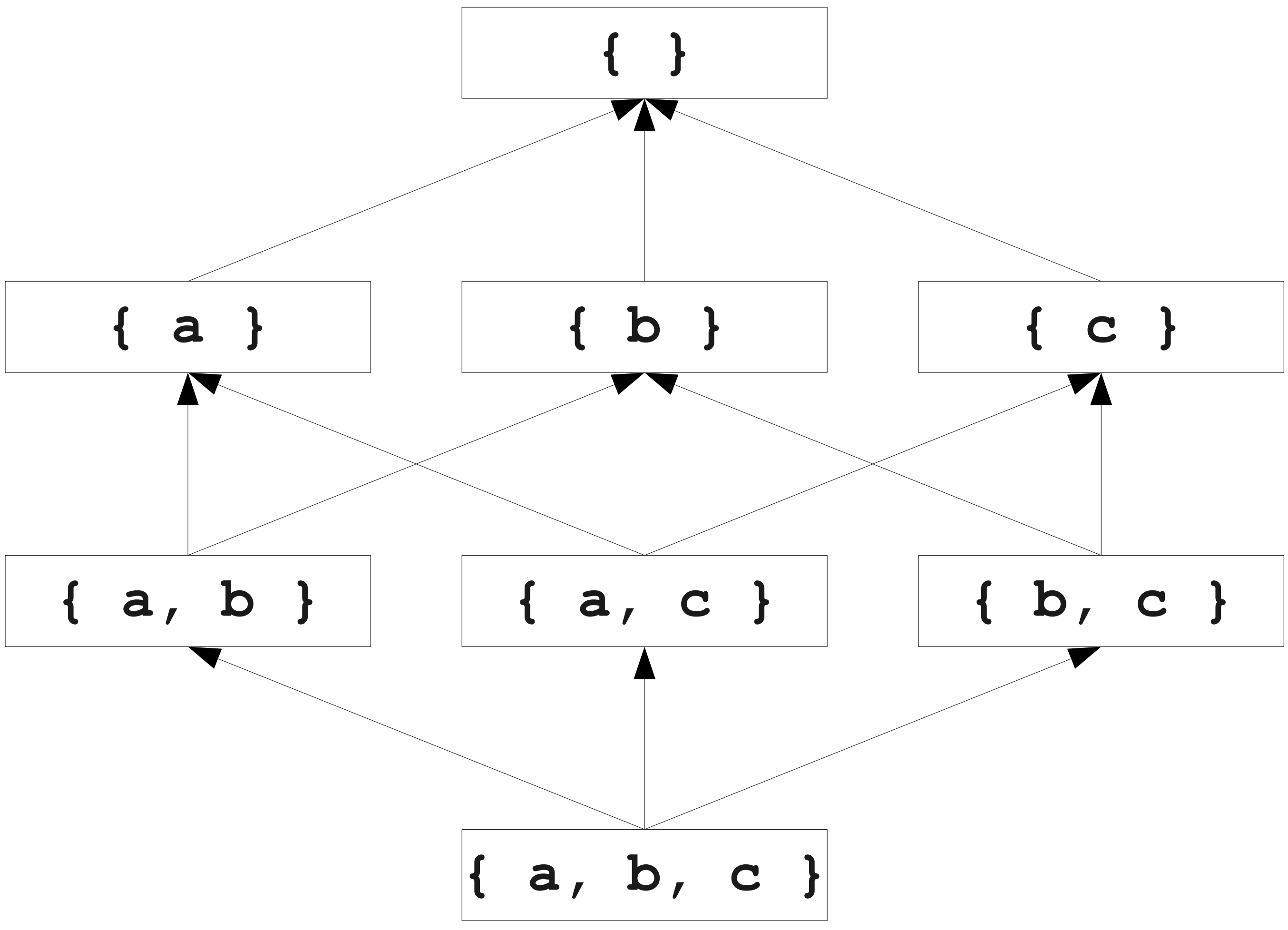
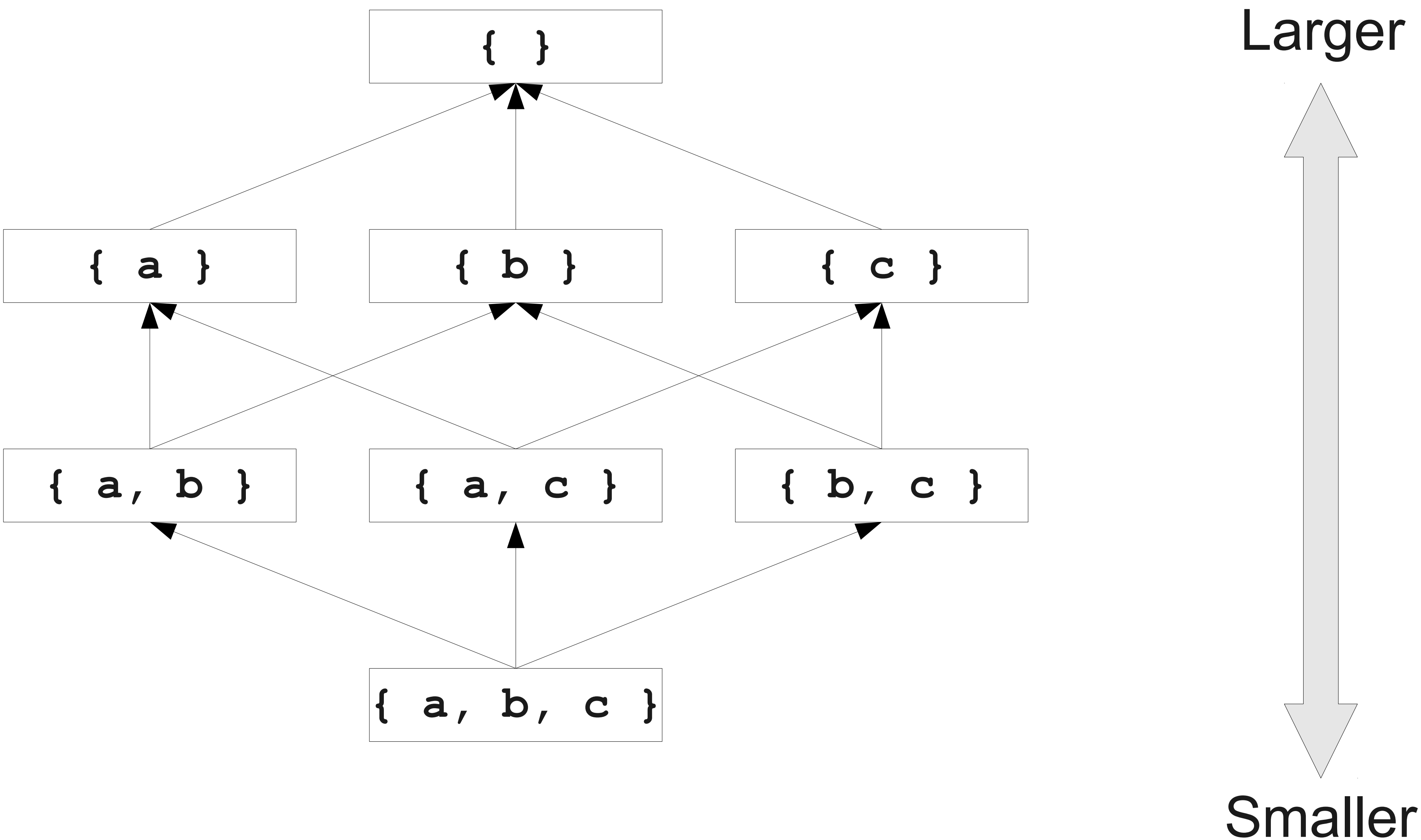


Review: Meet Semilattices

- A **meet semilattice** is a pair (D, ∧), where
  - D is a domain of elements.
  - ∧ is a **meet operator** that is
    - **idempotent:** $x \wedge x = x$
    - **commutative:** $x \wedge y = y \wedge x$
    - **associative:** $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
- If $x \wedge y = z$, we say that z is the **meet** or (**greatest lower bound**) of x and y.
- Every meet semilattice has a **top element** denoted $\top$ such that $\top \wedge x = x$ for all x.

Meet Semilattices and Orderings



Review: Orderings on Semilattices

- Every meet semilattice $(D, \wedge)$ induces an ordering relationship $\leq$ over its elements.
- Define $\mathbf{x \leq y}$ **iff** $\mathbf{x \wedge y = x}$

An Example Semilattice

- The set of natural numbers and the **max** function.
- Idempotent
  - **max**{a, a} = a
- Commutative
  - **max**{a, b} = **max**{b, a}
- Associative
  - **max**{a, **max**{b, c}} = **max**{**max**{a, b}, c}
- Top element is 0:
  - **max**{0, a} = a
- Ordering relationship over this lattice:
  - $x \leq y$ iff $x \wedge y = x$ iff **max**{x, y} = x iff x is larger than y.

A Semilattice for Liveness

- Sets of live variables and the set union operation.
- Idempotent:
  - $x \cup x = x$
- Commutative:
  - $x \cup y = y \cup x$
- Associative:
  - $(x \cup y) \cup z = x \cup (y \cup z)$
- Top element:
  - The empty set: $\{ \} \cup x = x$
- Ordering relationship over this lattice:
  - $x \leq y$ iff $x \wedge y = x$ iff $x \cup y = x$ iff $x \supseteq y$.

Semilattices and Program Analysis

- Semilattices naturally solve many of the problems we encounter in global analysis.
- How do we combine information from multiple basic blocks?
  - Use the meet of all of those blocks' information.
- What value do we give to basic blocks we haven't seen yet?
  - Use the top element.
- How do we know that the algorithm always terminates?
  - Actually, we still don't! More on that later.

A General Framework

- A global analysis is a tuple (D, V, ∧, F, I), where
  - **D** is a direction (forward or backward)
    - The order to visit statements **within** a basic block, not the order in which to visit the basic blocks.
  - **V** is a set of values.
  - **∧** is a meet operator over those values.
  - **F** is a set of transfer functions $f : V \to V$
  - **I** is an initial value.
- The only difference from local analysis is the introduction of the meet operator.

Running Global Analyses

- Assume that this is a forward analysis.
- Set OUT[**s**] = ⊤ for all statements **s**.
- Set OUT[**begin**] = I.
- Repeat until no values change:
  - For each statement **s** with predecessors $\mathbf{p_1}$, $\mathbf{p}_2$, … , $\mathbf{p_n}$:
    - Set IN[**s**] = $\mathbf{p_1} \wedge \mathbf{p_2} \wedge \ldots \wedge \mathbf{p_n}$
    - Set OUT[**s**] = f_s (IN[**s**])
- The order of this iteration does not matter.

For Comparison

- Set OUT[**s**] = ⊤ for all stmts **s**.
- Set OUT[**begin**] = I.
- Repeat until no values change:
  - For each statement **s** with predecessors $\mathbf{p_1}$, $\mathbf{p}_2$, … , $\mathbf{p_n}$:
    - Set IN[**s**] = $\mathbf{p_1} \wedge \mathbf{p_2} \wedge \ldots \wedge \mathbf{p_n}$
    - Set OUT[**s**] = f_s (IN[**s**])

- Set IN[**s**] = { } for each stmt **s**.
- Set IN[**exit**] to the set of variables known to be live on exit.
- Repeat until no changes occur:
  - For each statement **s** of the form **a** = **b** + **c**
    - Set OUT[**s**] to set union of IN[**p**] for each successor **p** of **s**.
    - Set IN[**s**] to (OUT[**s**] – **a**) ∪ {**b**, **c**}.

The Dataflow Framework

- This form of analysis is called the **dataflow framework**.
- Can be used to easily prove an analysis is sound.
- With certain restrictions, can be used to prove that an analysis eventually terminates.
  - Again, more on that later.

Global Constant Propagation

- **Constant propagation** is an optimization that replaces each variable that is known to be a constant value with that constant.
- An elegant example of the dataflow framework.

Properties of Constant Propagation

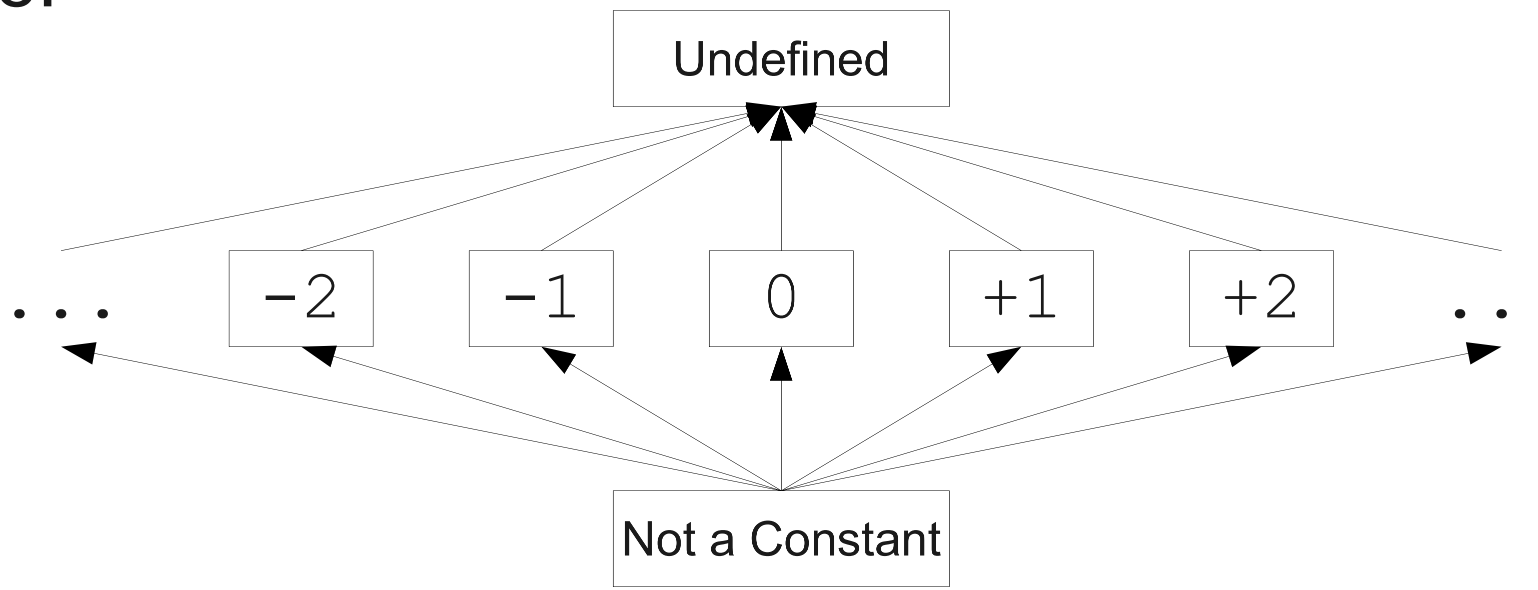
- For now, consider just some single variable **x**.
- At each point in the program, we know one of three things about the value of **x**:
  - **x** is definitely not a constant, since it's been assigned two values or assigned a value that we know isn't a constant.
  - **x** is definitely a constant and has value **k**.
  - We have never seen a value for **x**.
- Note that the first and last of these are **not** the same!
  - The first one means that there may be a way for **x** to have multiple values.
  - The last one means that **x** never had a value at all.

Defining a Meet Operator

- The meet of **Undefined** and any other value is that other value.
  - (If **x** has no value on some path and does have a value on some other path, we can just pretend it always had the assigned value.)
- The meet of **Not a Constant** and any other value **Not a Constant**.
  - (If on some path the value is known not to be a constant, then on entry to a statement its value can't possibly be a constant.)
- The meet of any two different constants is **Not a Constant**.
  - (If the variable might have two different values on entry to a statement, it cannot be a constant.)

A Semilattice for Constant Propagation

- One possible semilattice for this analysis is shown here:



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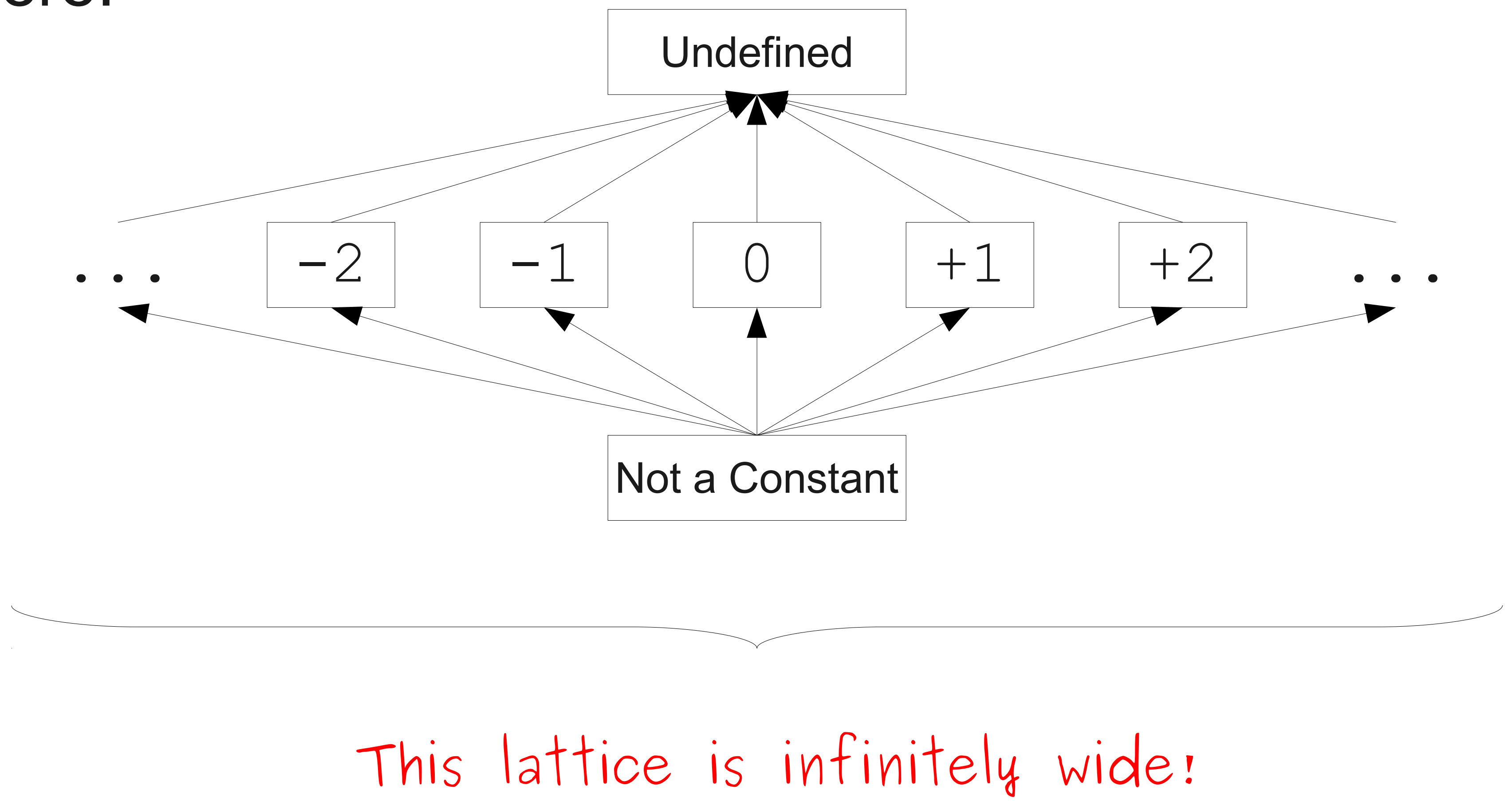

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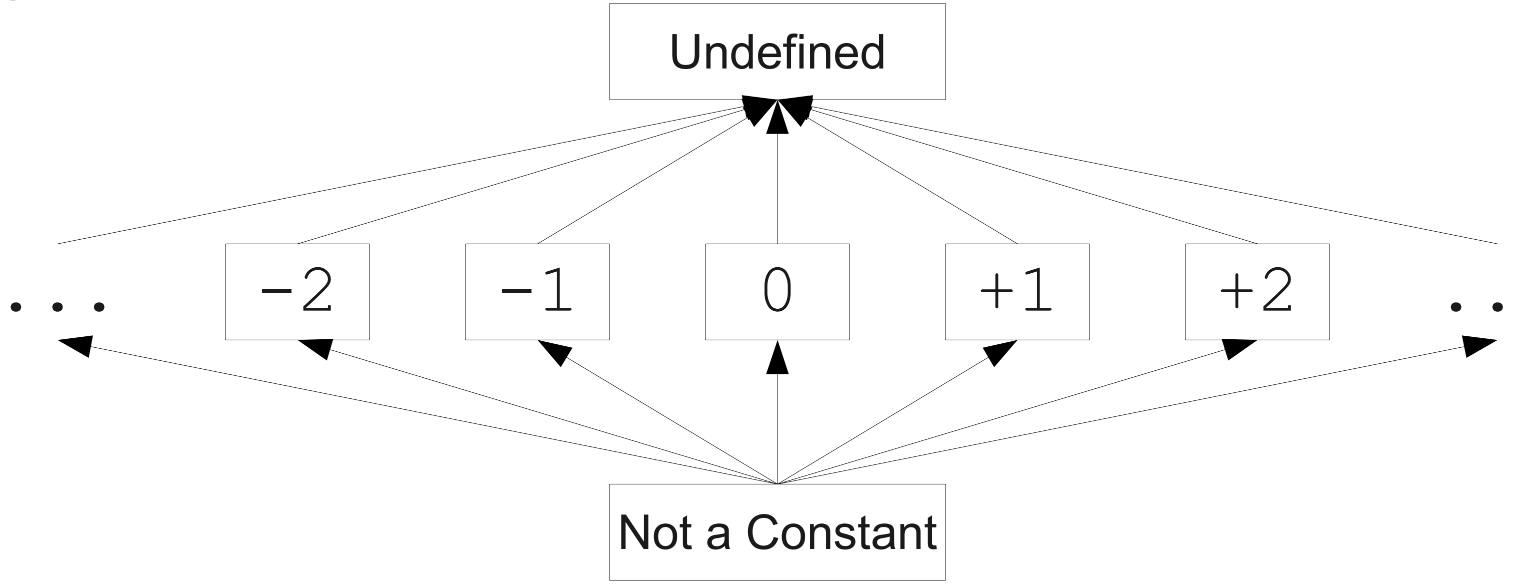

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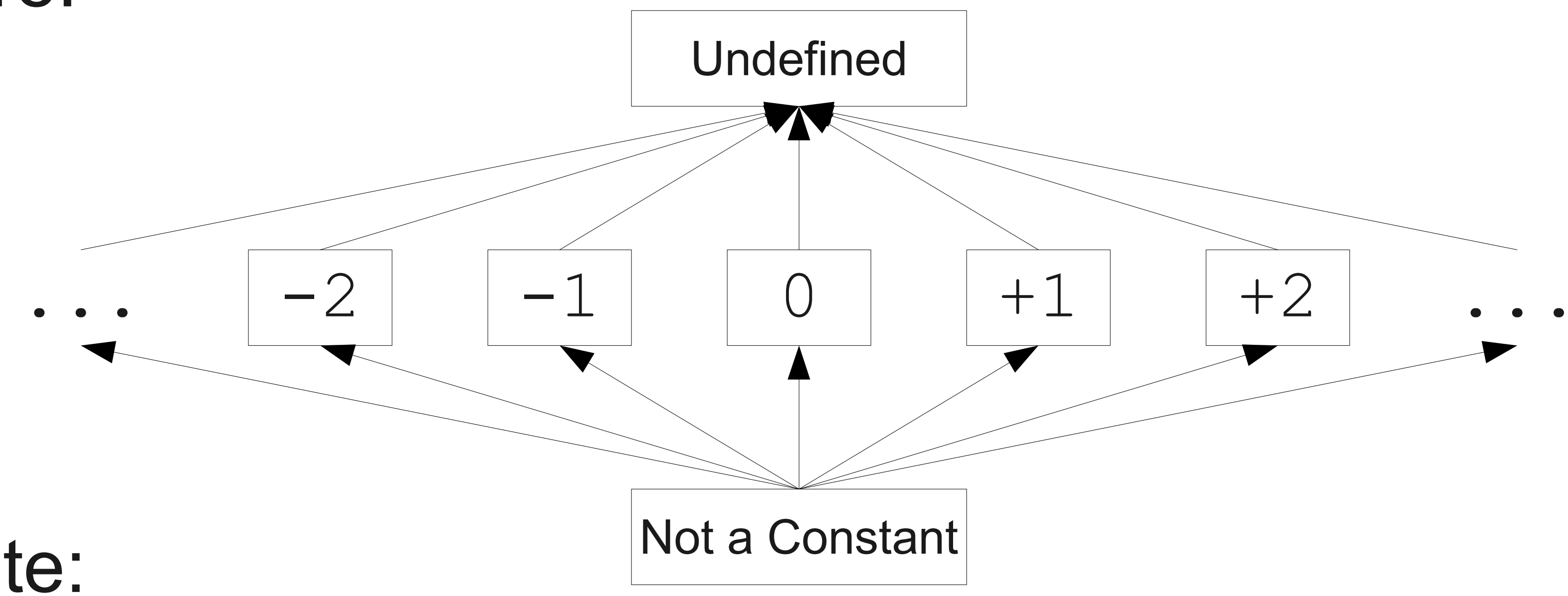
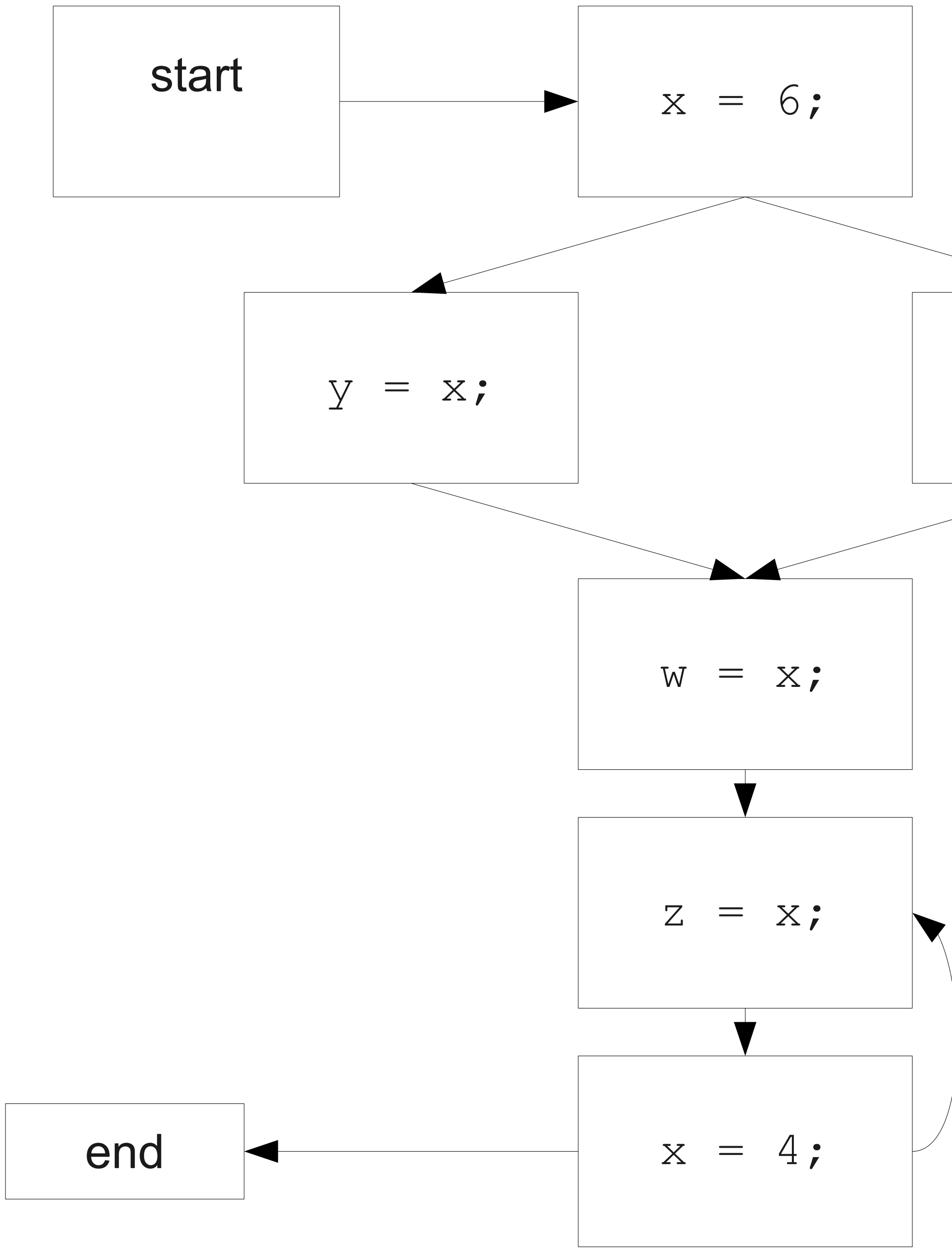


- Note:
  - The meet of any two different constants is **Not a Constant**.
  - The meet of **Undefined** and any value is that value.
  - The meet of **Not a Constant** and any value is **Not a Constant**.

Dataflow for Constant Propagation

- Direction: **Forward**
- Semilattice: **Defined on previous slide**
- Transfer functions:
  - $f_{x=k}(V) = k$ *(assigning a constant)*
  - $f_{x=a+b}(V) =$ Not a Constant *(assigning a non-constant)*
  - $f_{y=a+b}(V) = V$ *(unrelated assignment)*
- Initial value: **x is Undefined**
  - (When might we use some other value?)

Global Constant Propagation



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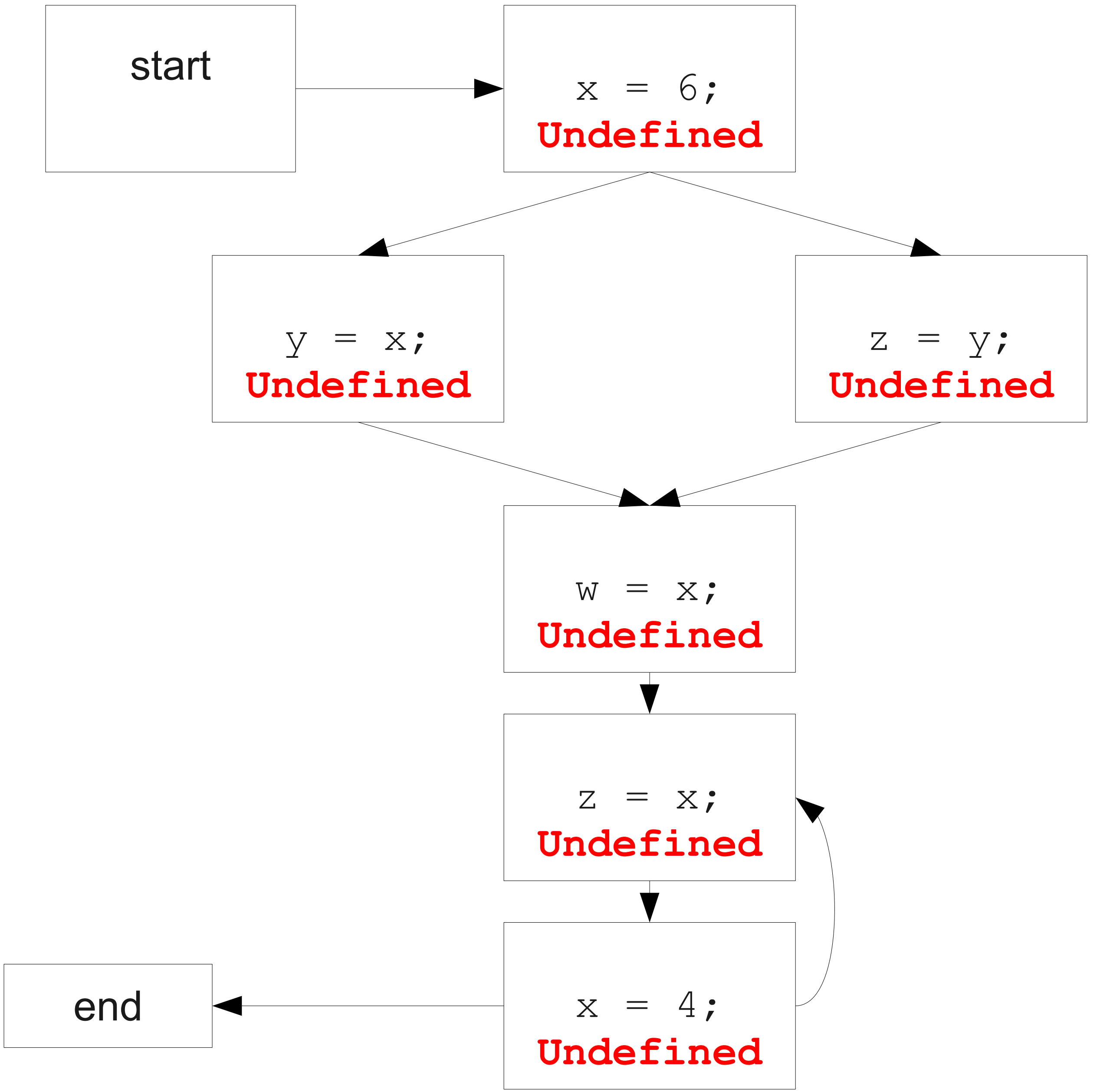

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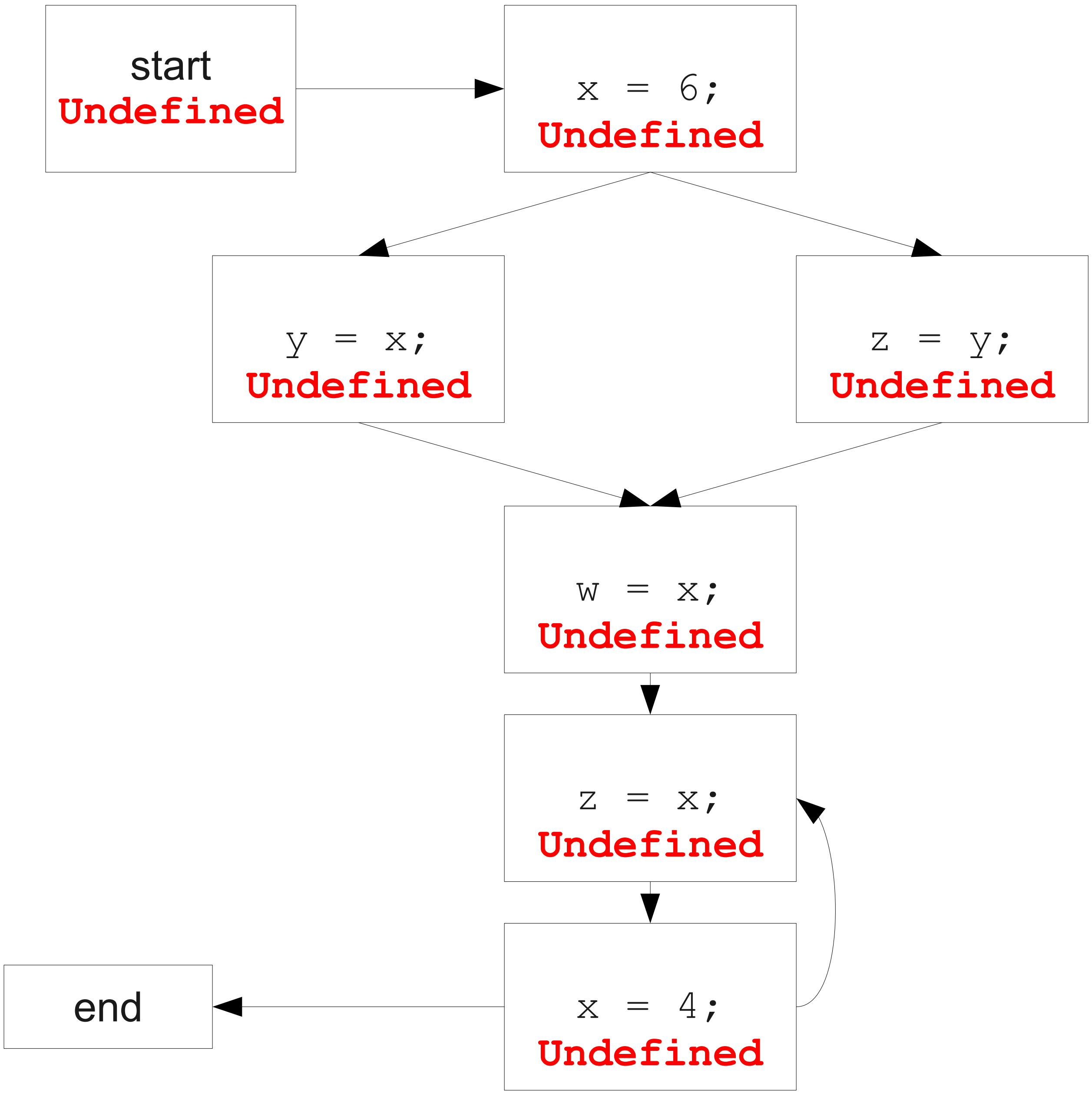

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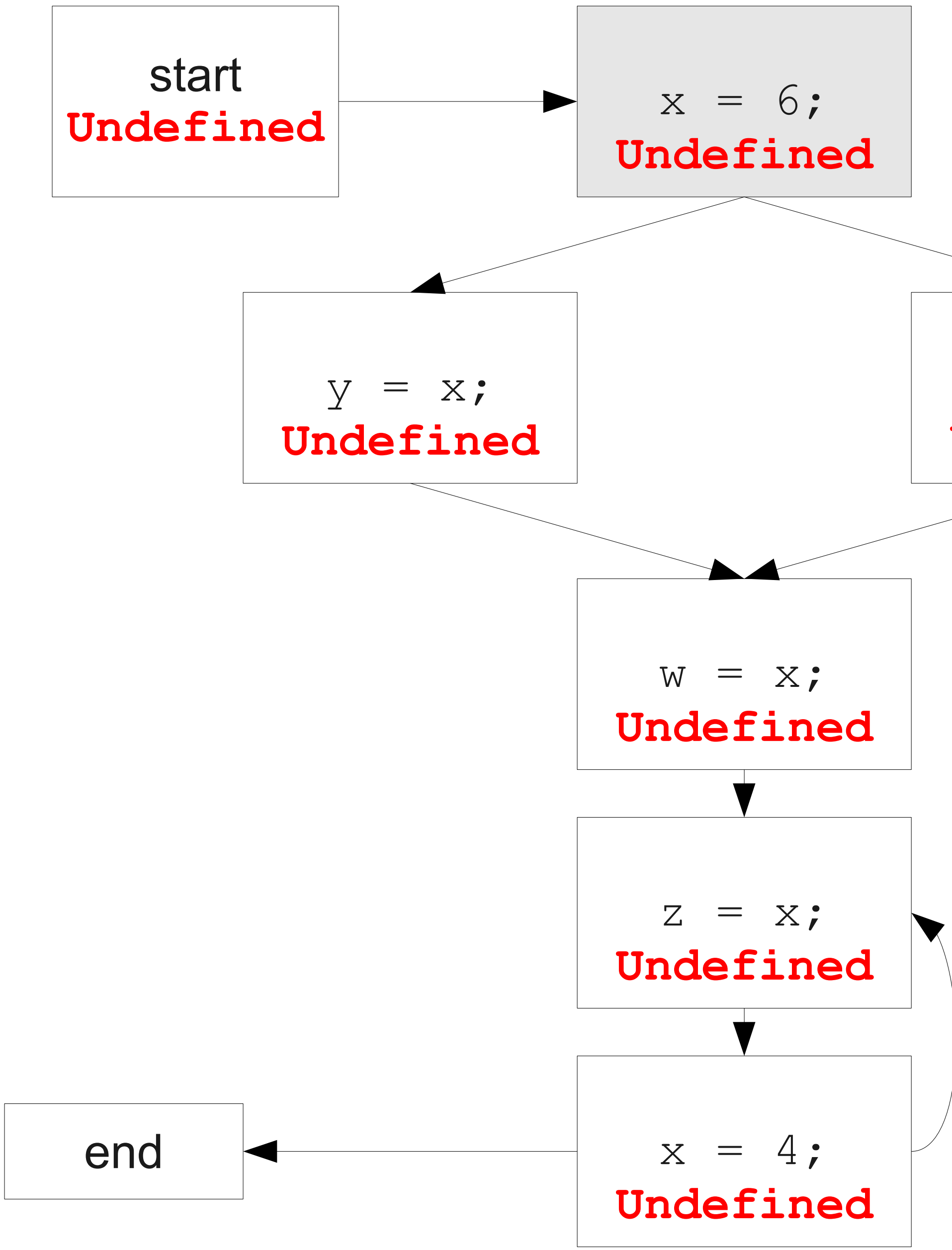

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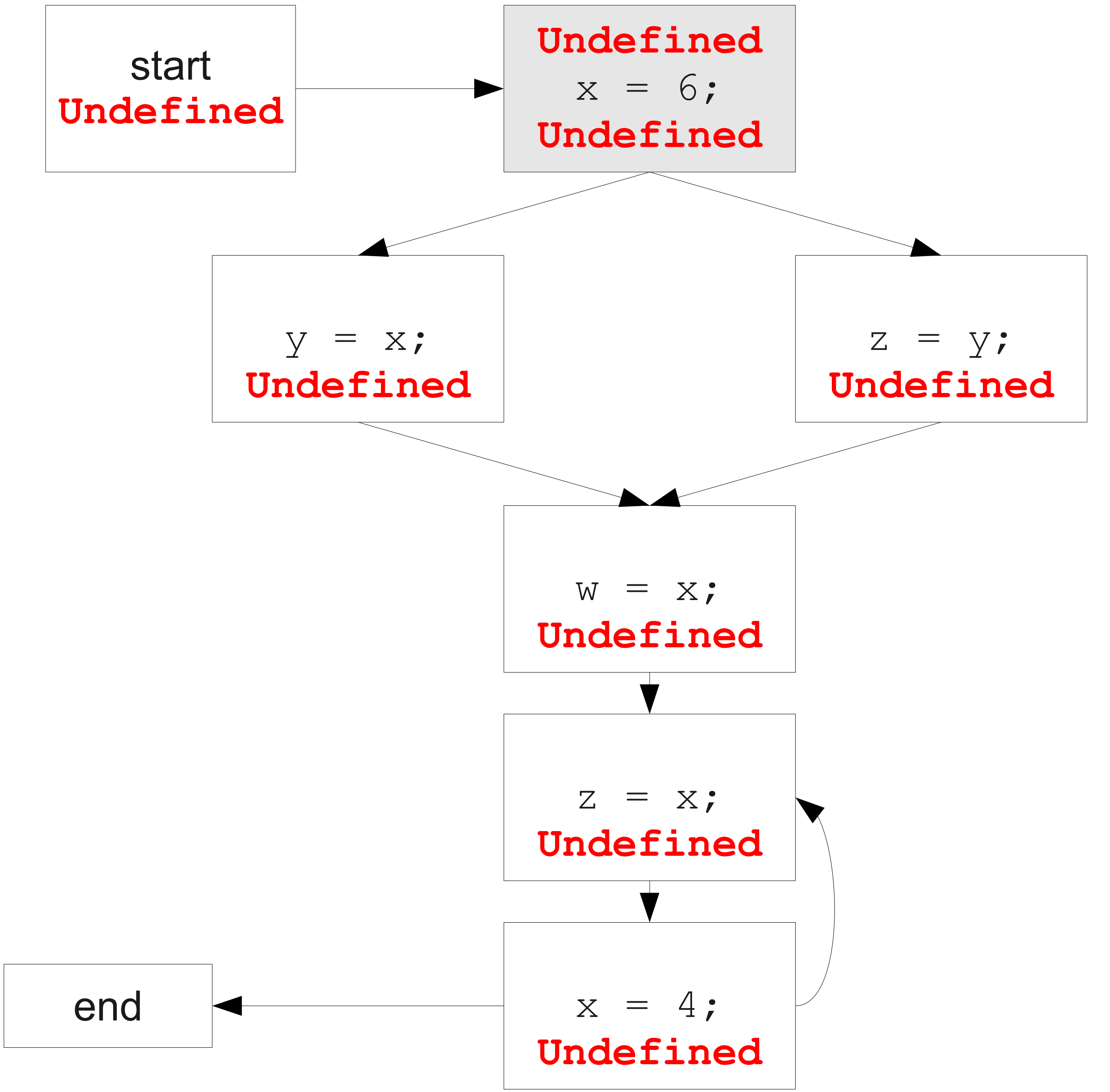

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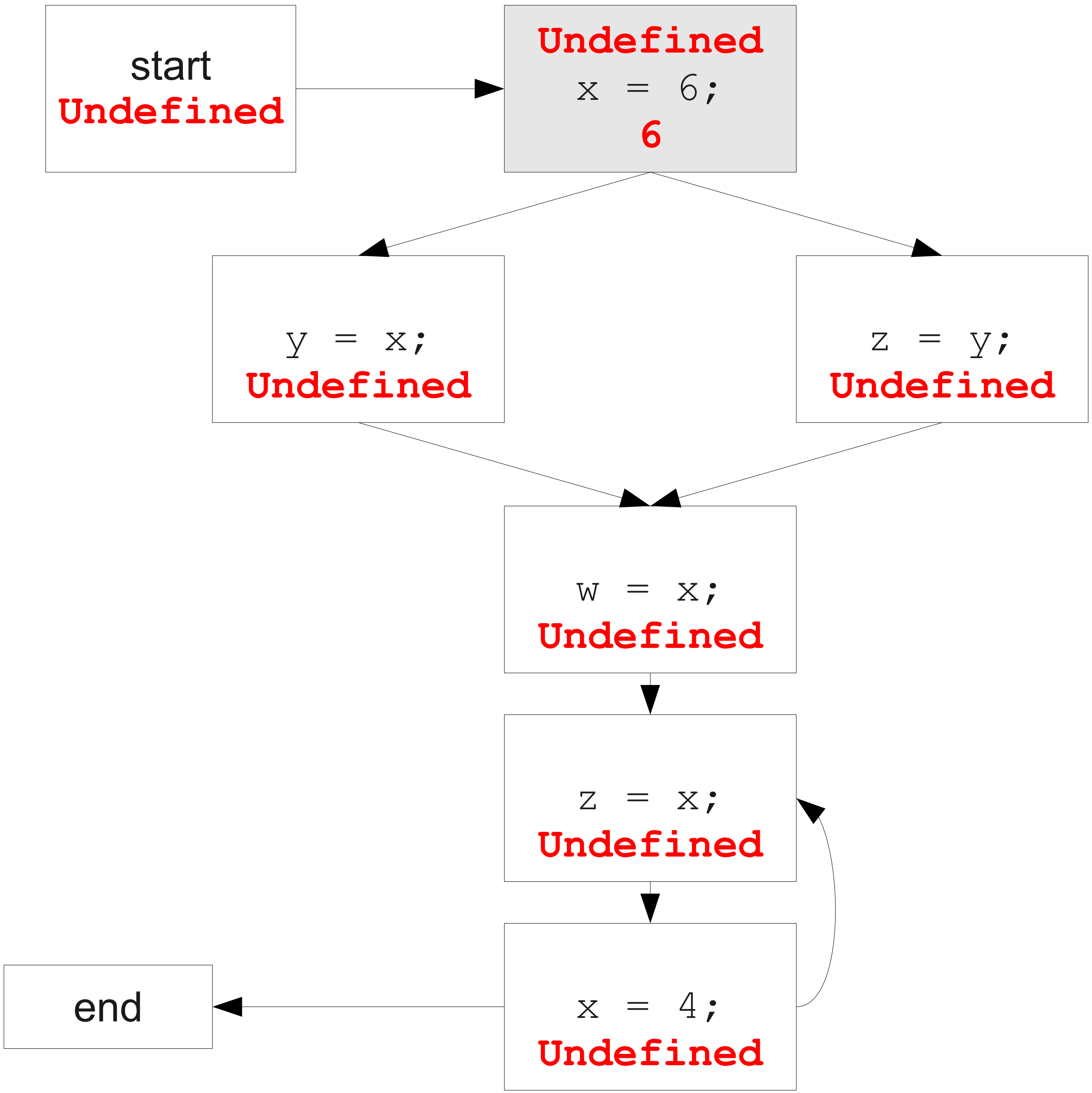

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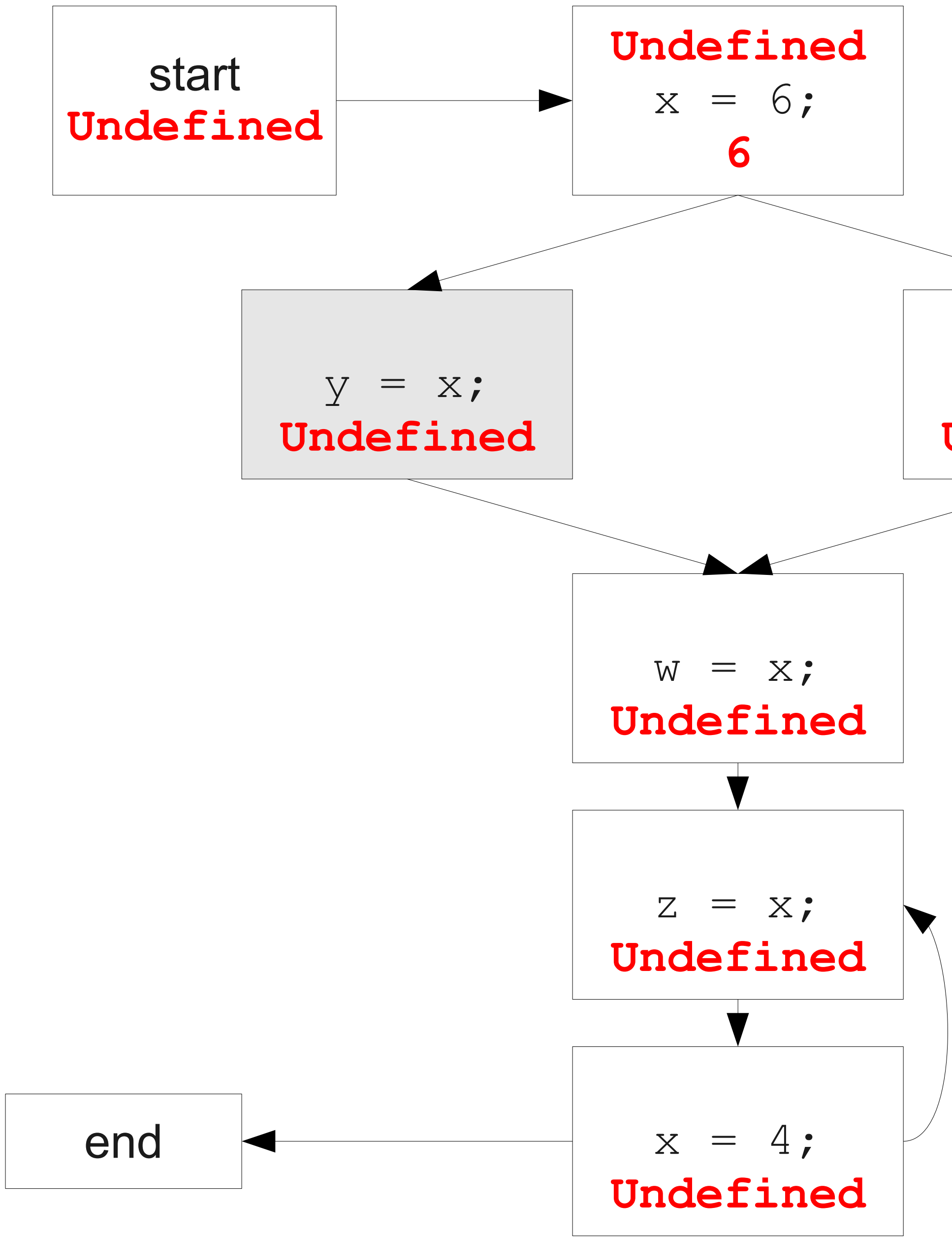

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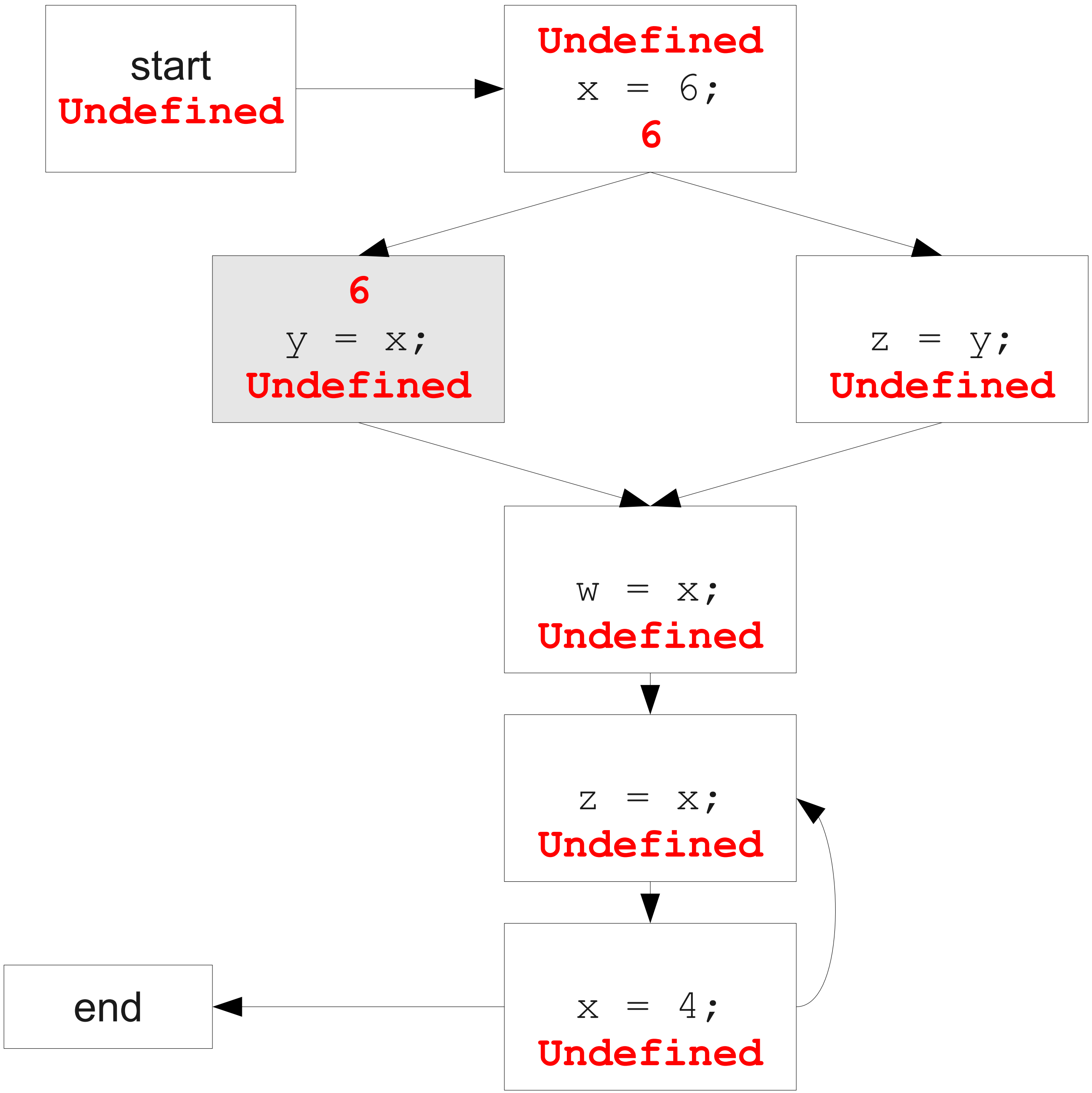

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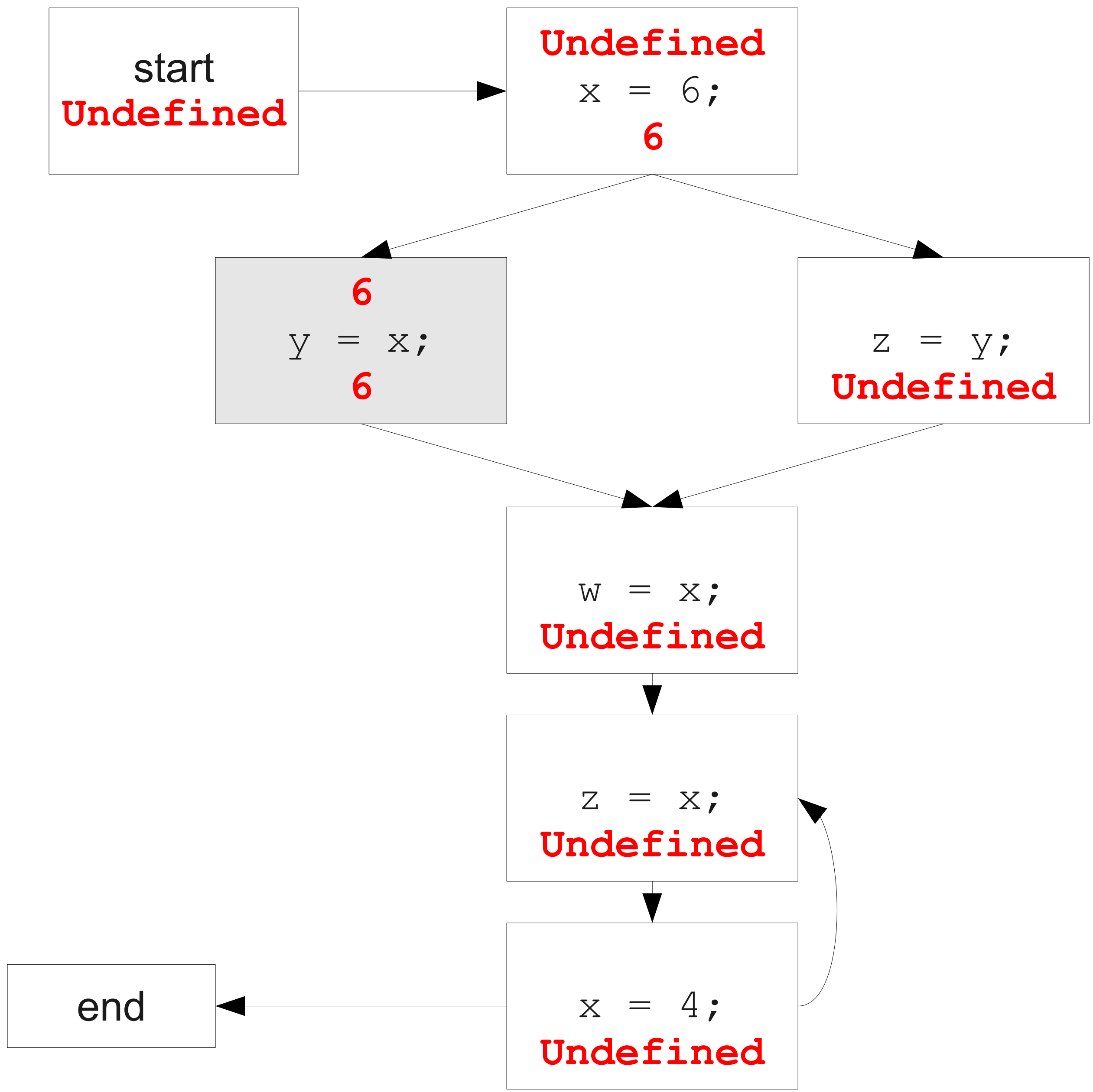

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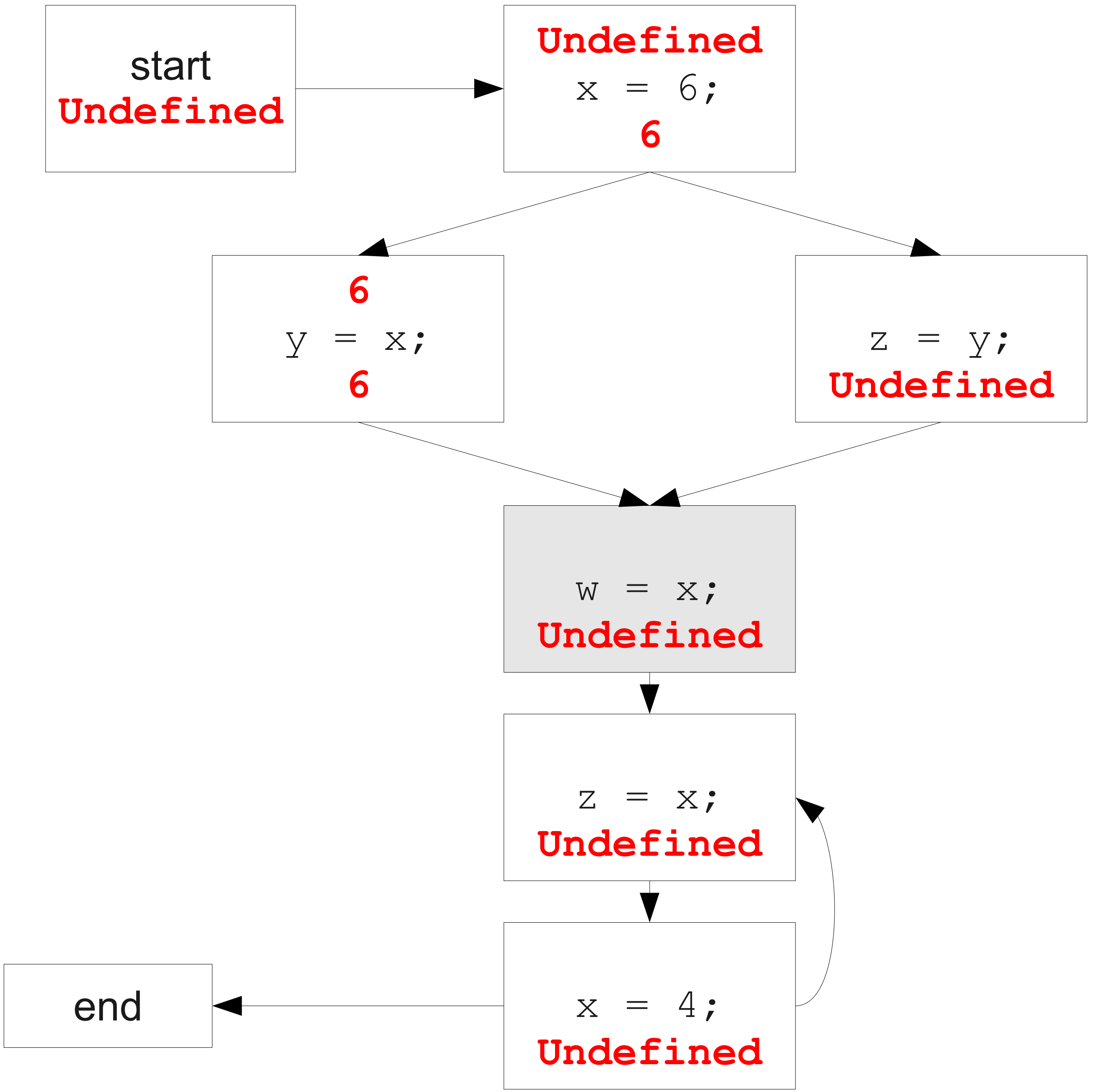

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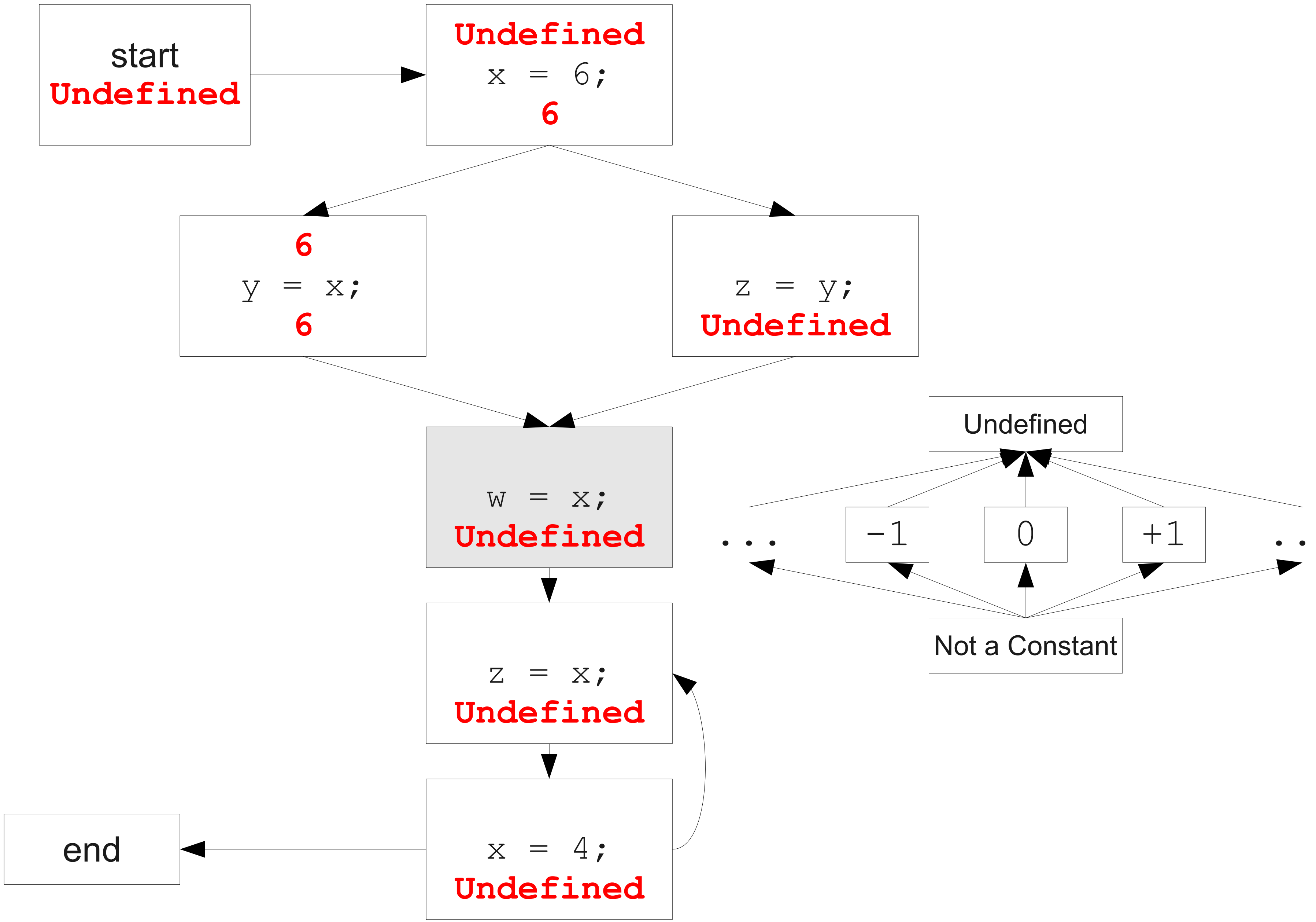

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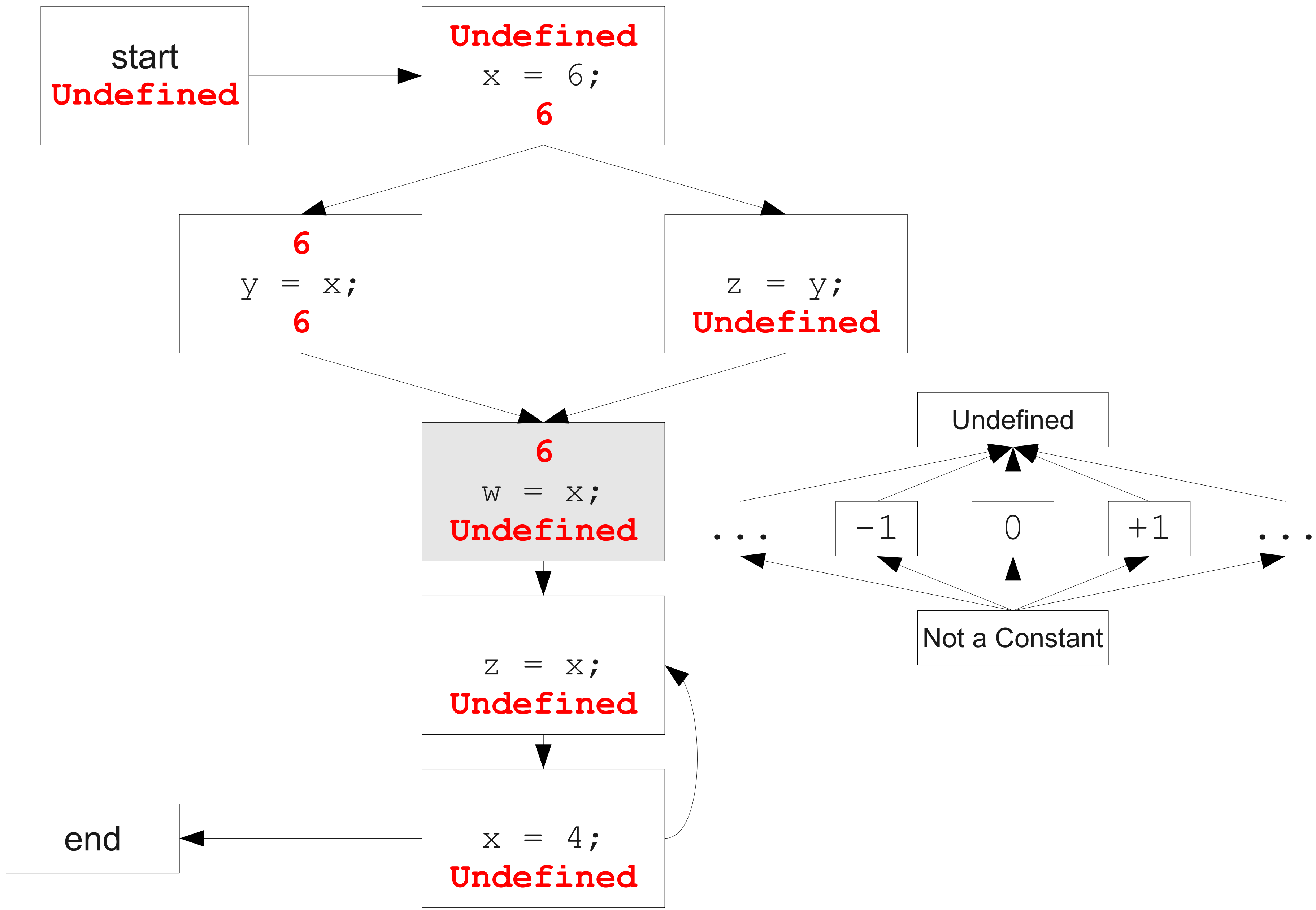

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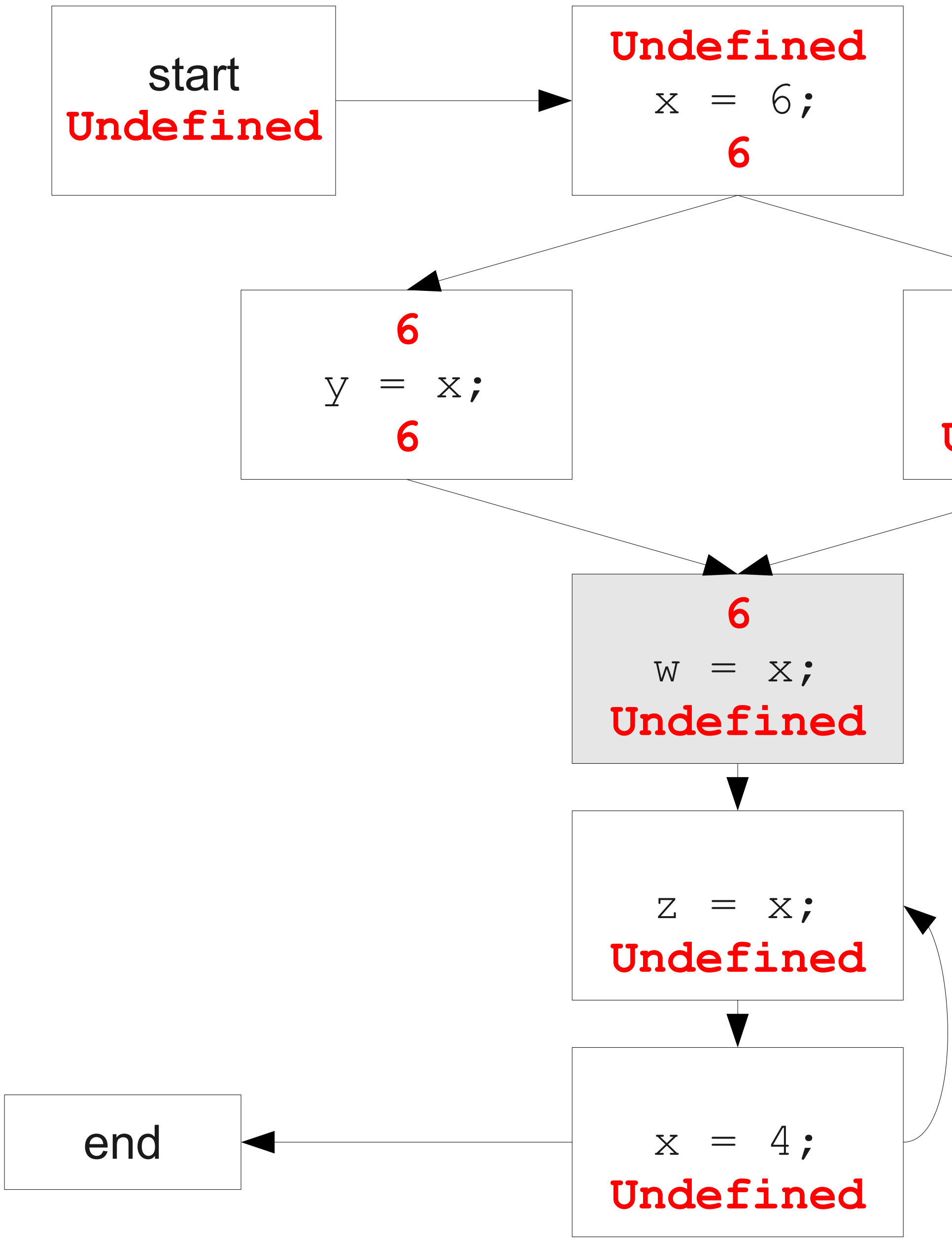

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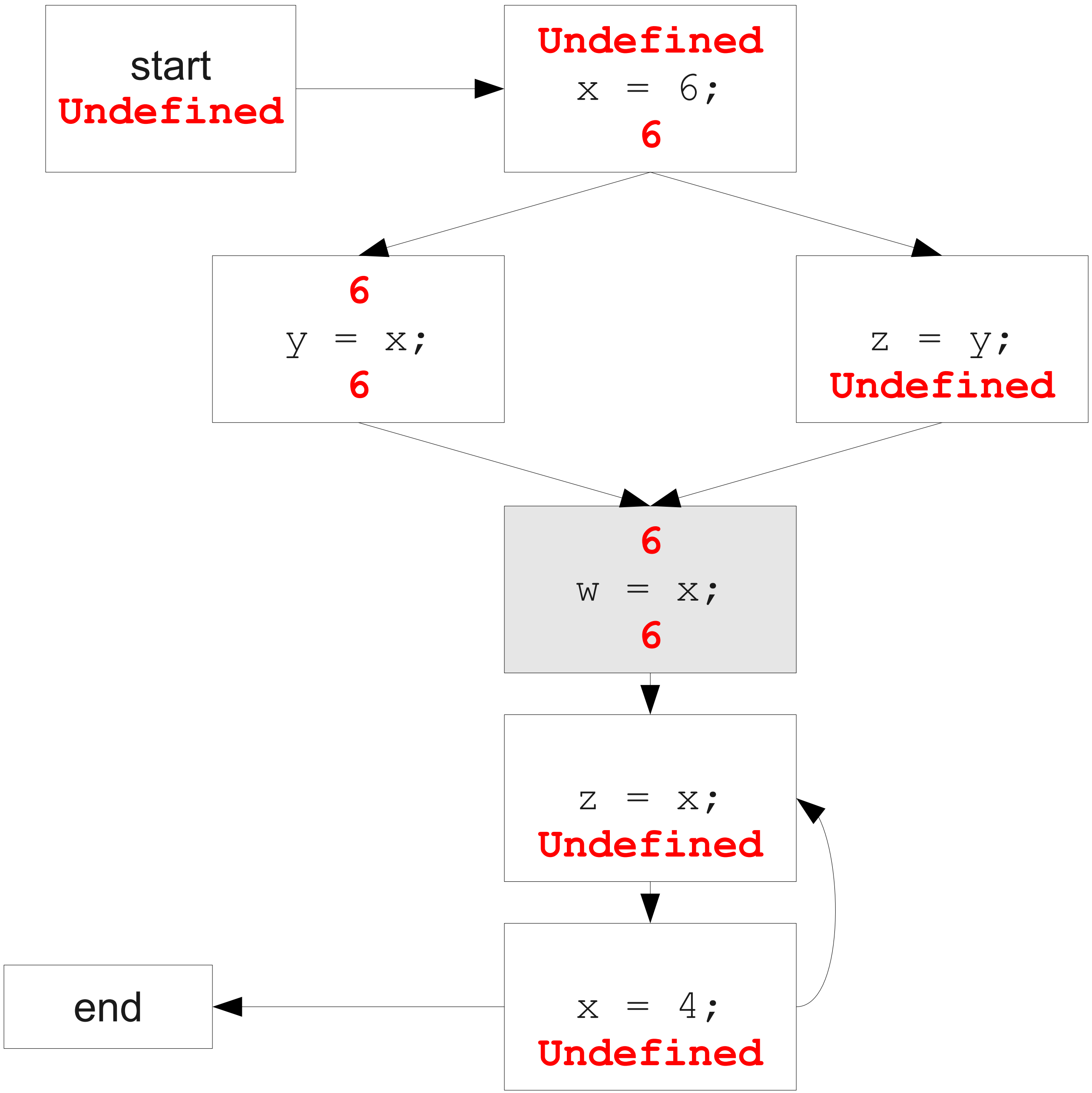

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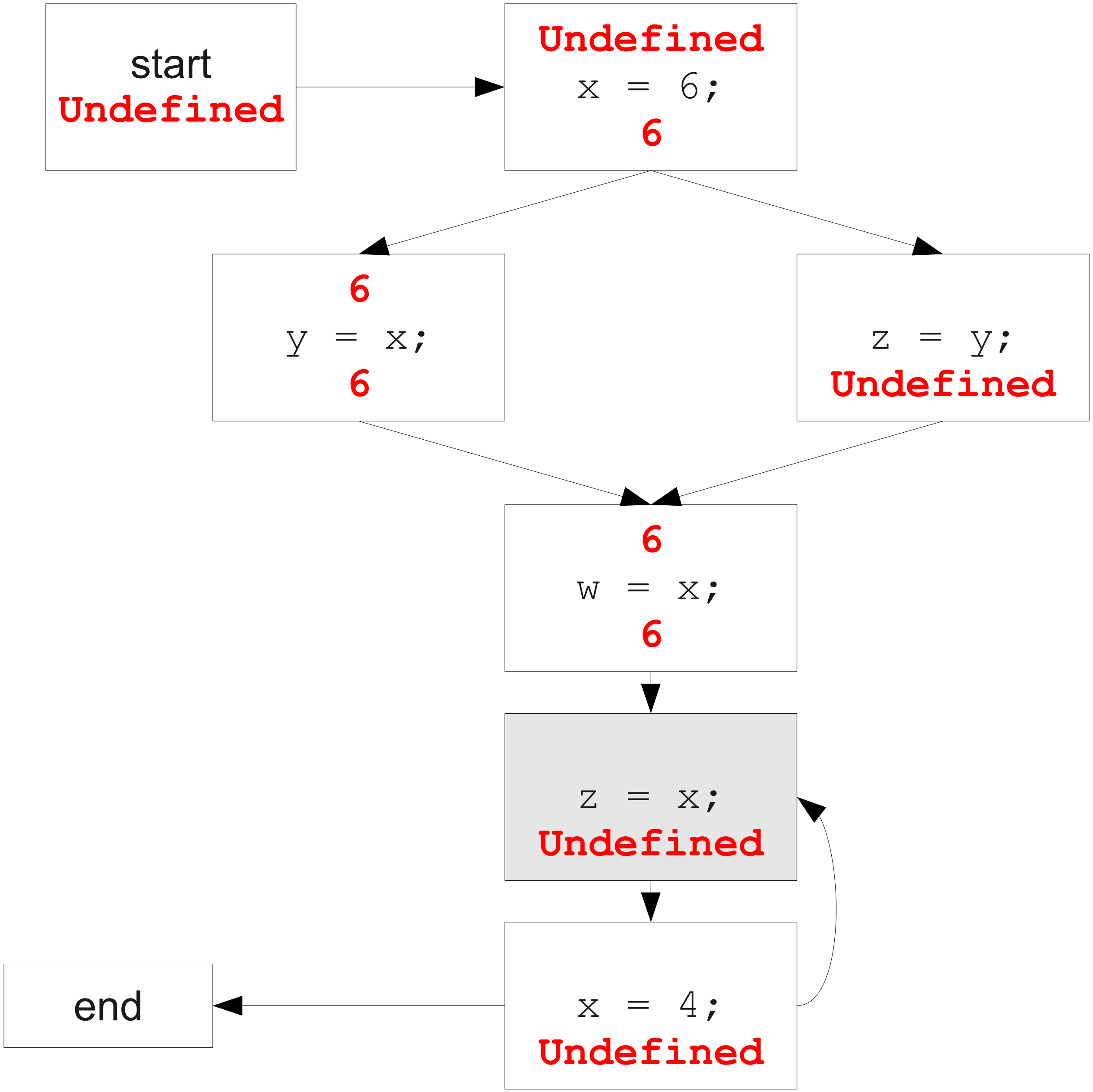

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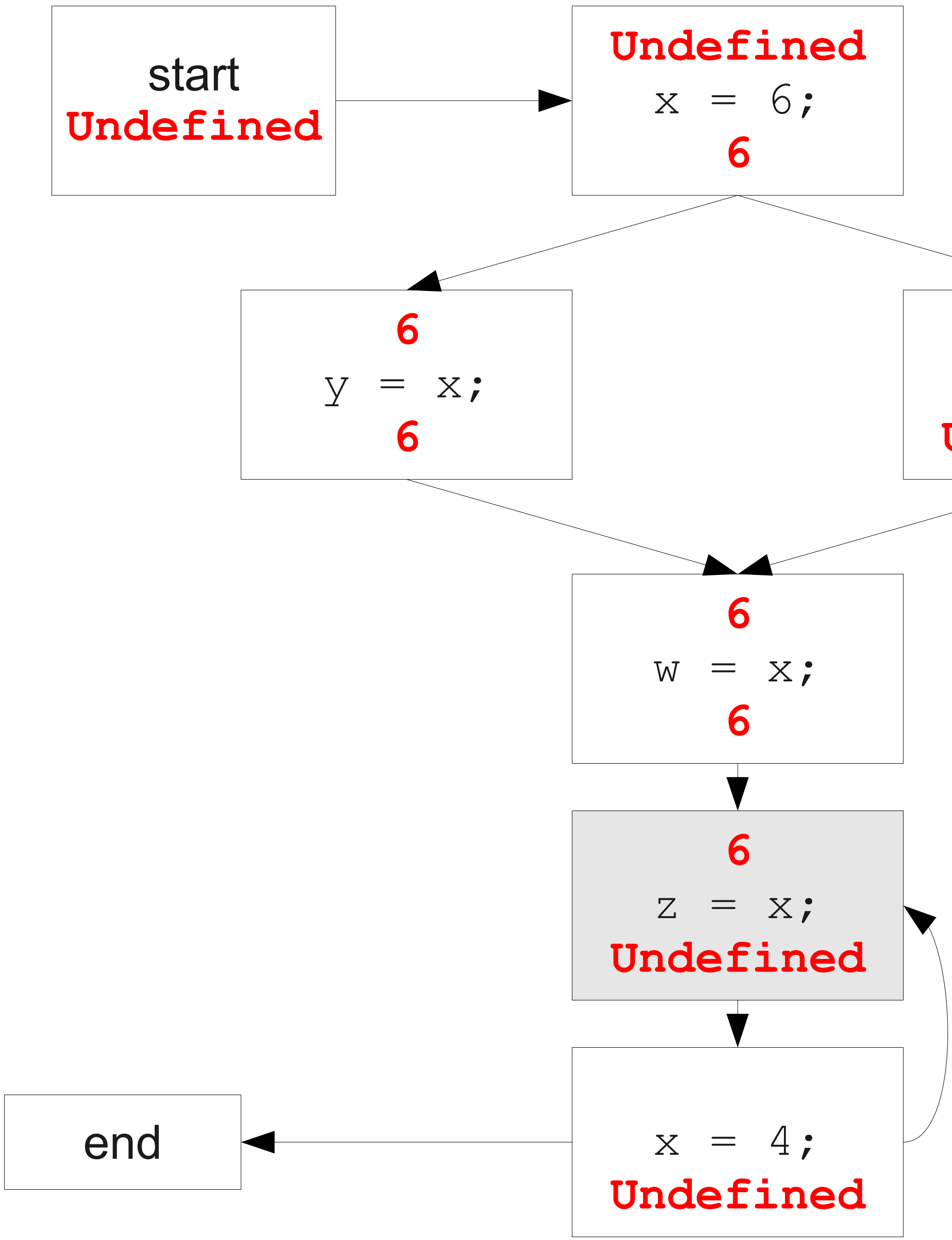

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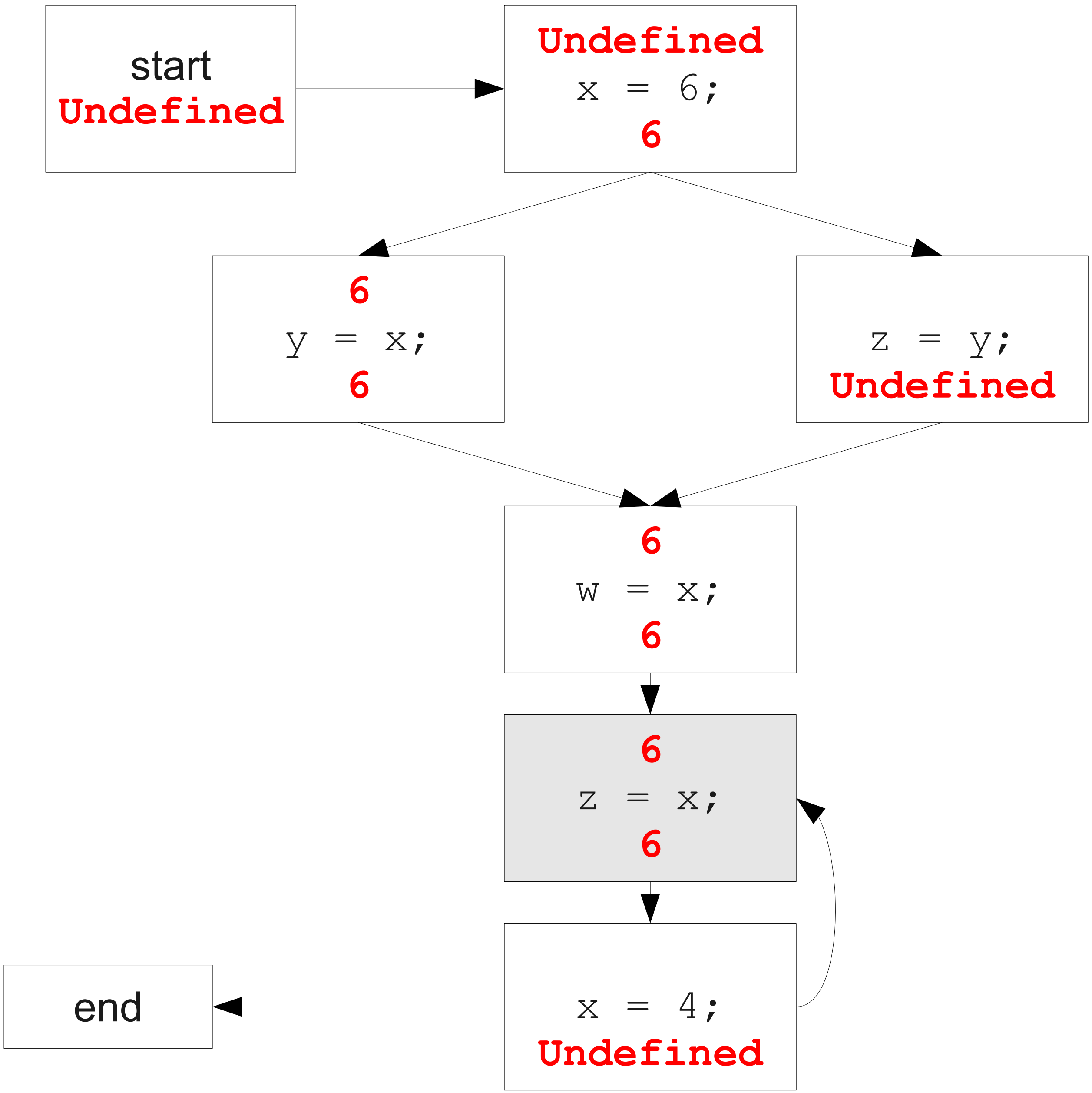

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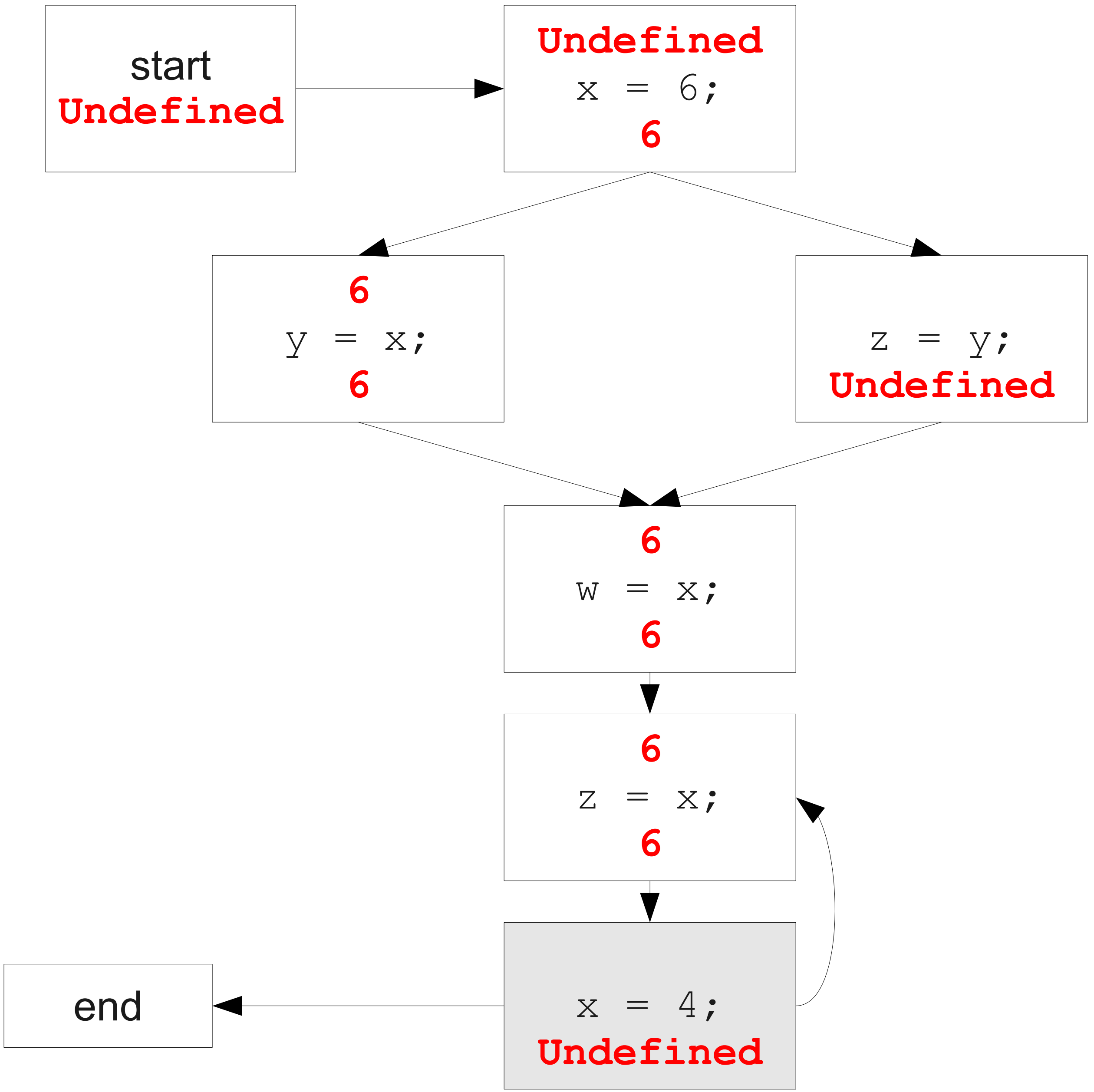

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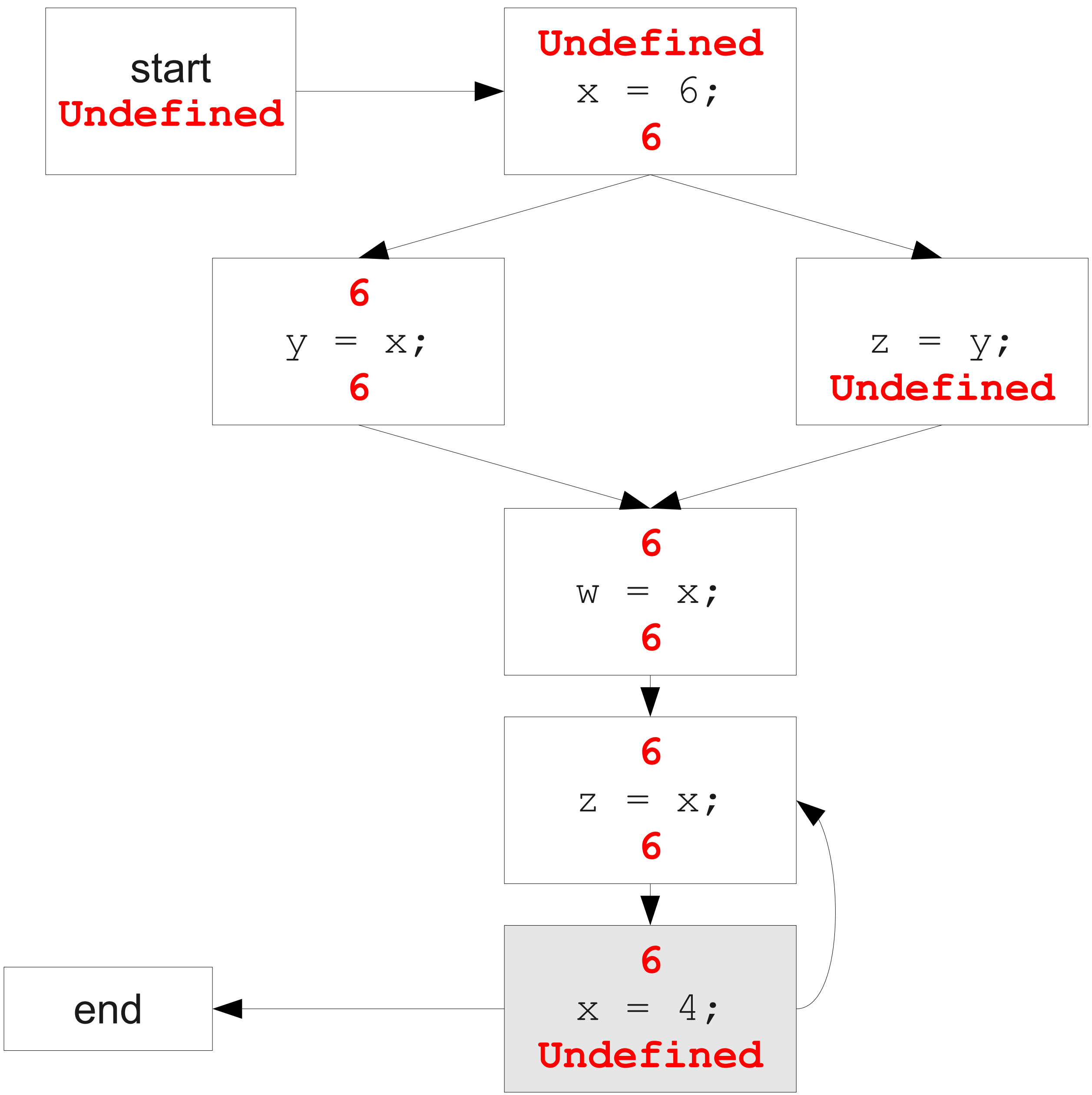

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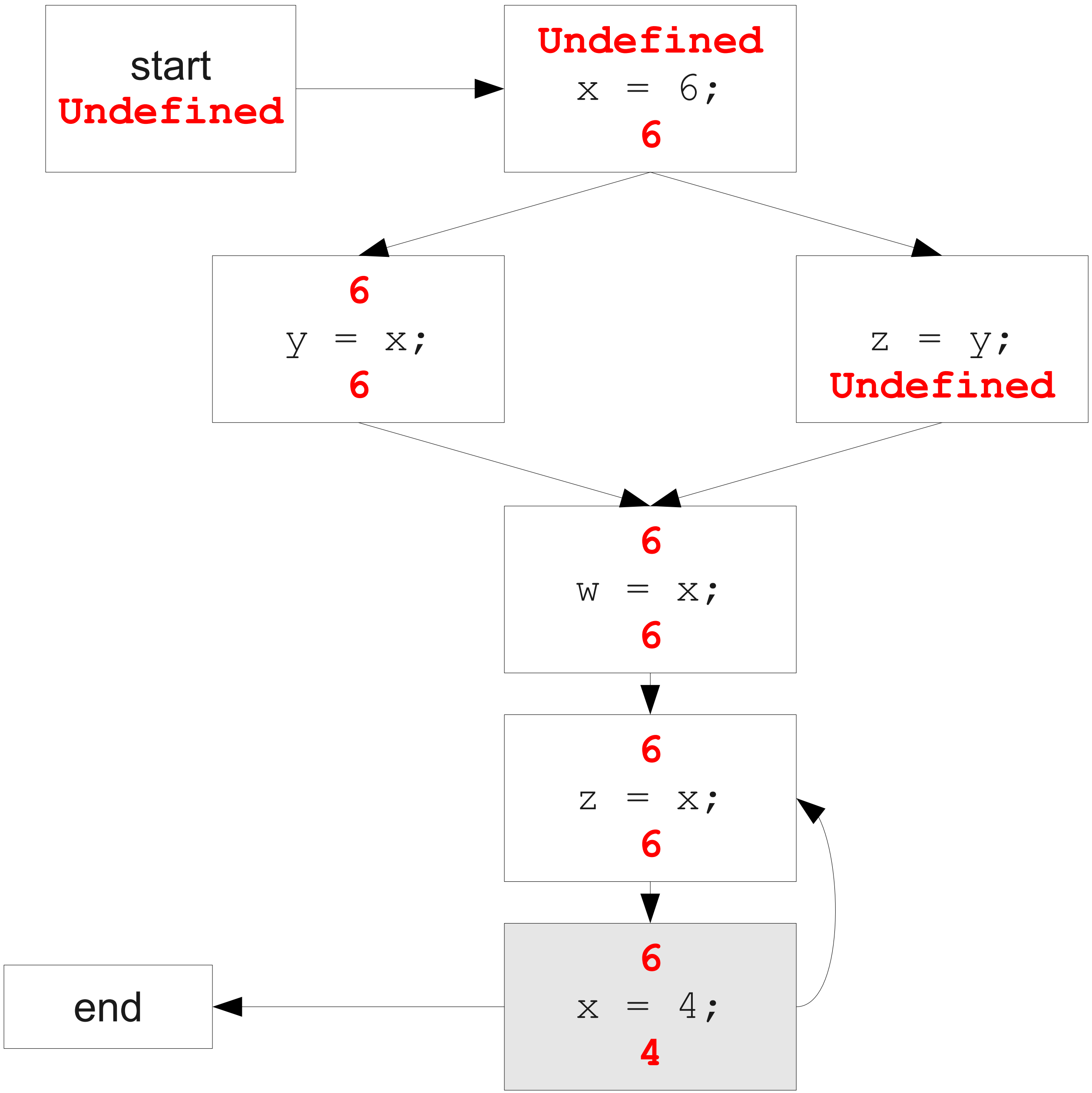

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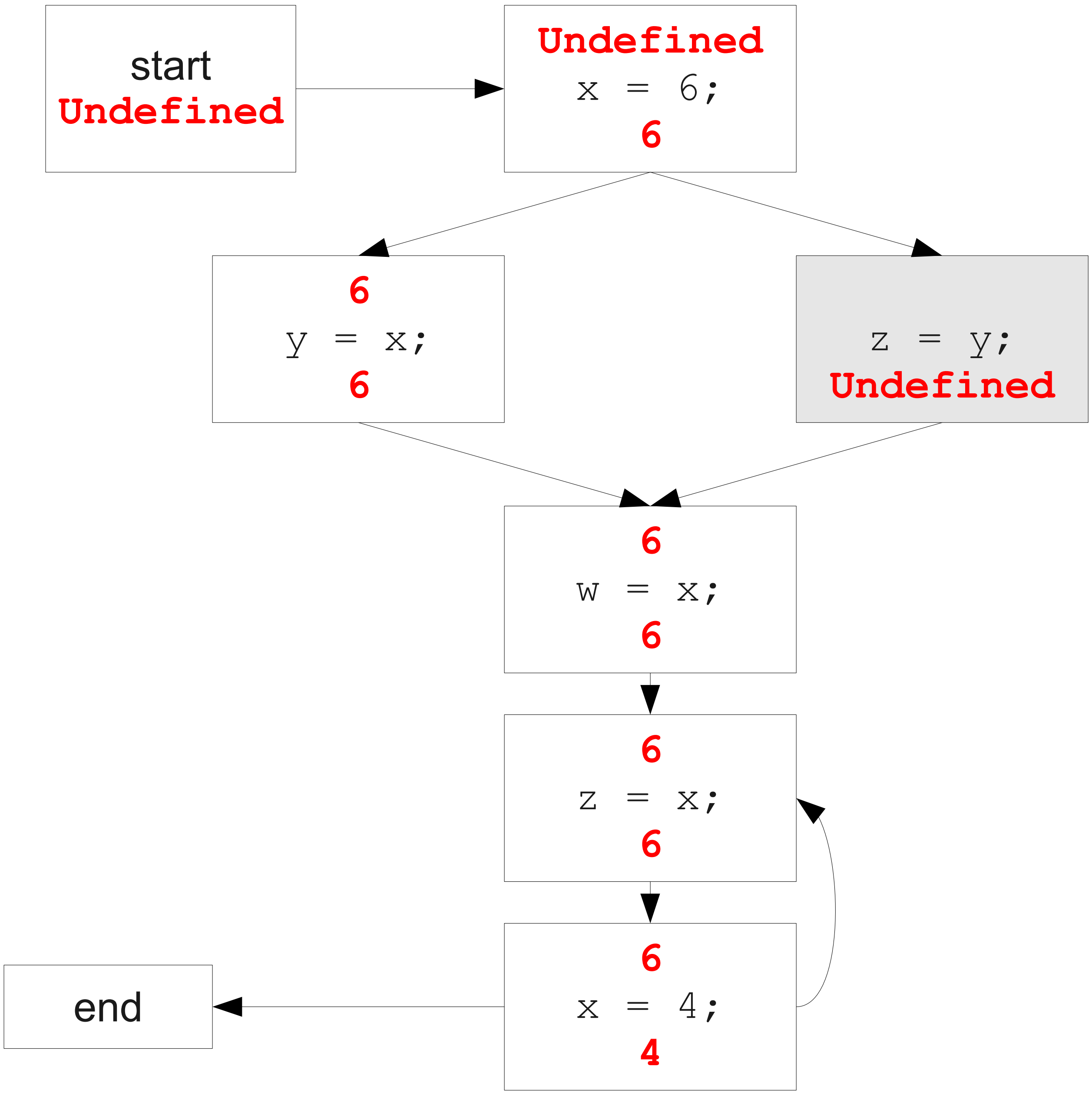

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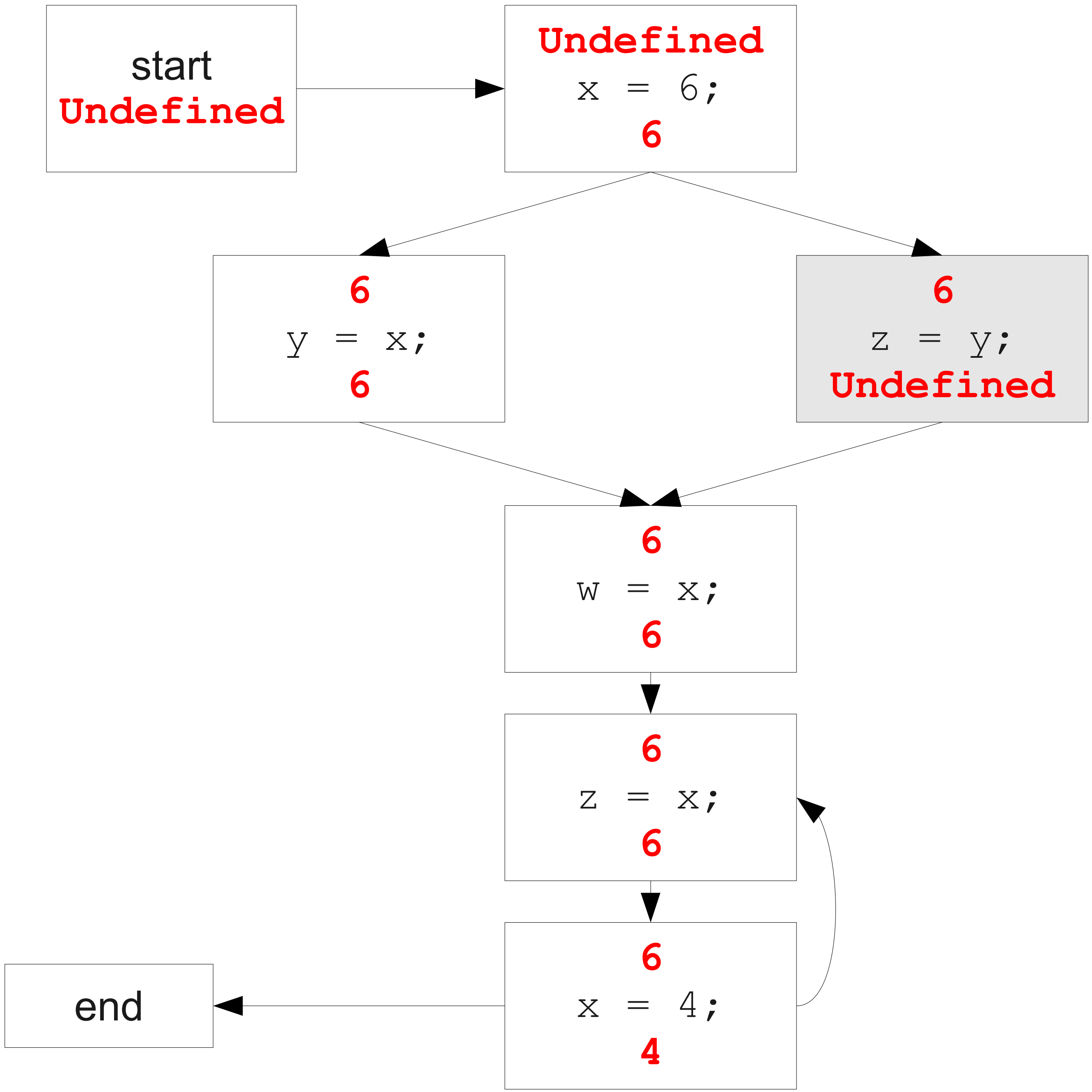

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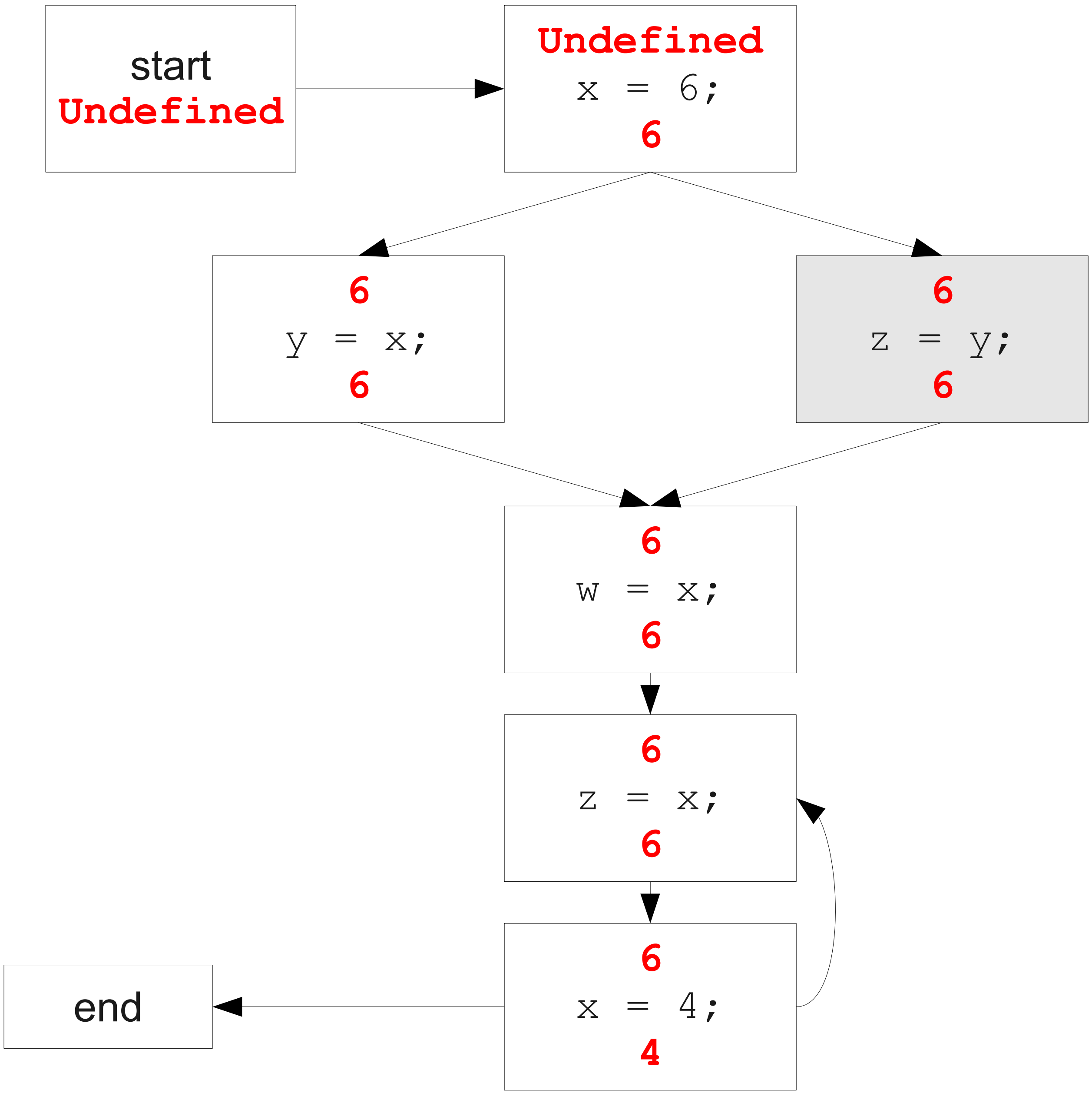

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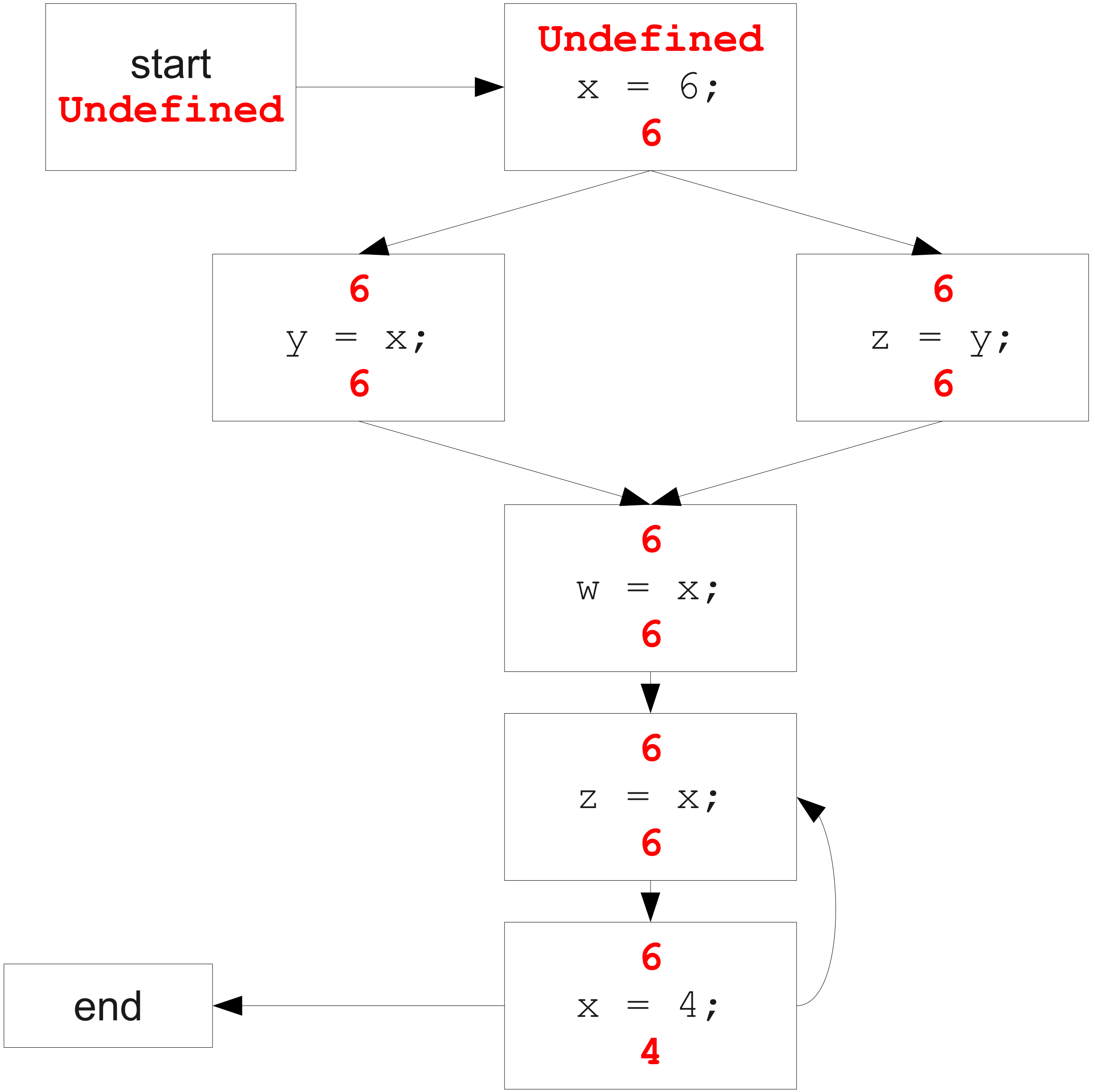

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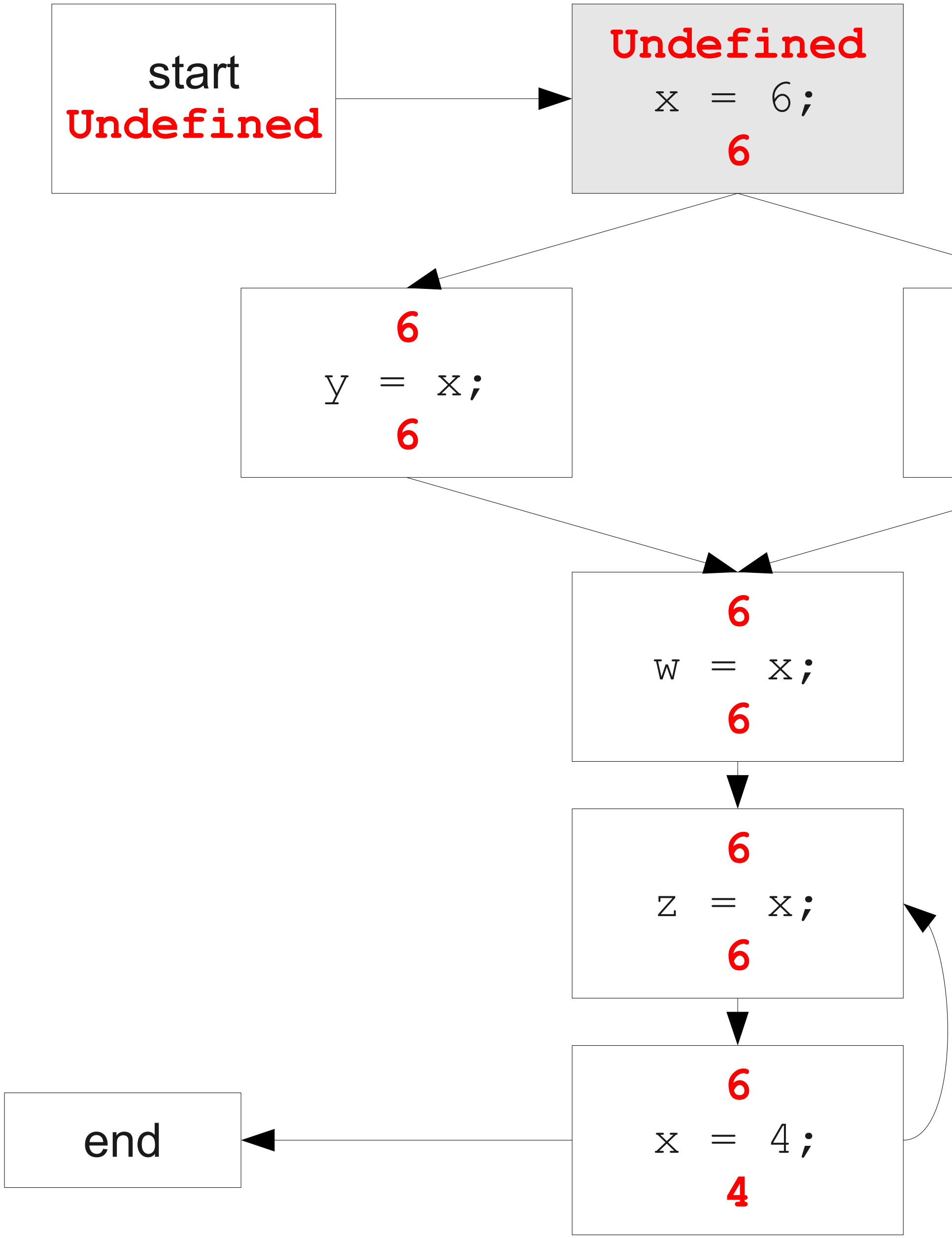

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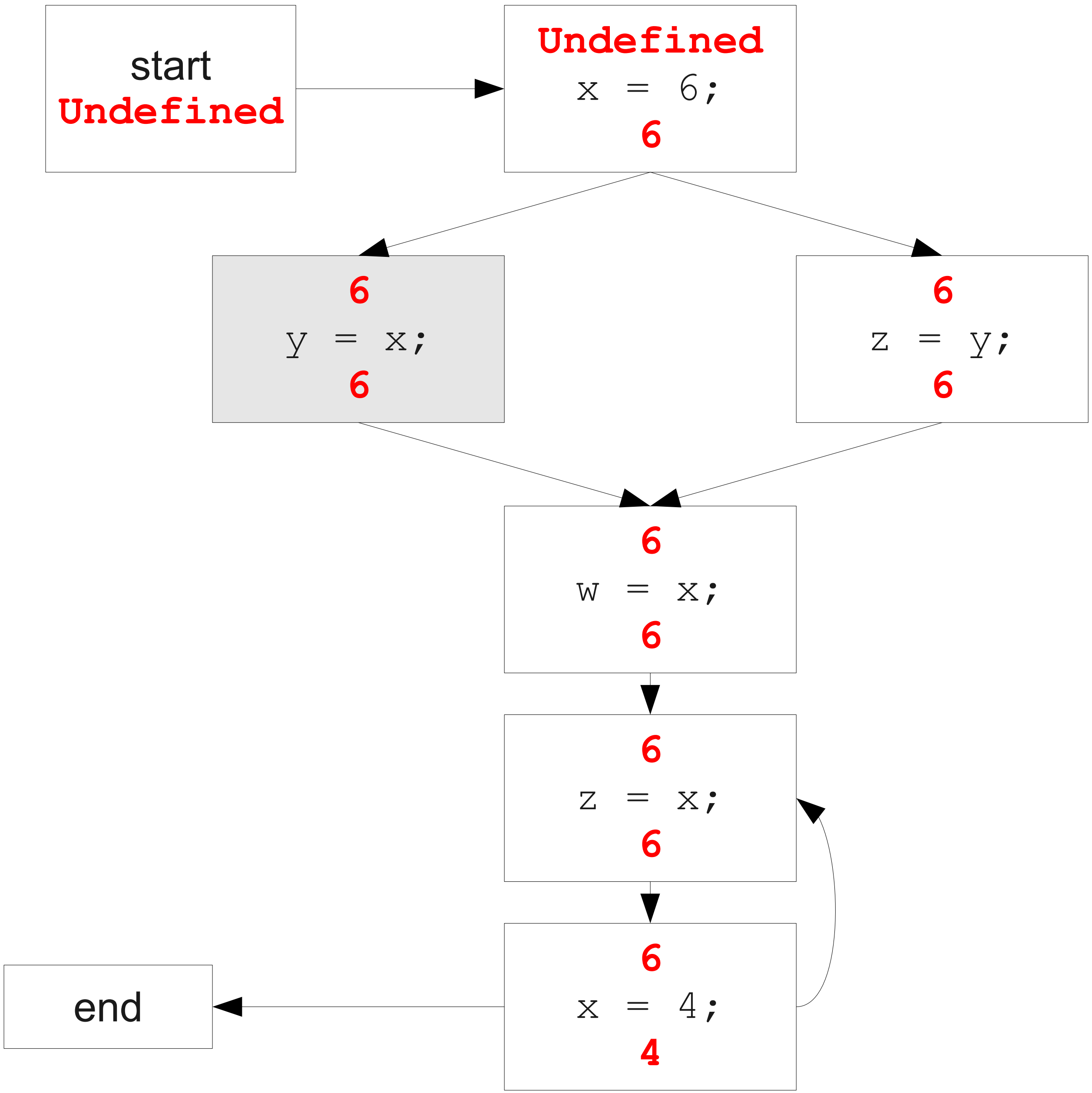

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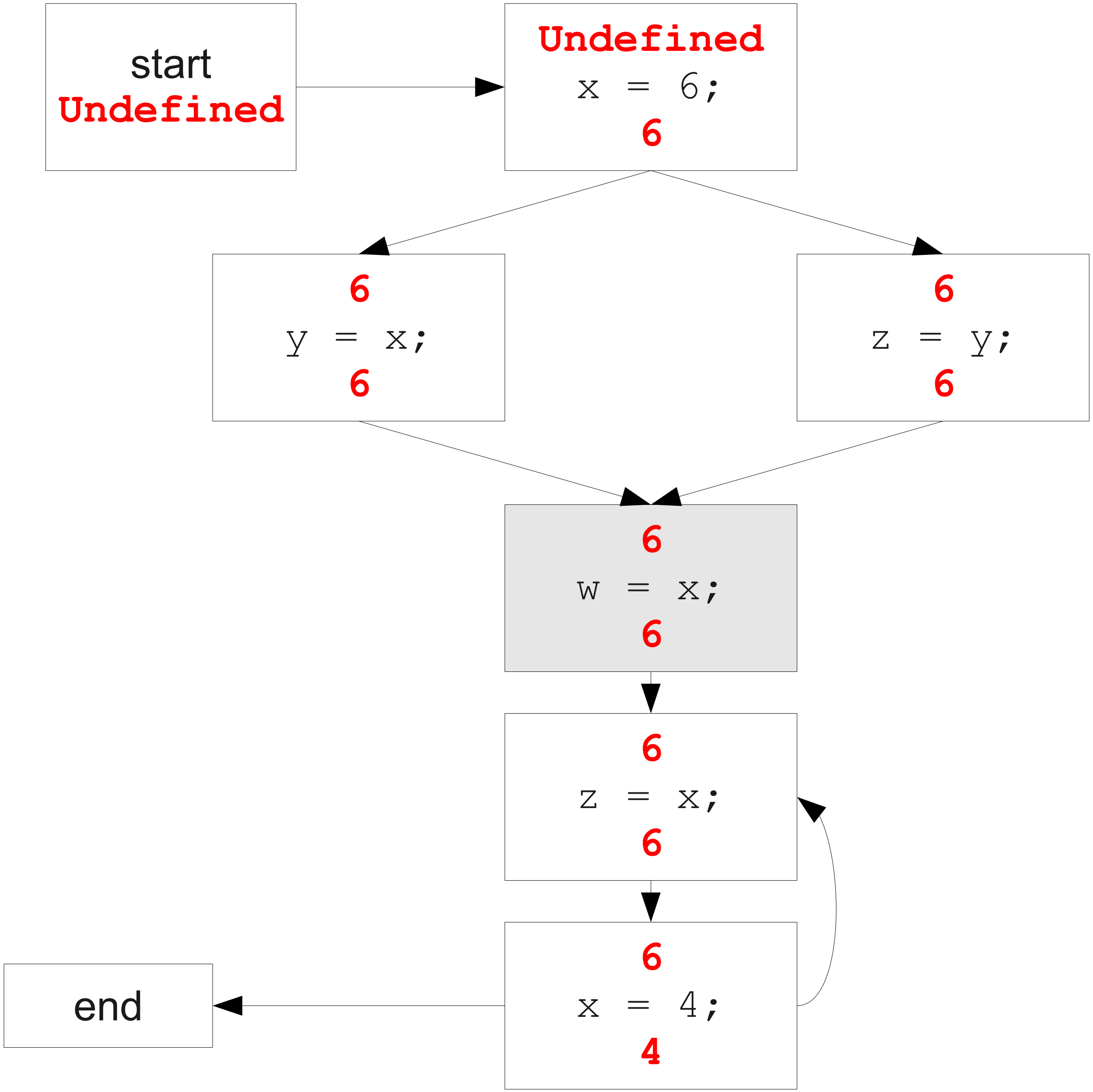

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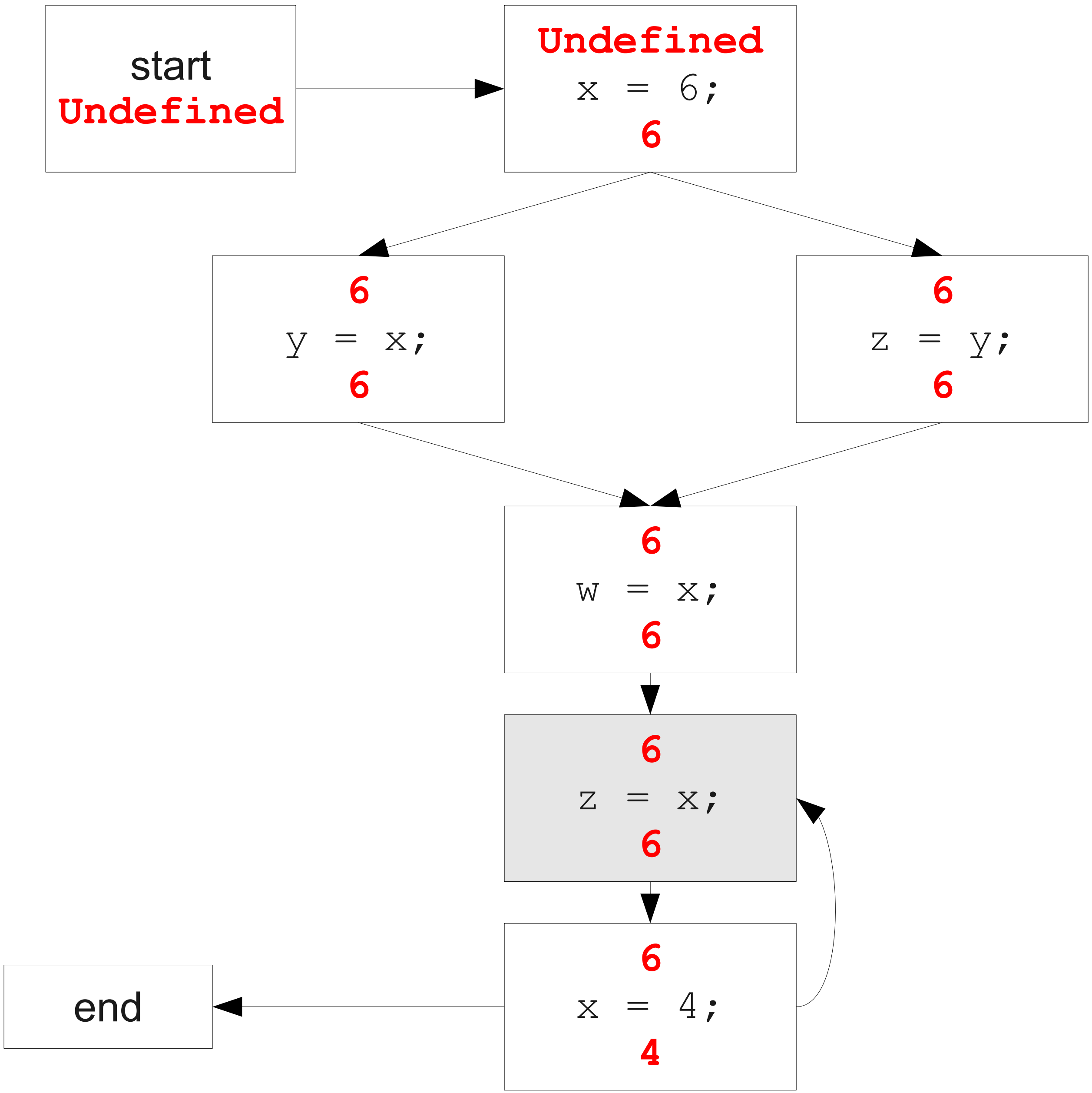

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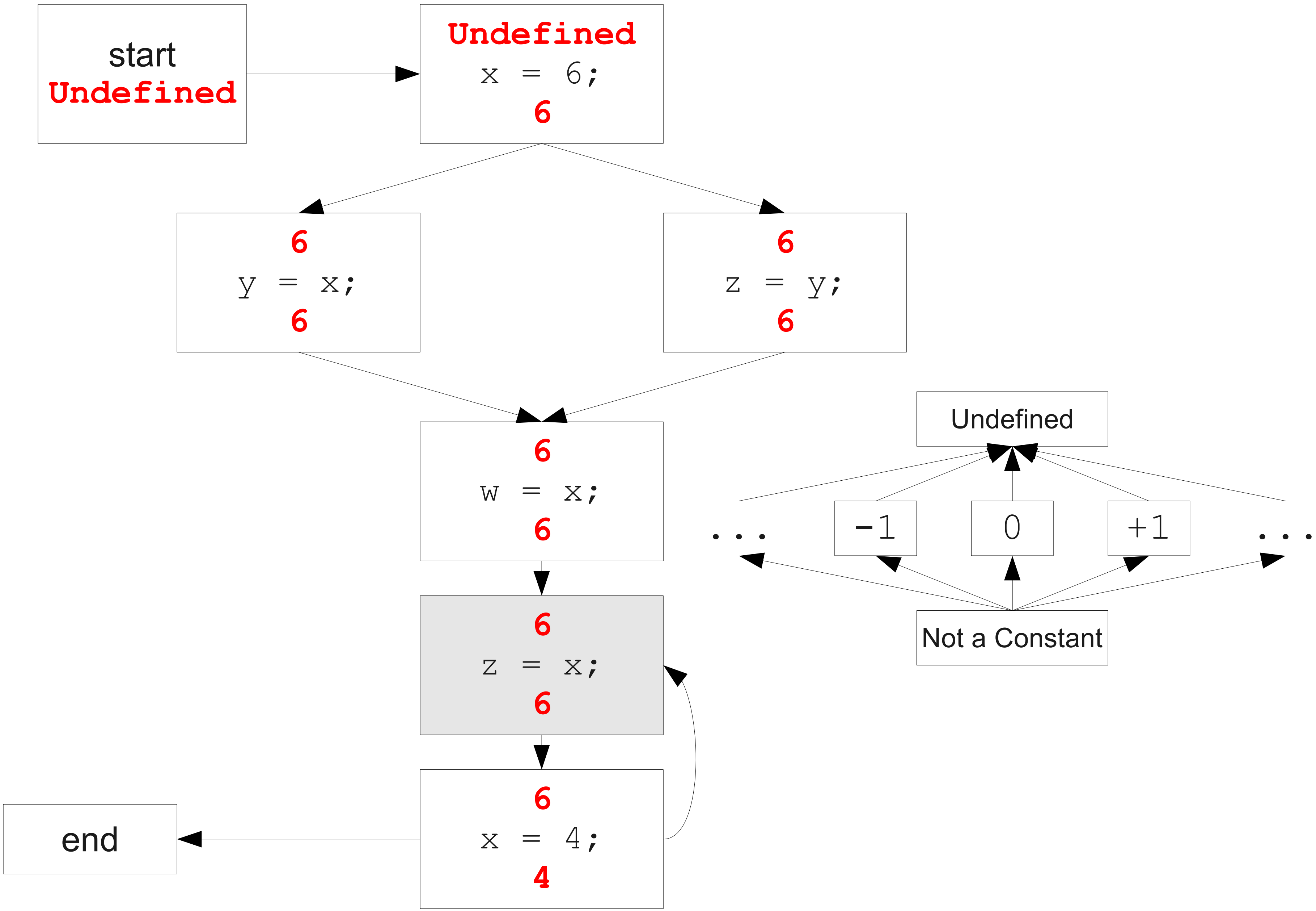

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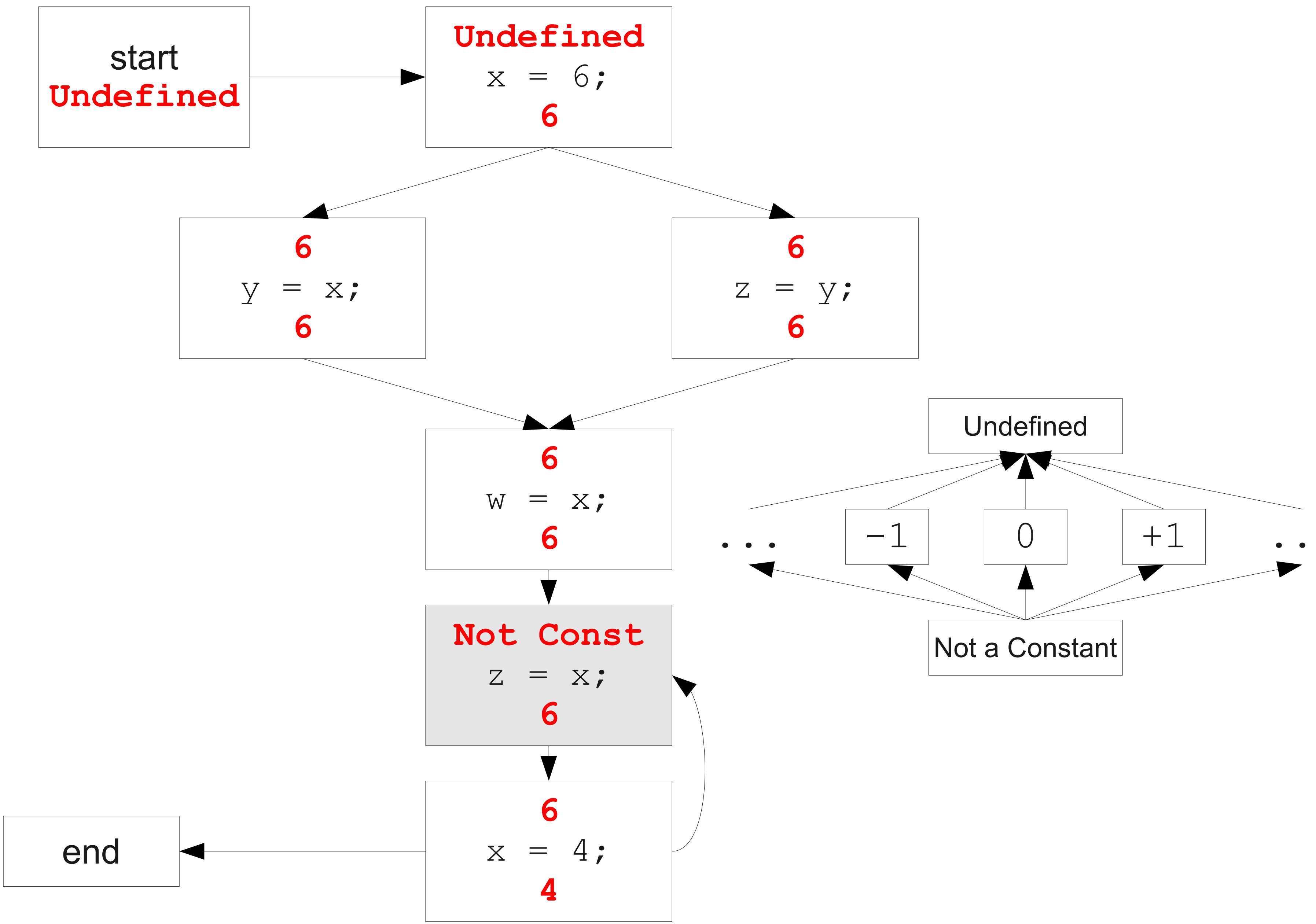

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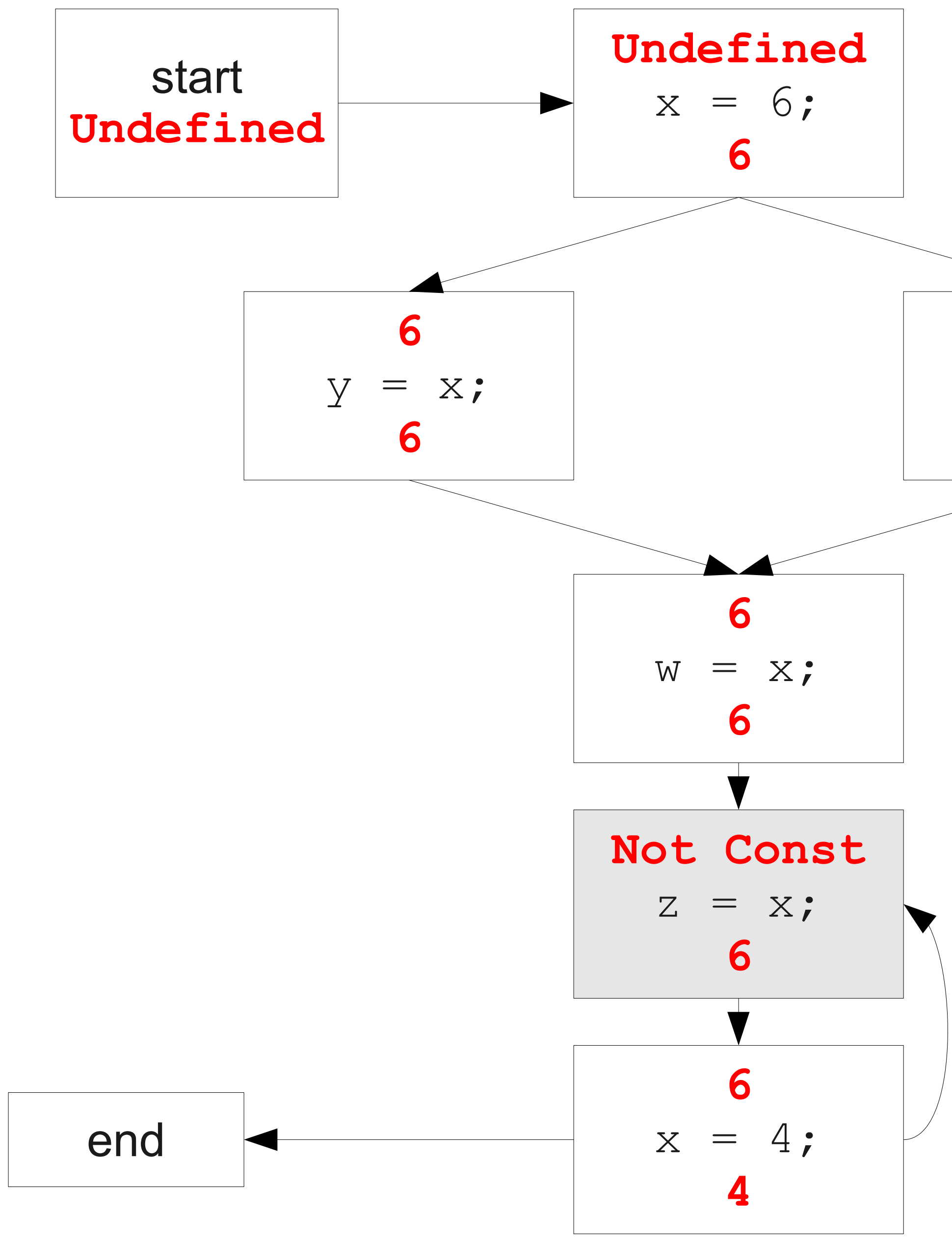

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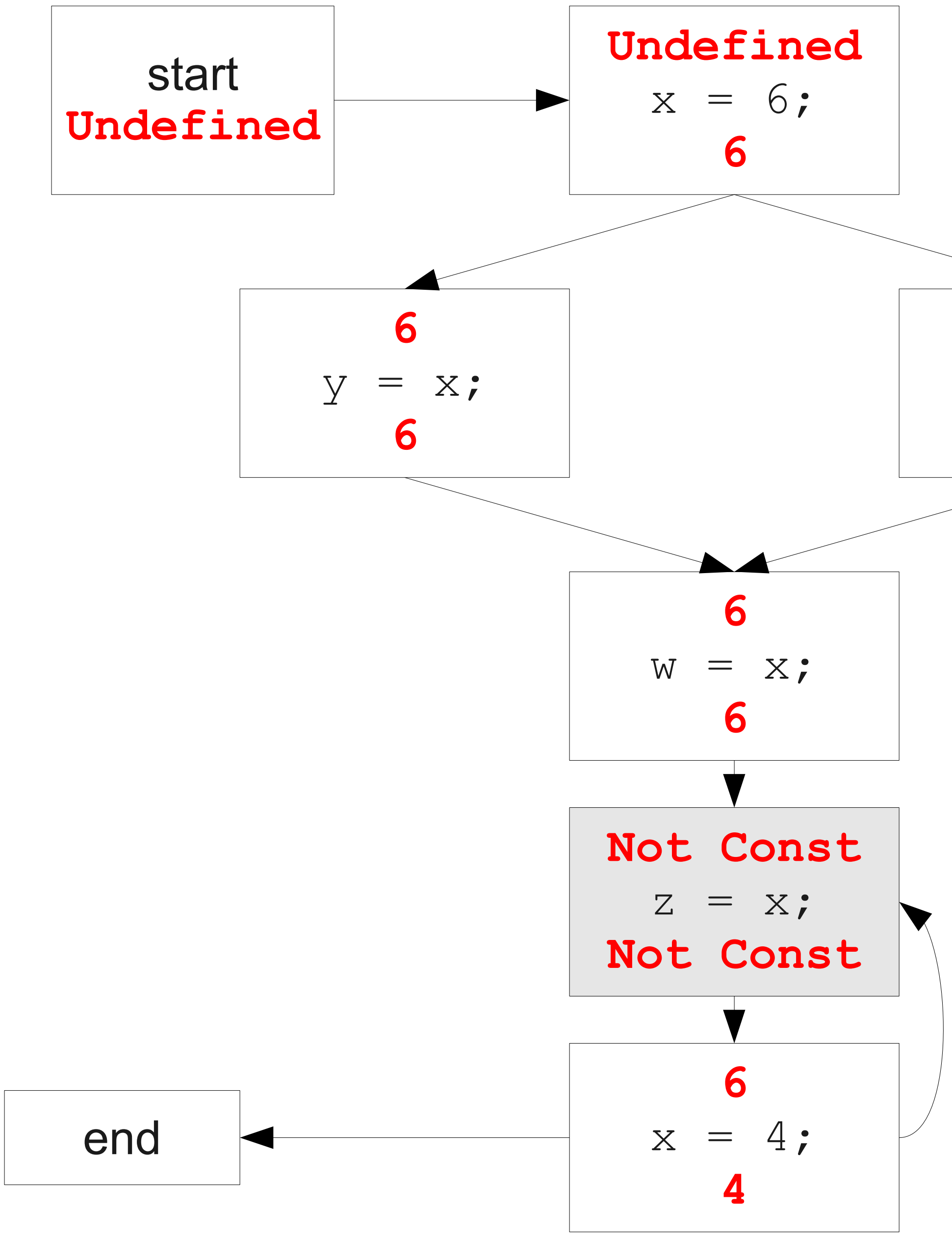

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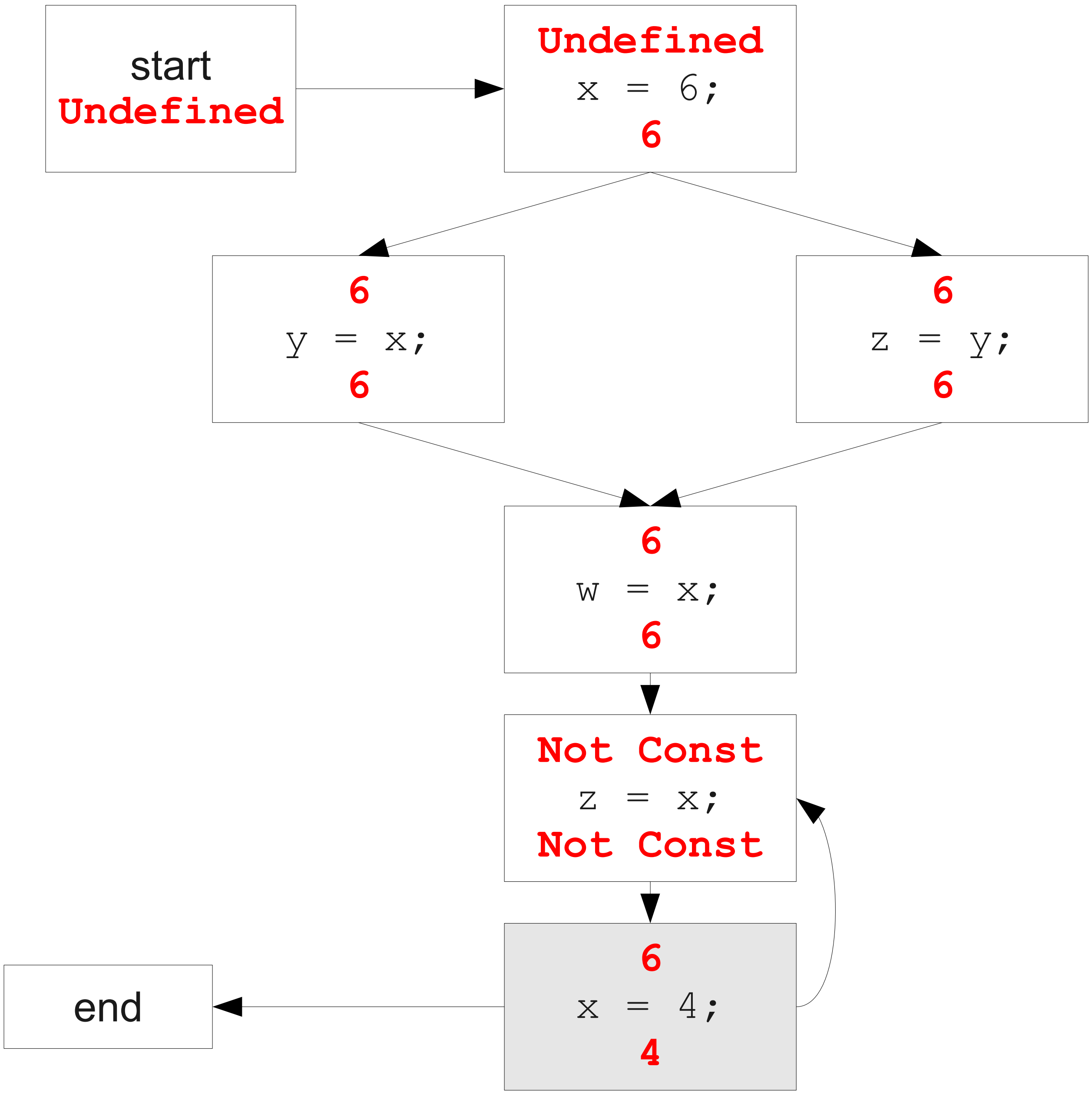

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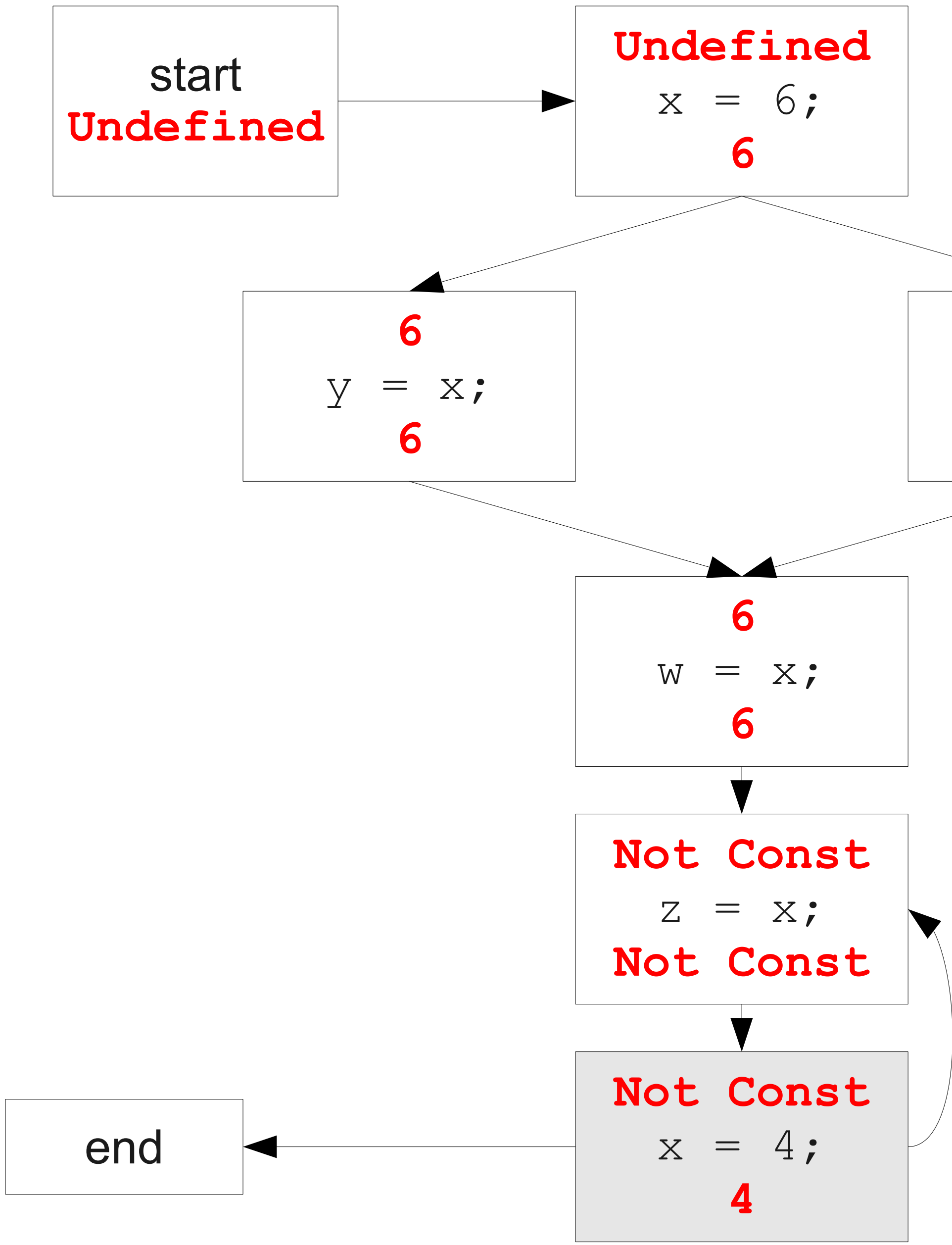

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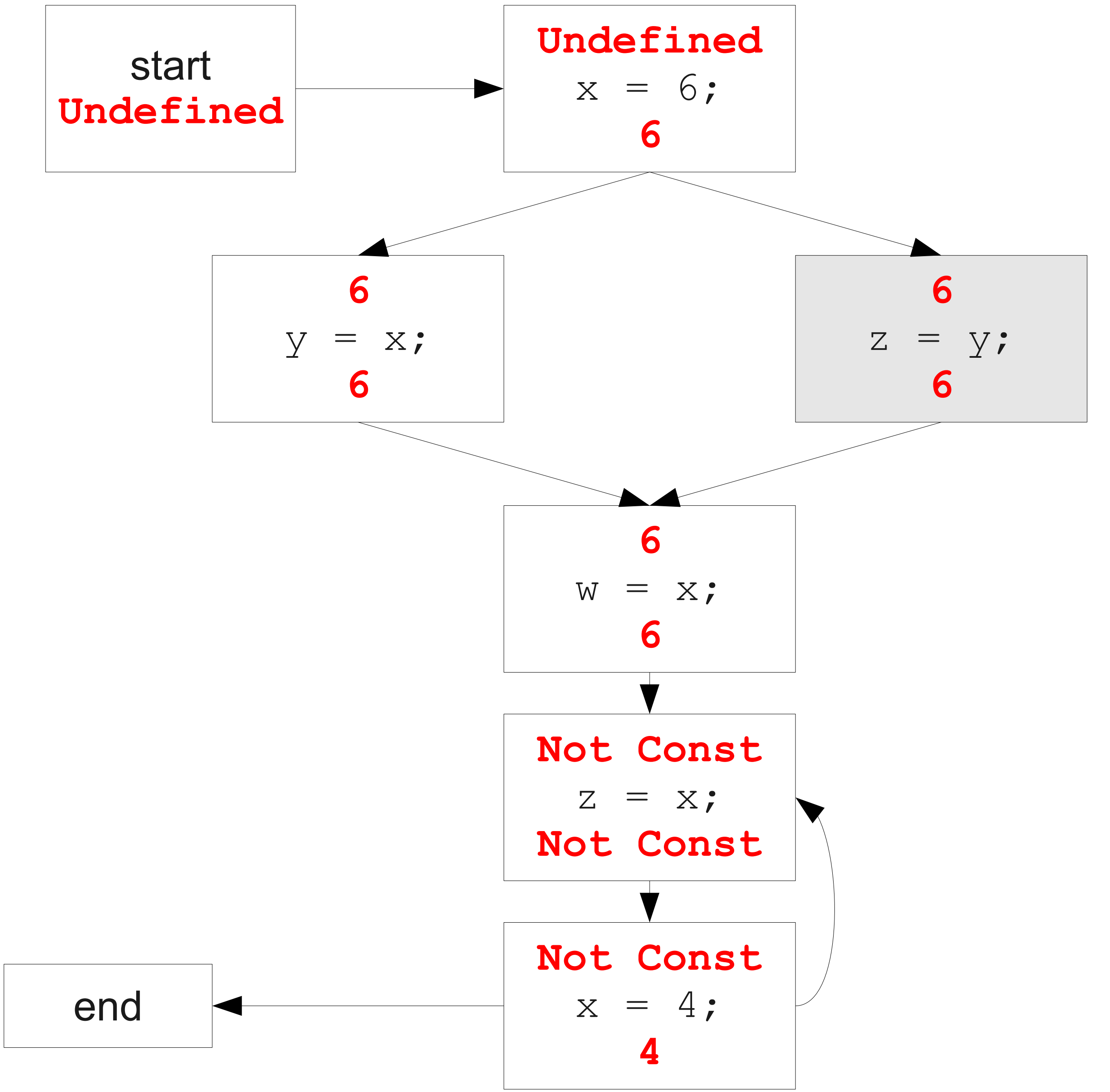

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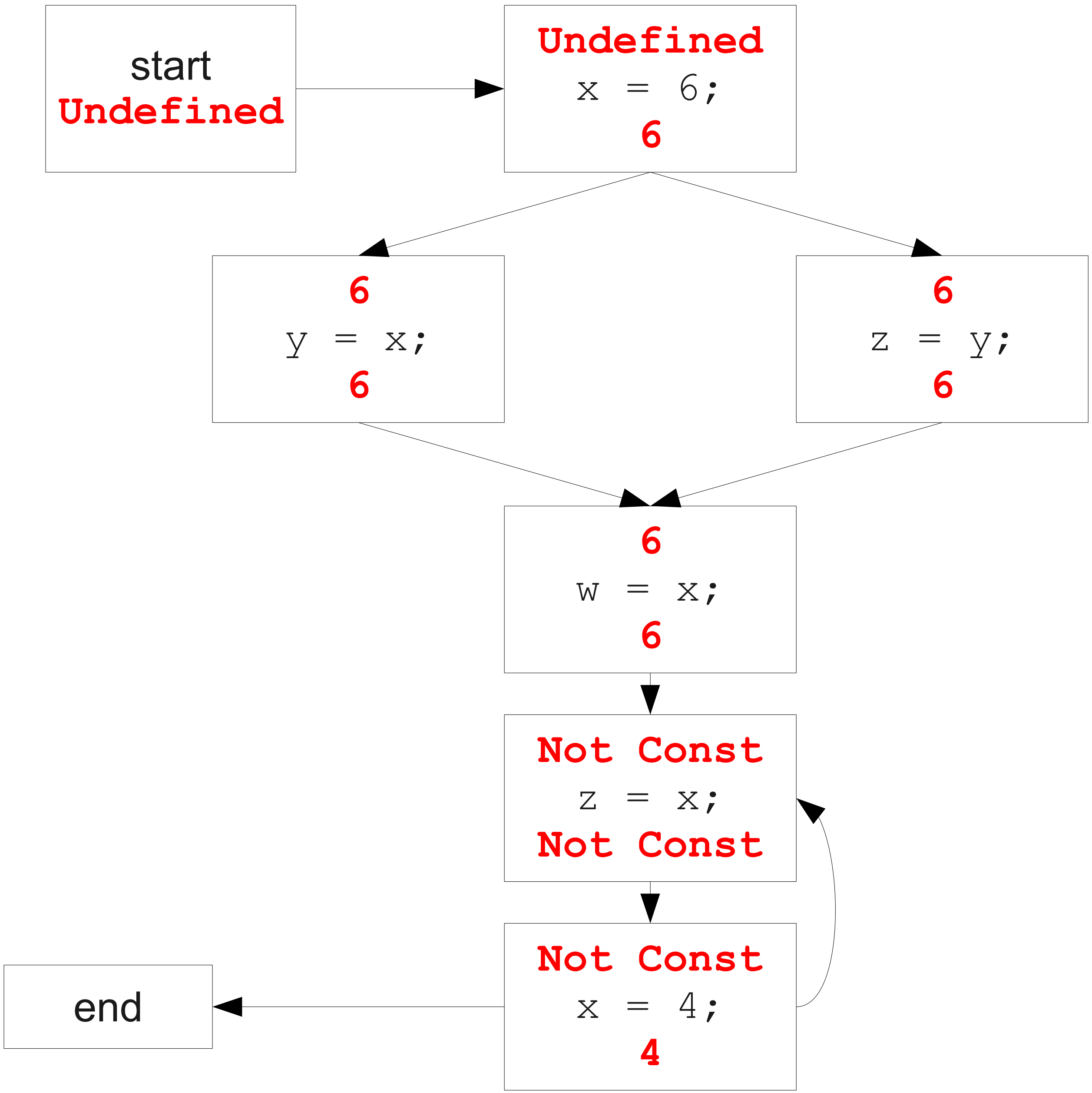

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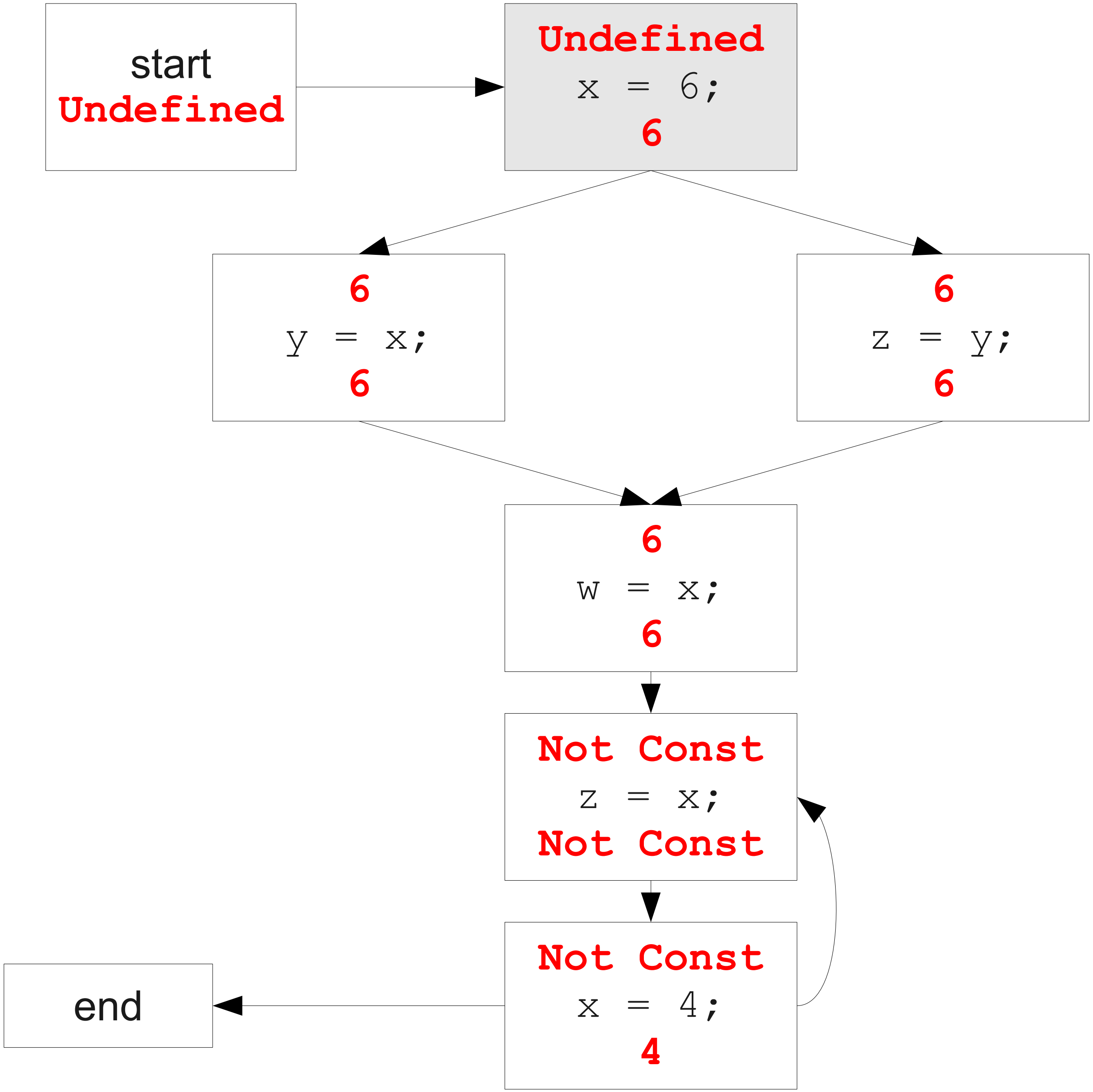

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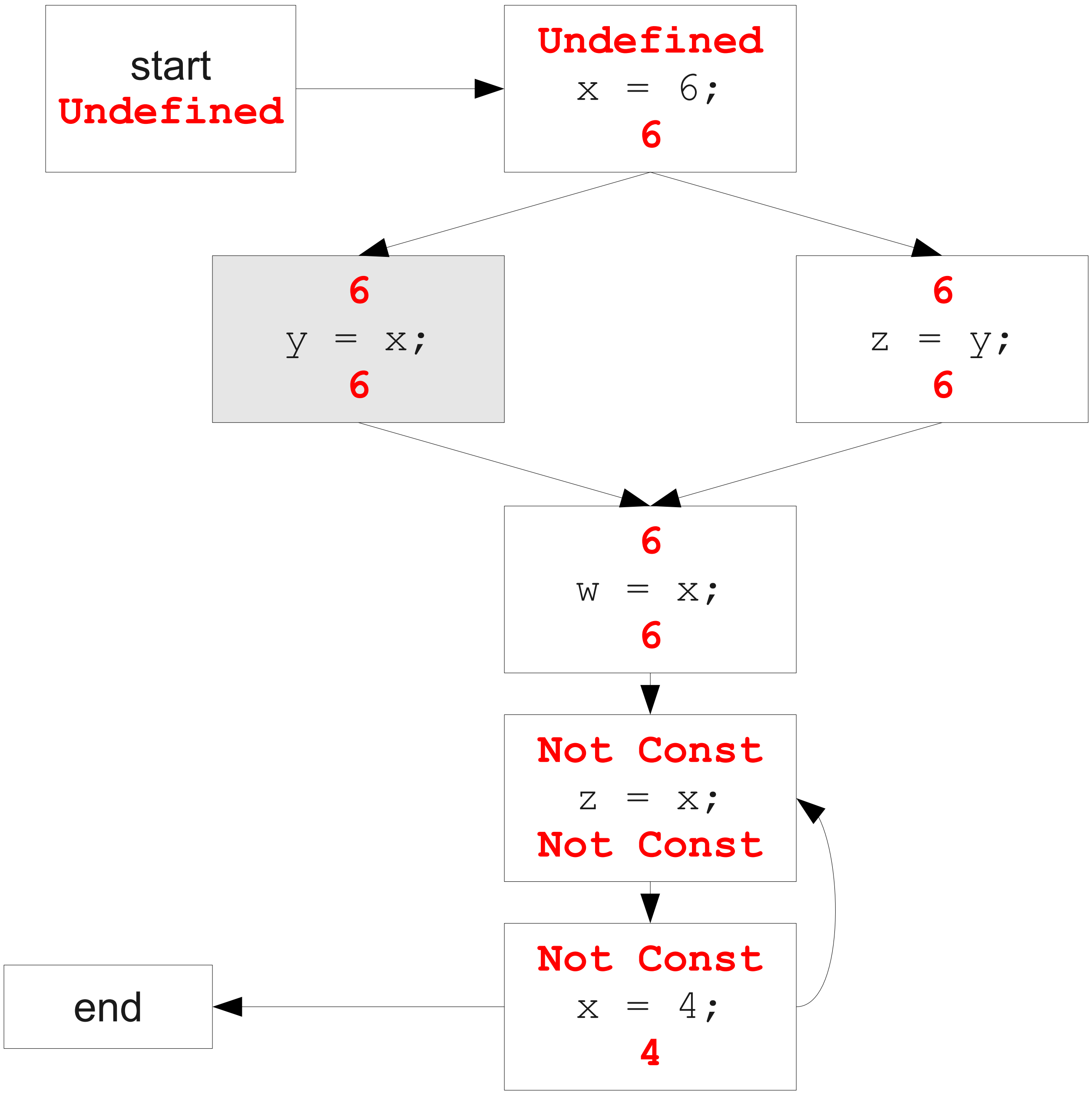

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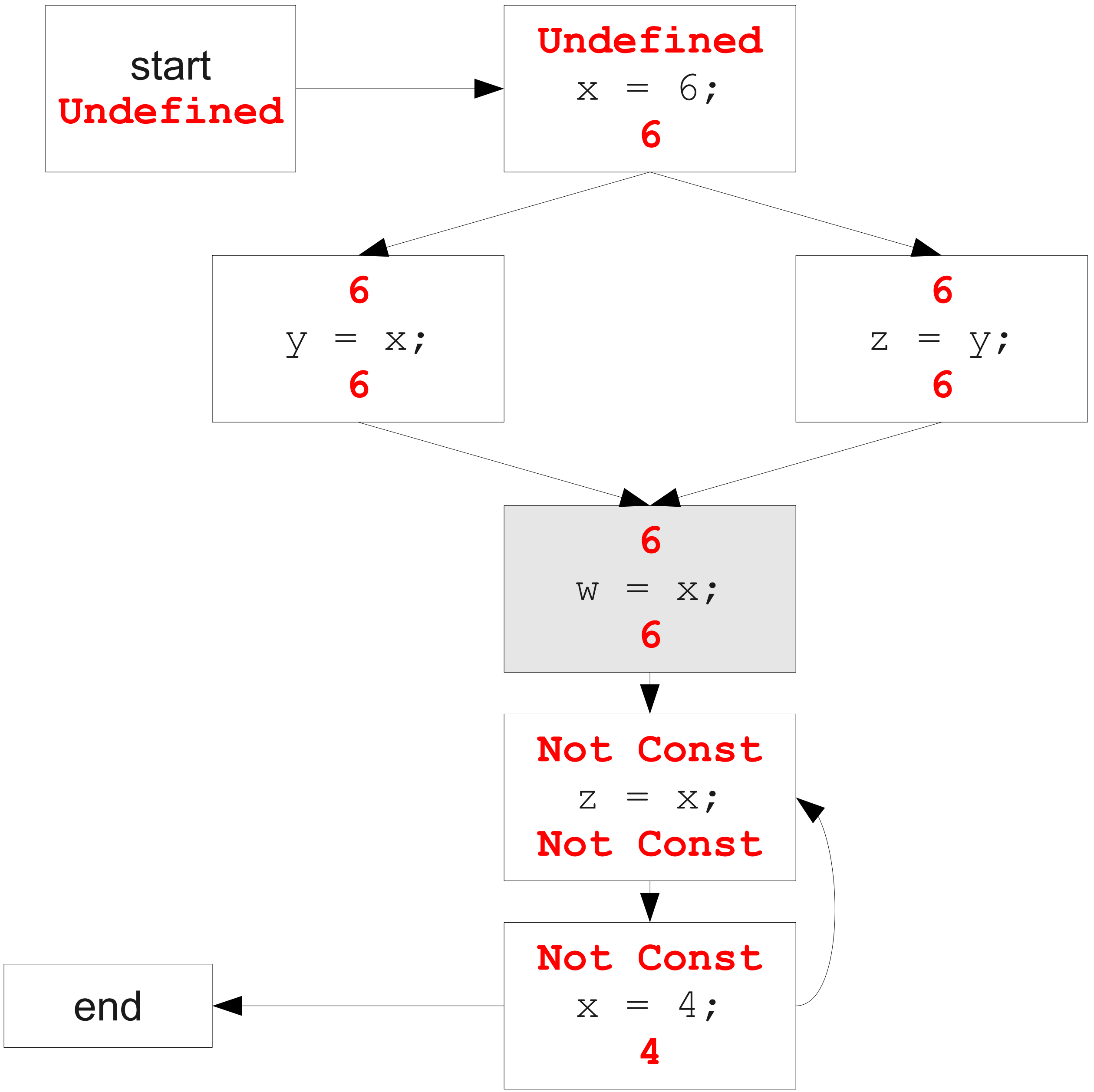

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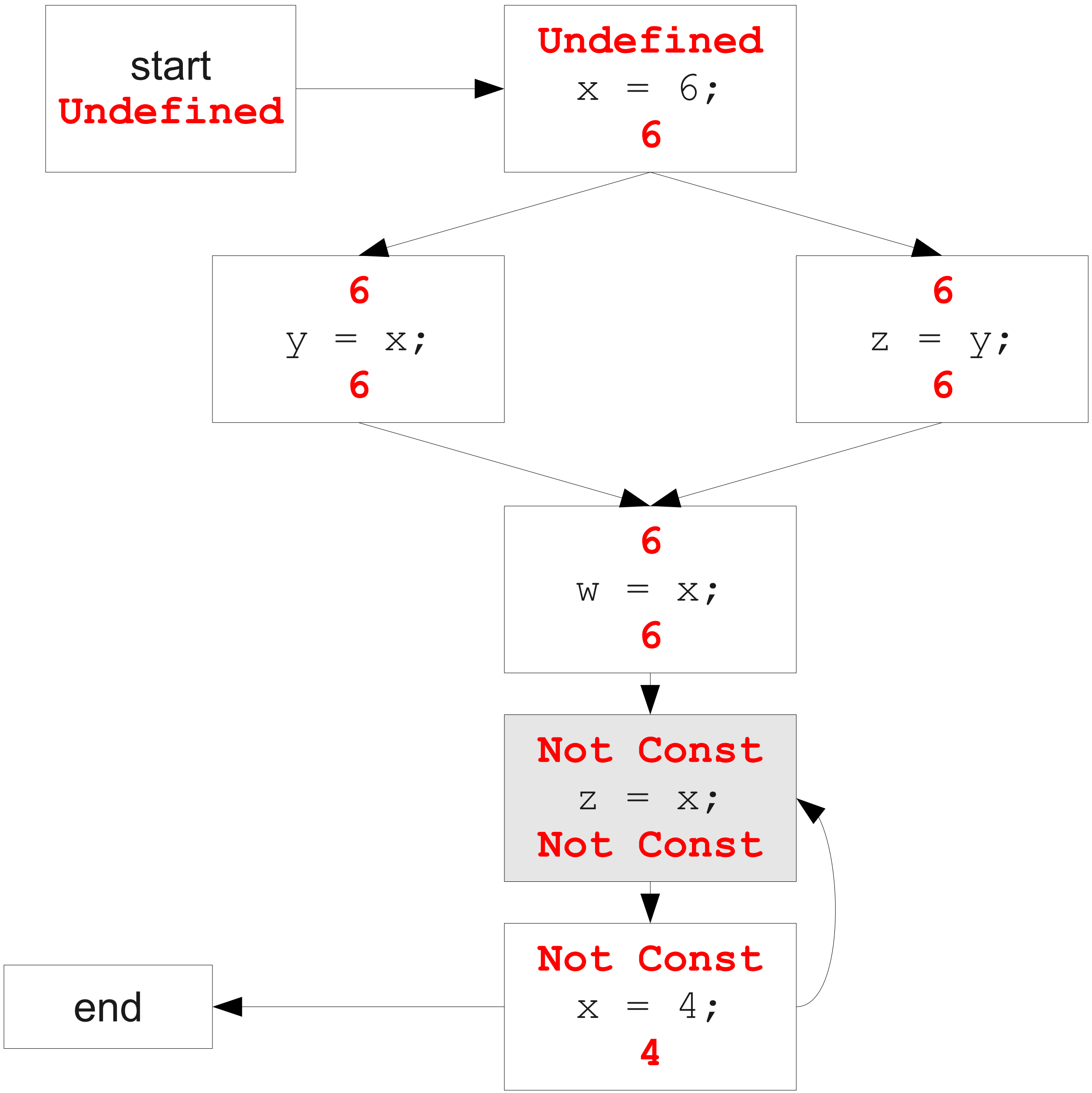

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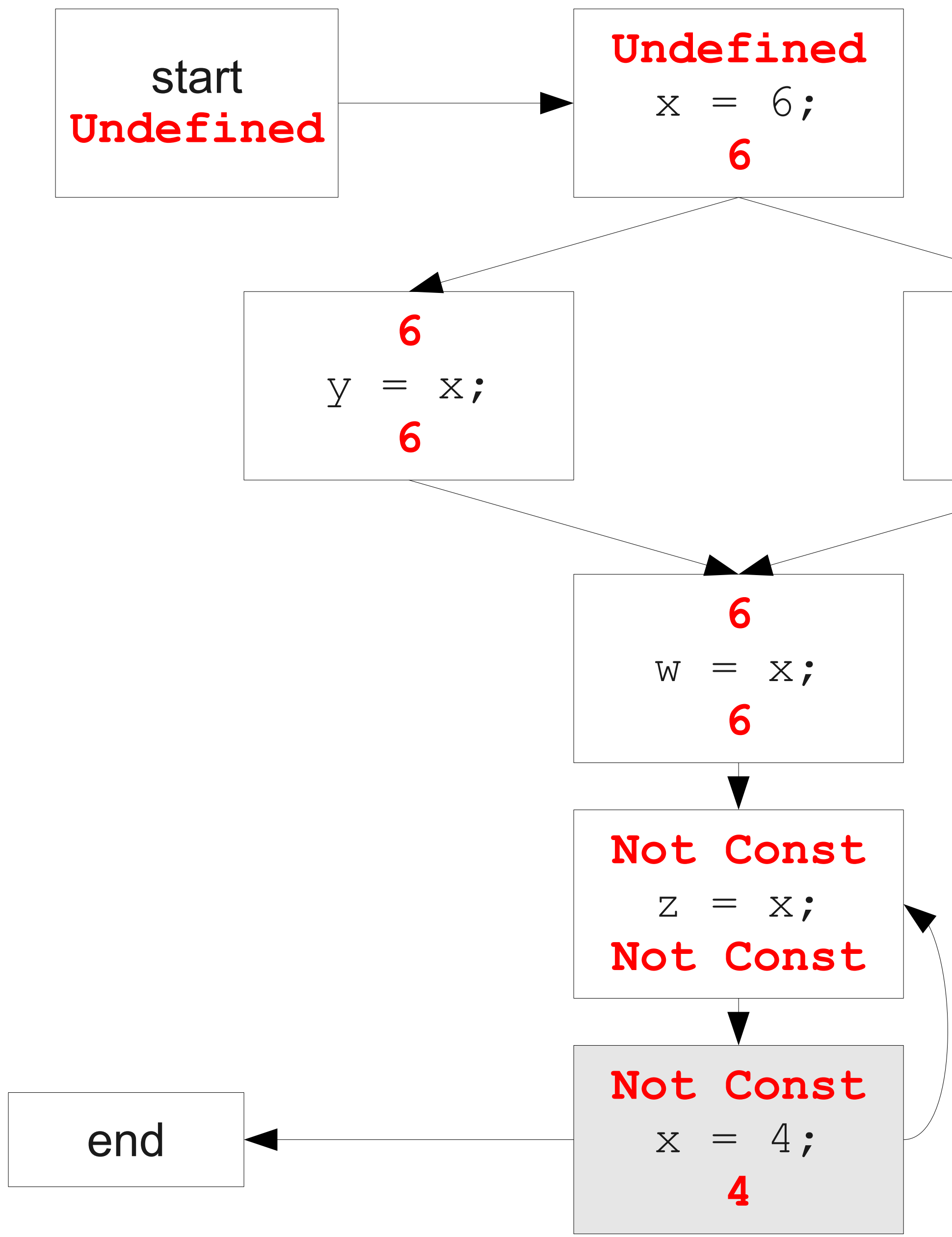

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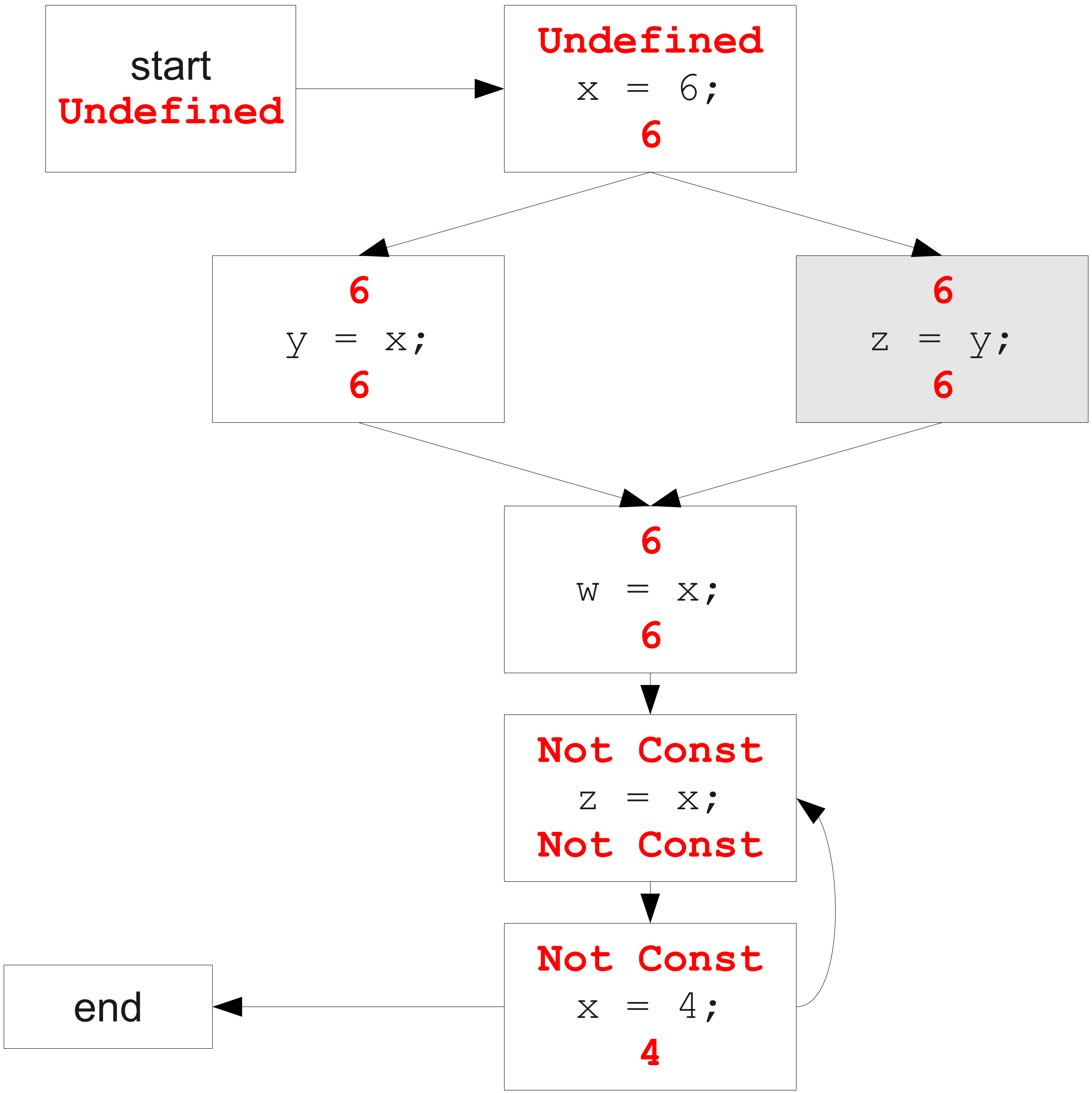

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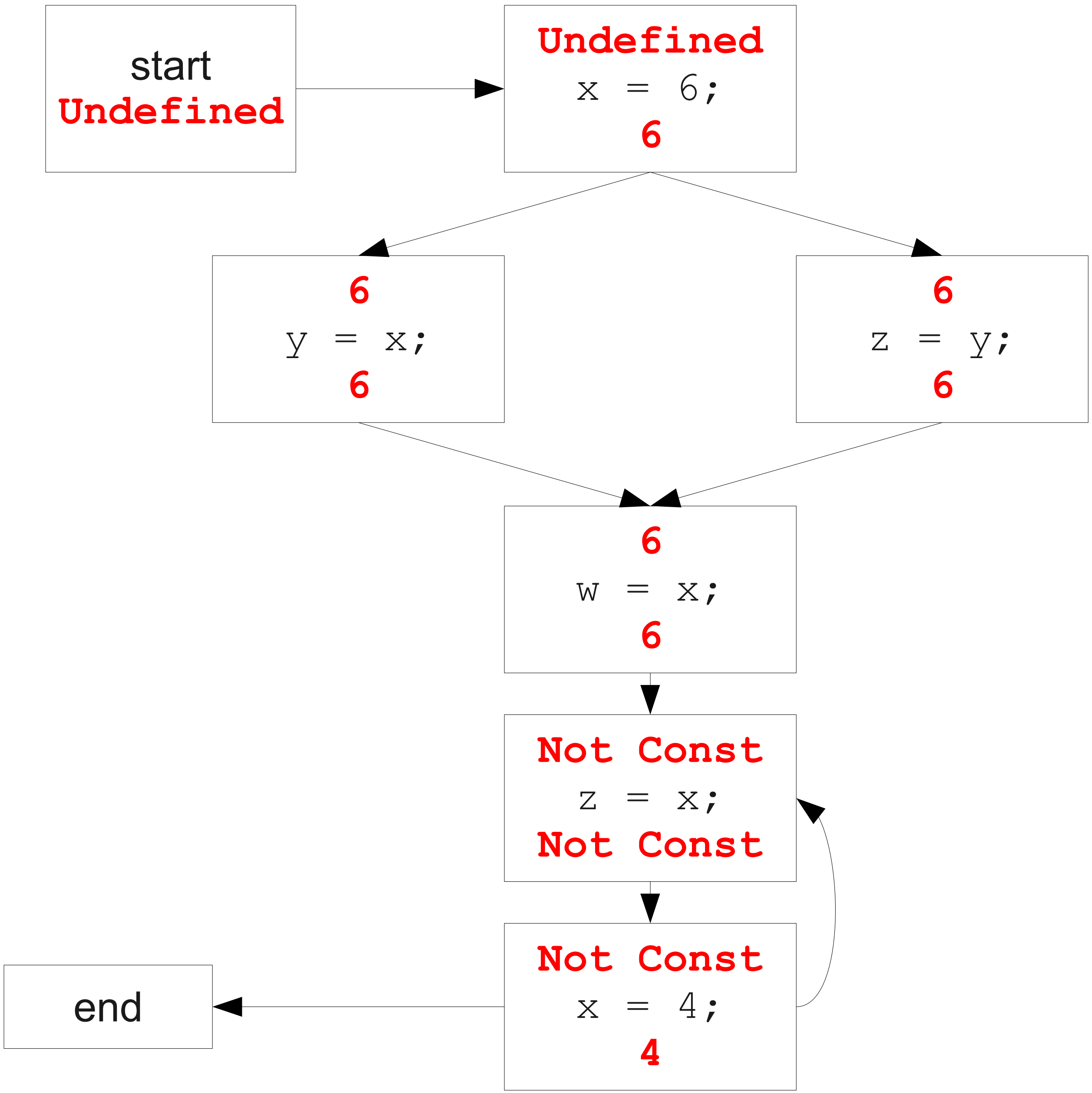
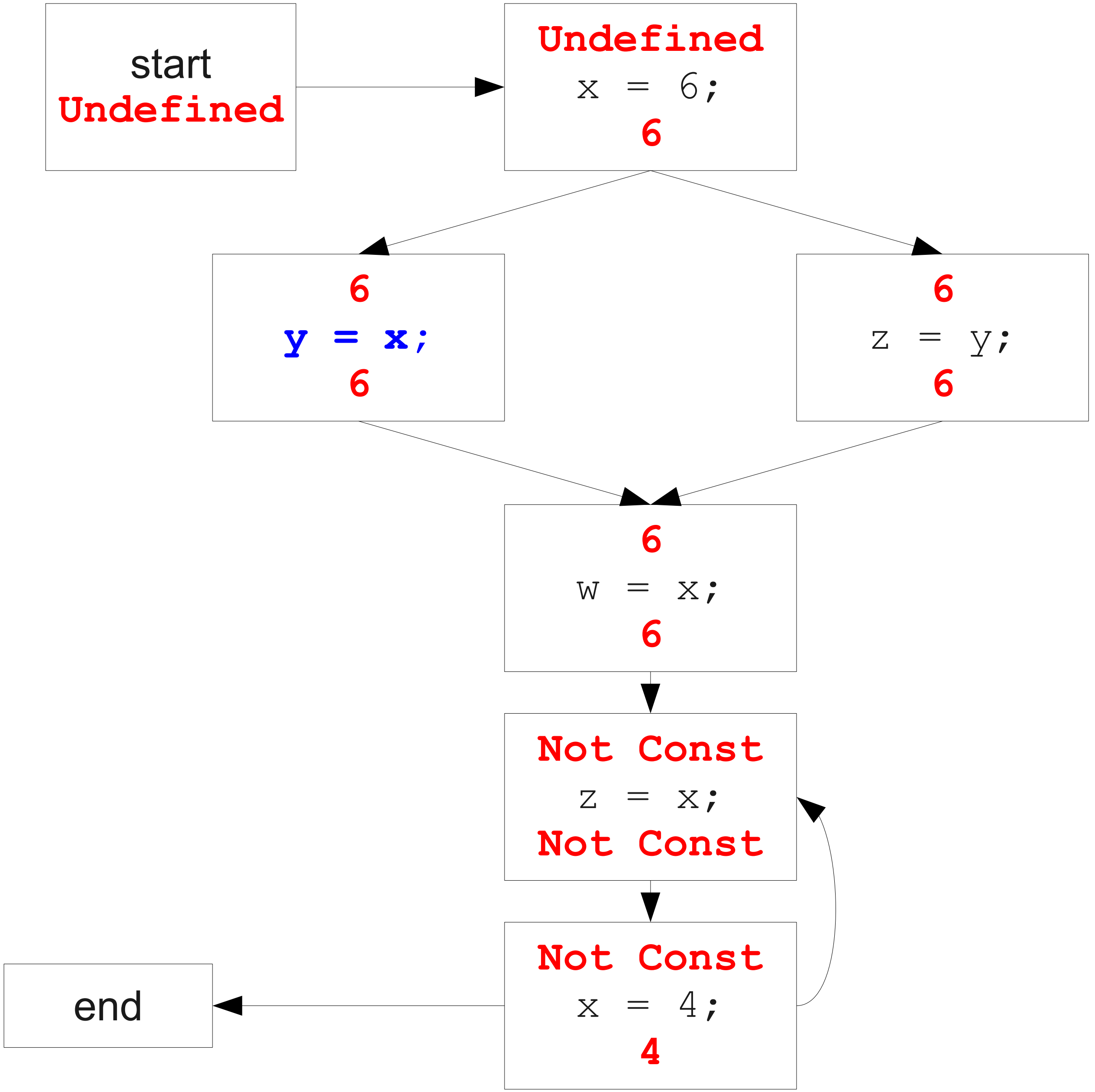


Global Constant Propagation



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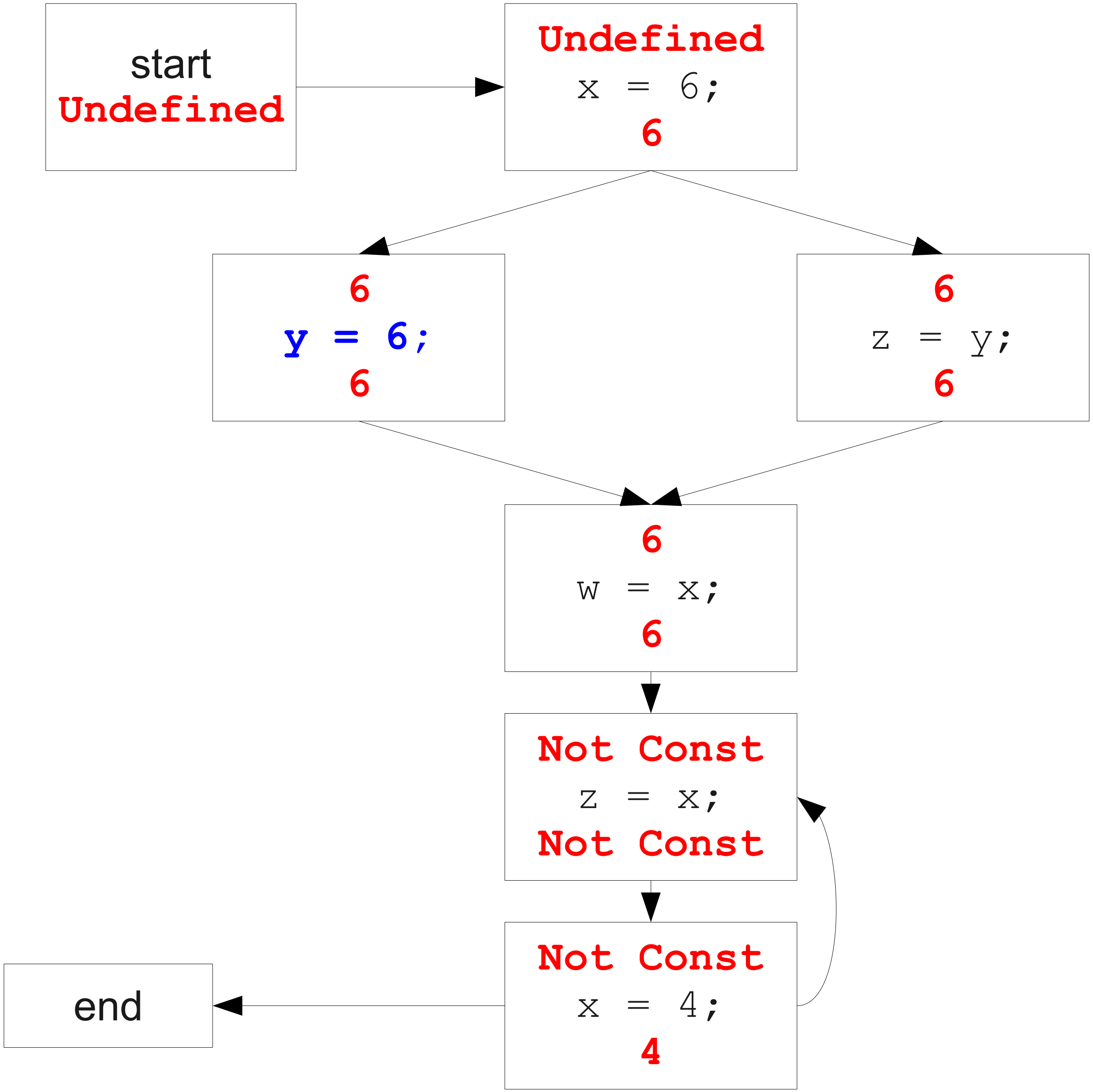

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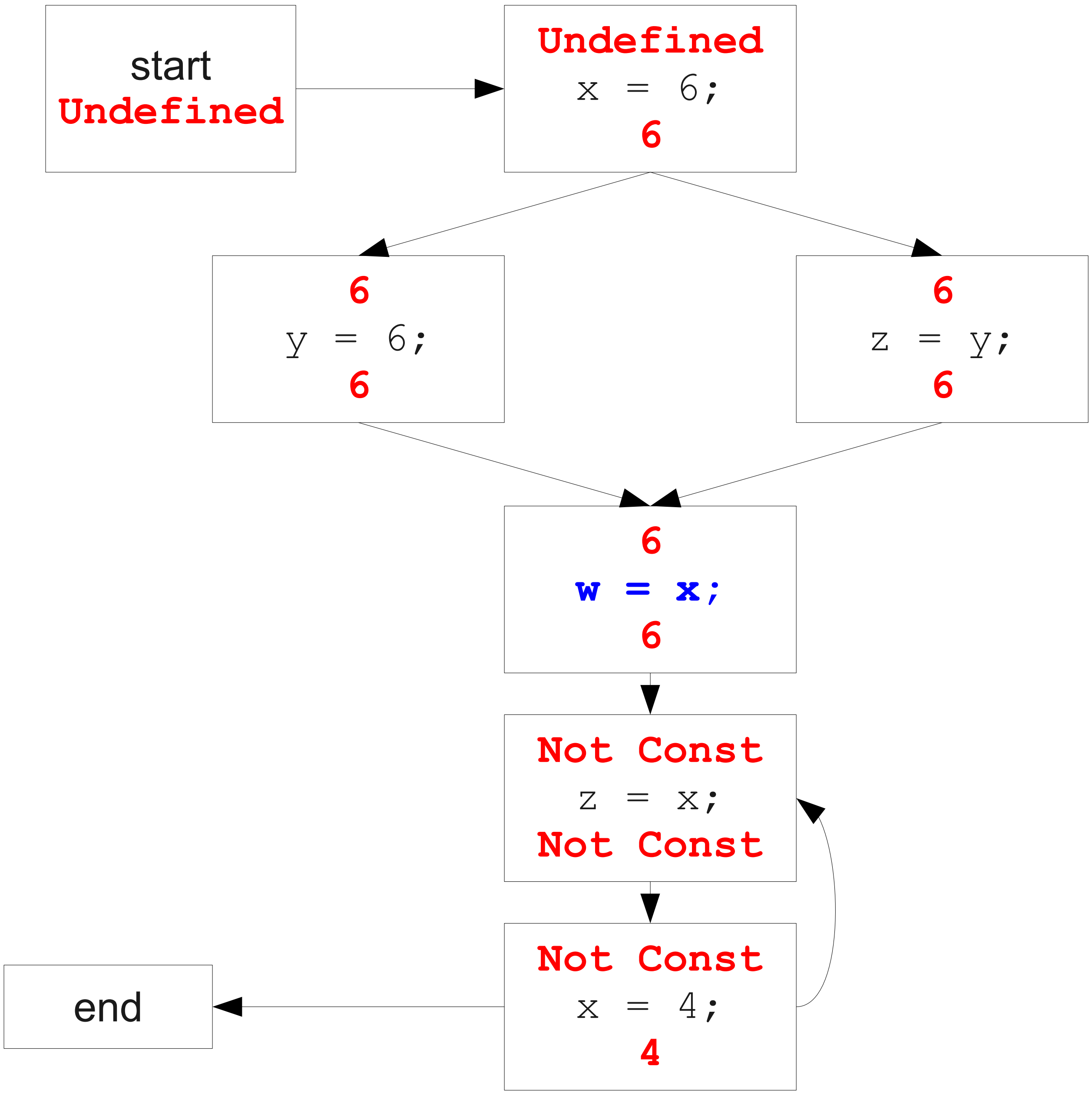

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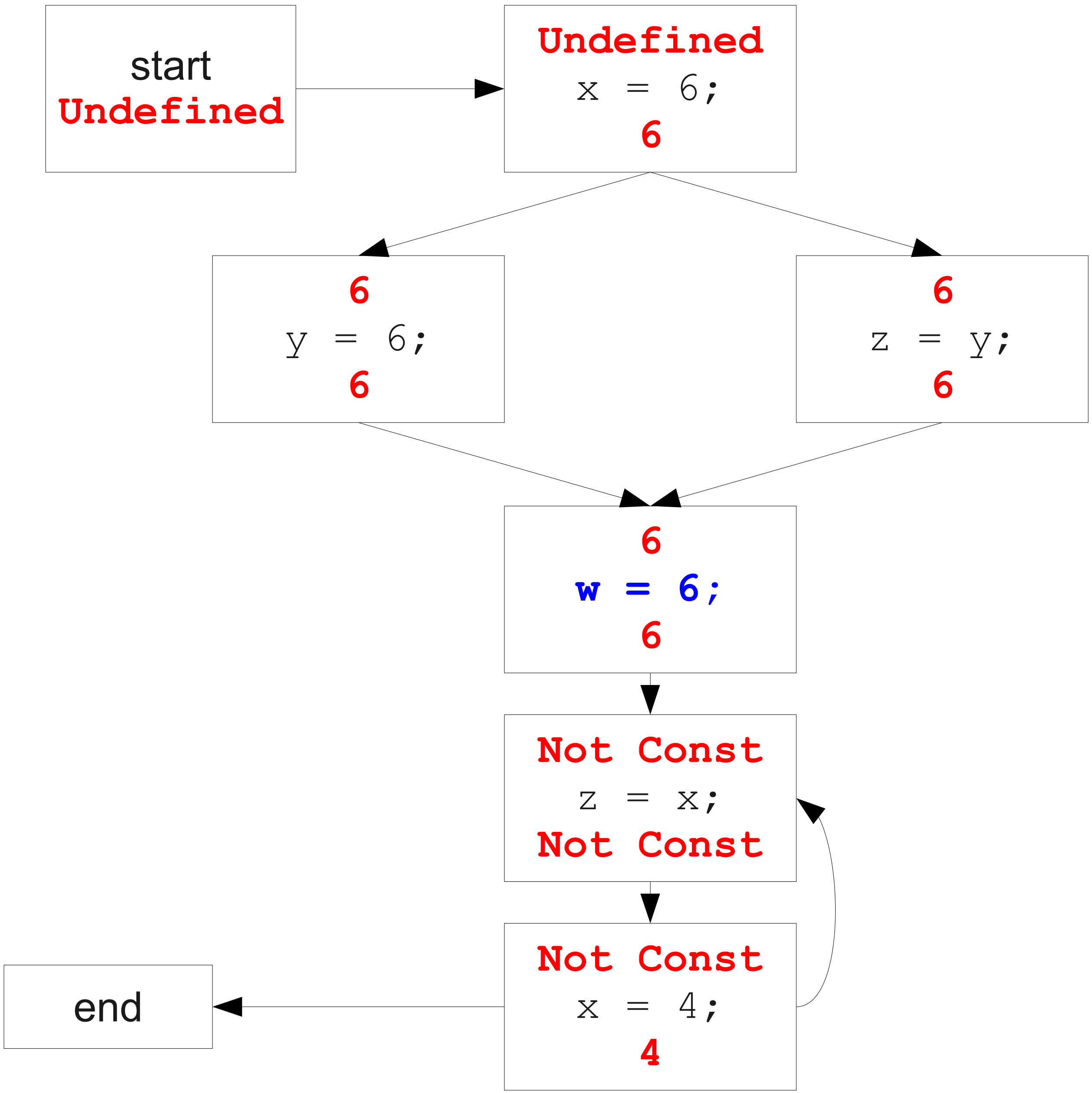

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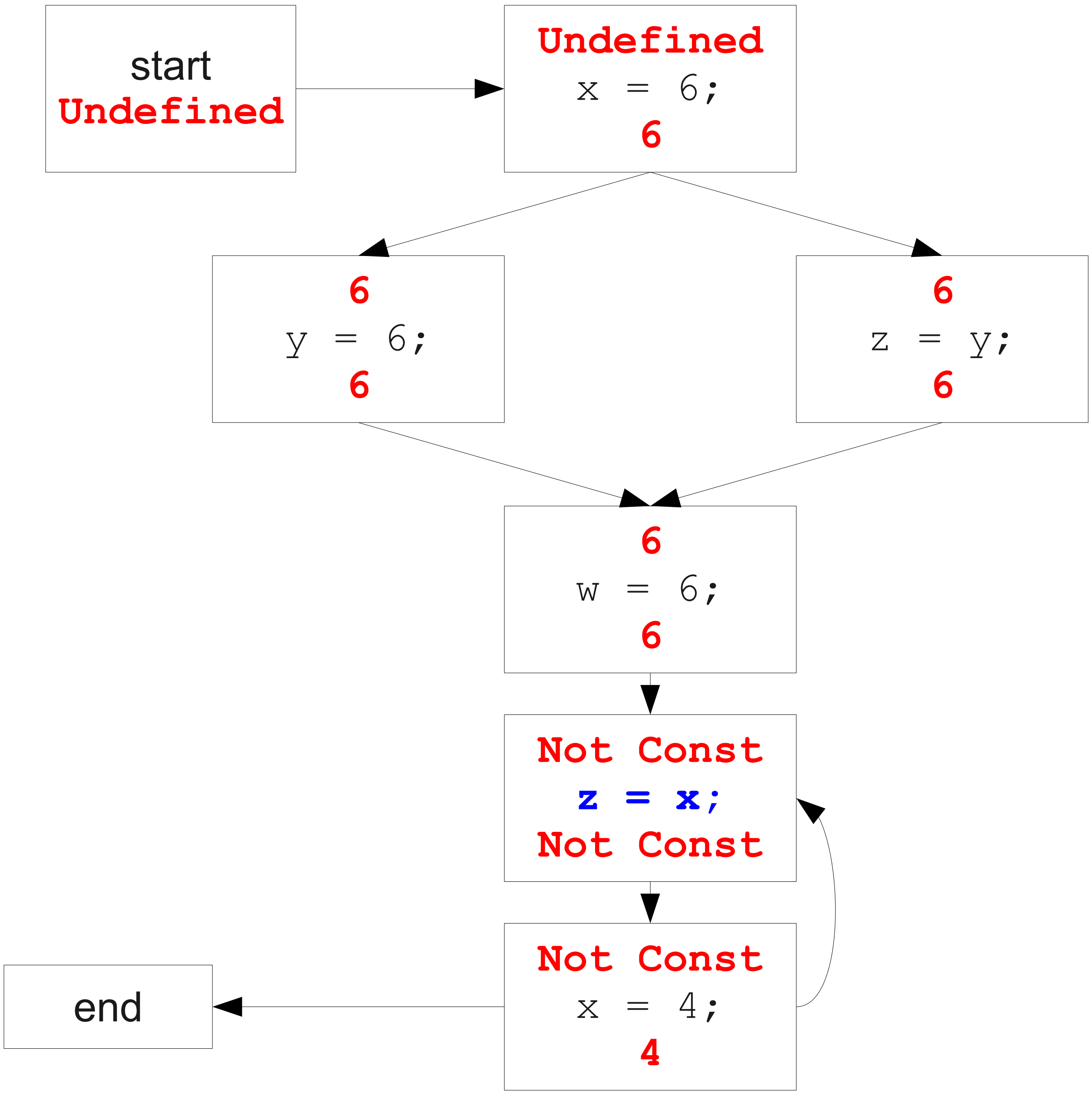

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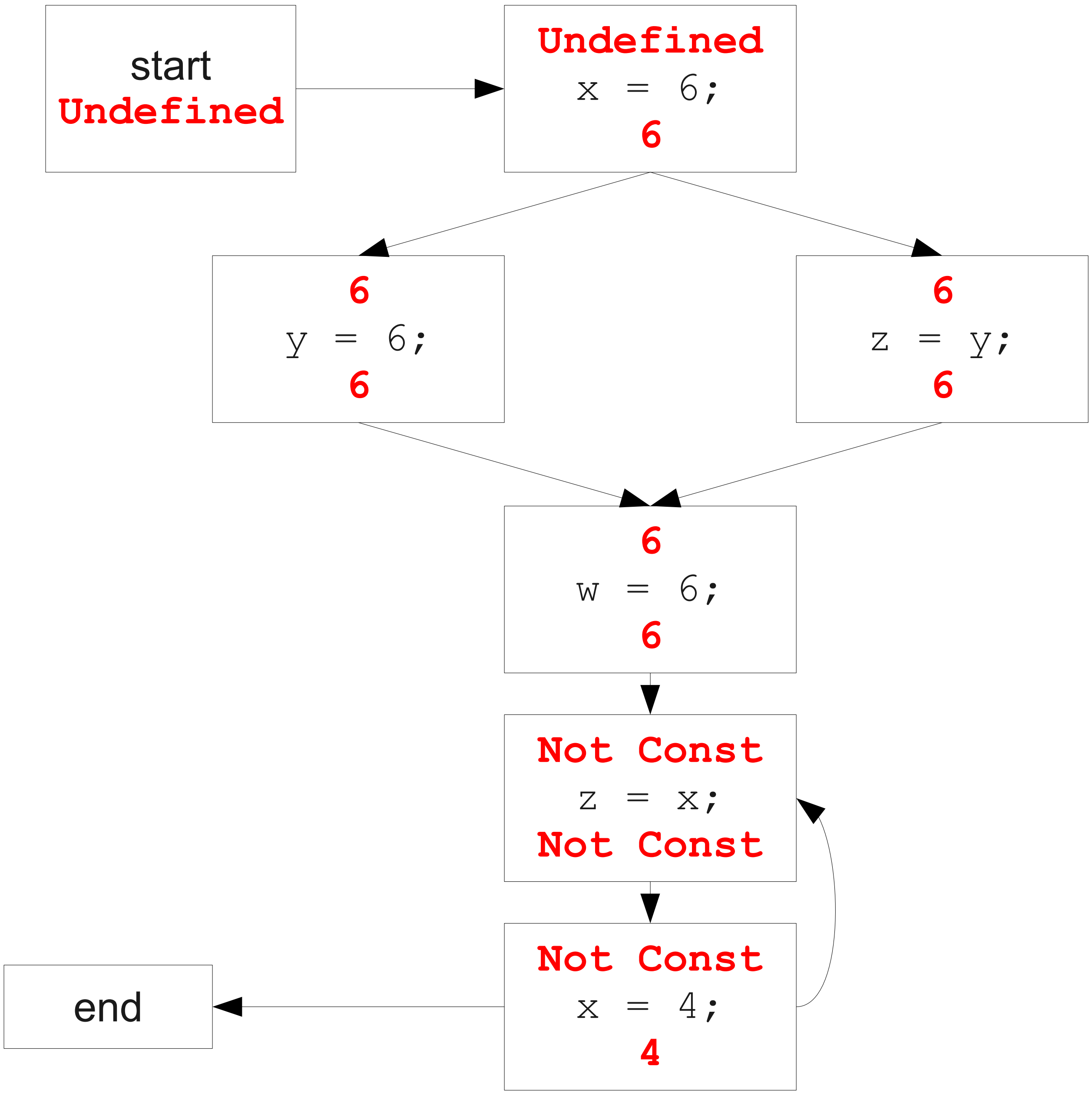

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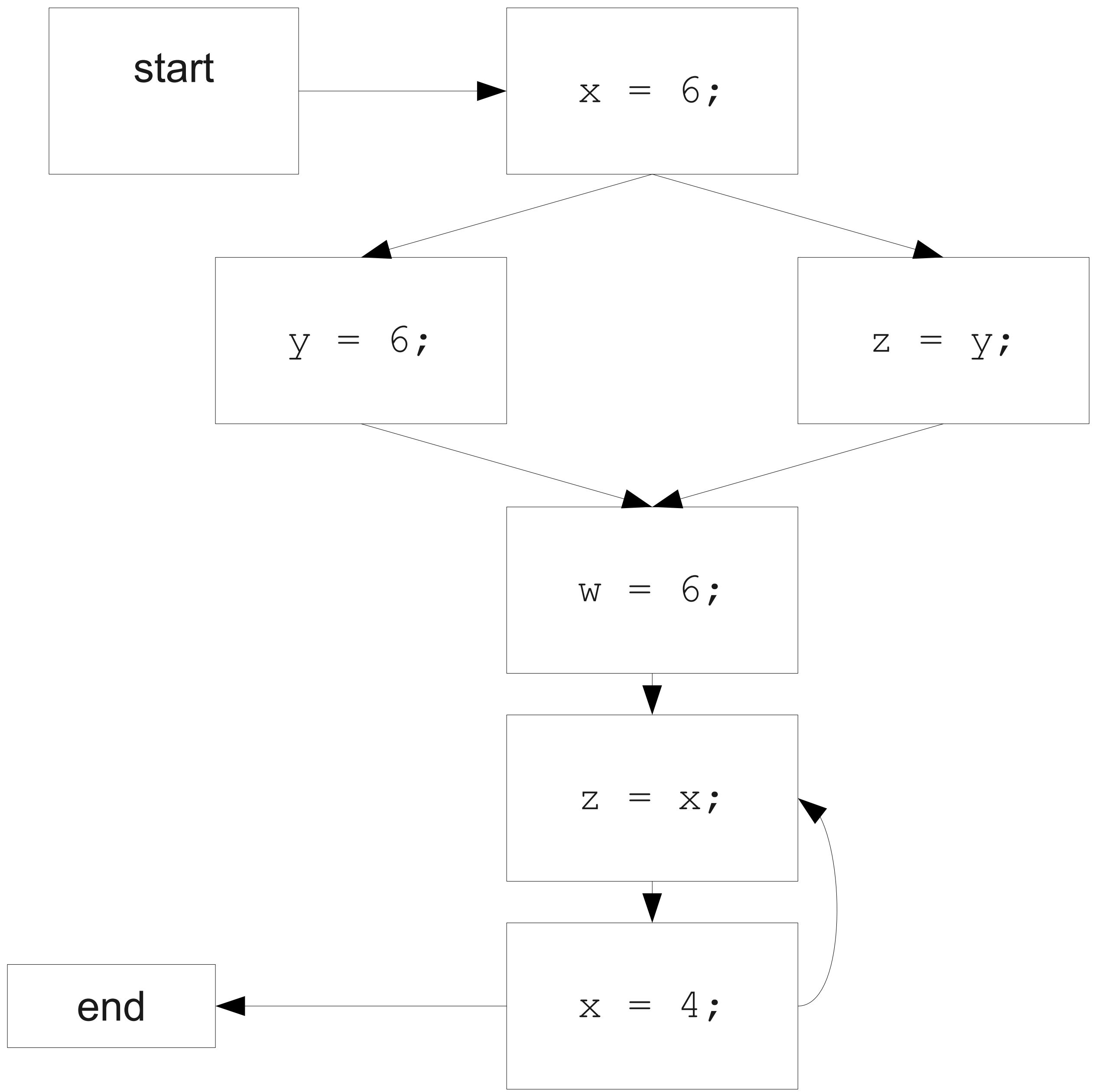


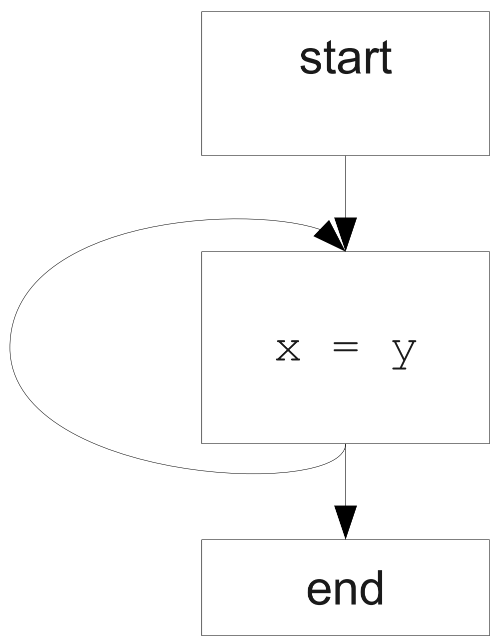
Proving Termination

- Our algorithm for running these analyses continuously loops until no changes are detected.
- Given this, how do we know the analyses will eventually terminate?
- In general, **we don't**.

A Nonterminating Analysis

- The following analysis will loop infinitely on any CFG containing a loop:
- Direction: **Forward**
- Domain: **The natural numbers 0, 1, 2, …**
- Meet operator: **max**
- Transfer function: **f(n) = n + 1**
- Initial value: **0**

A Nonterminating Analysis



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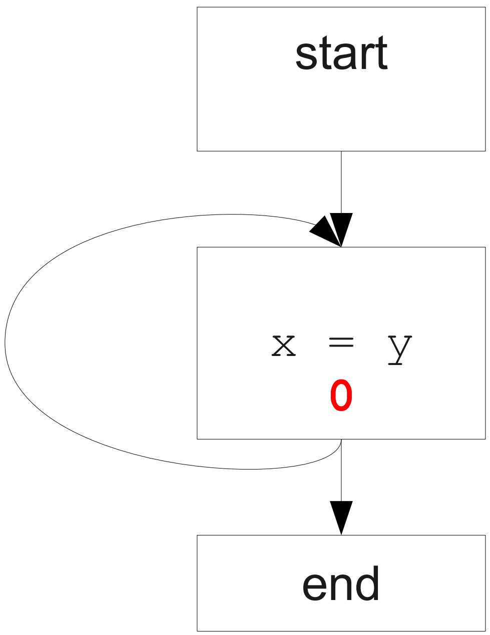

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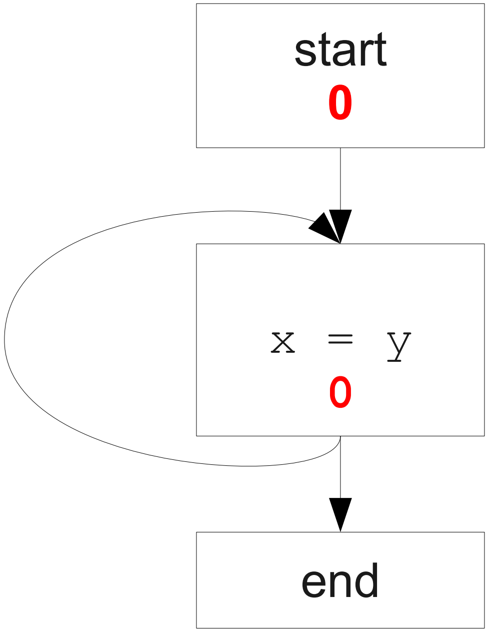

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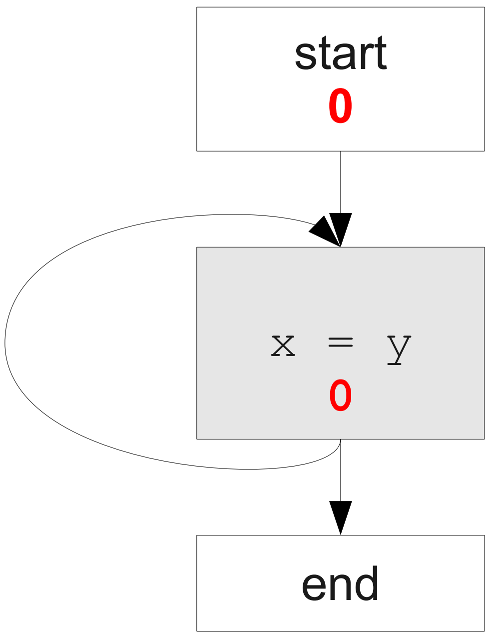

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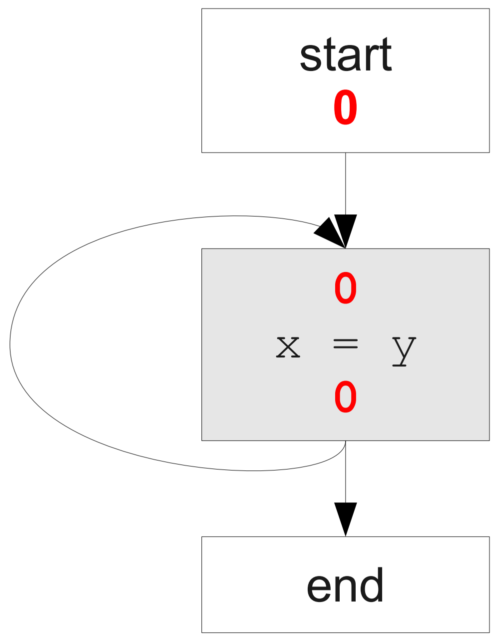

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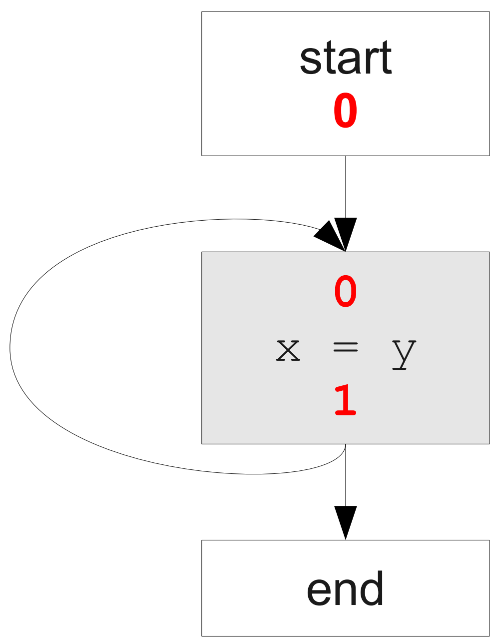

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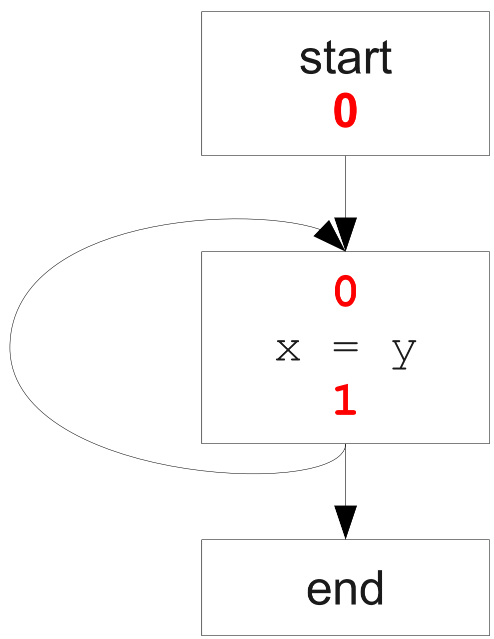

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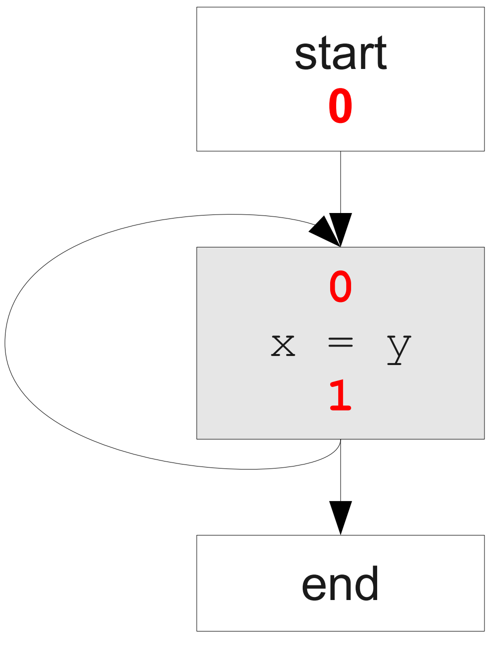

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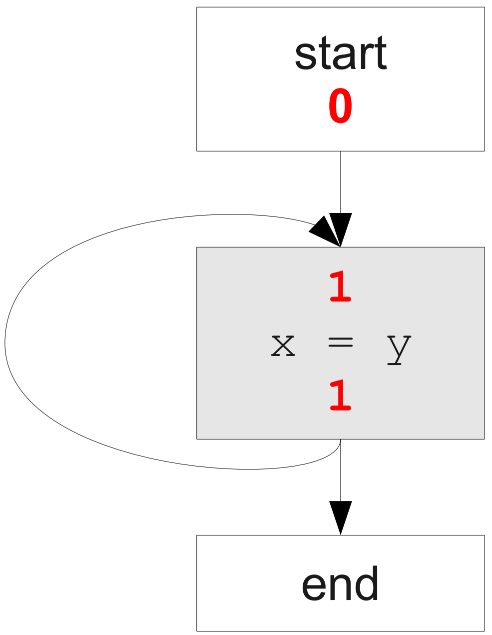

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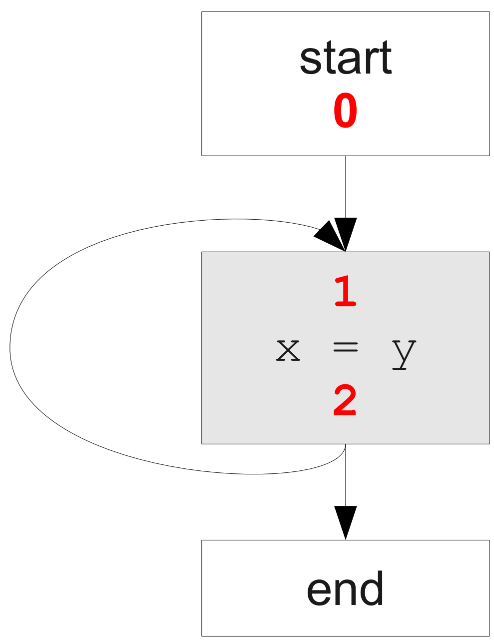

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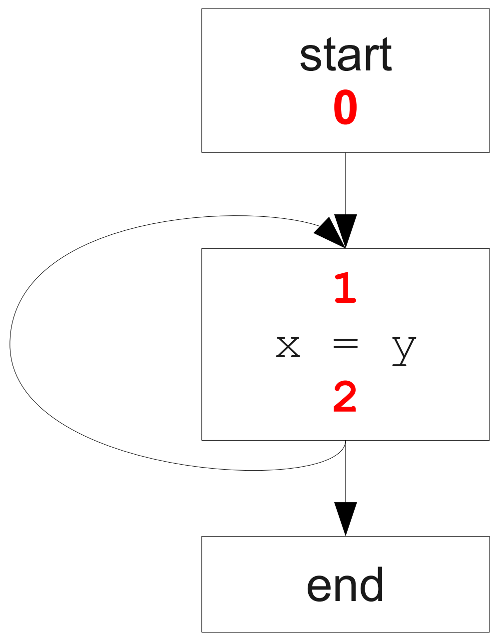

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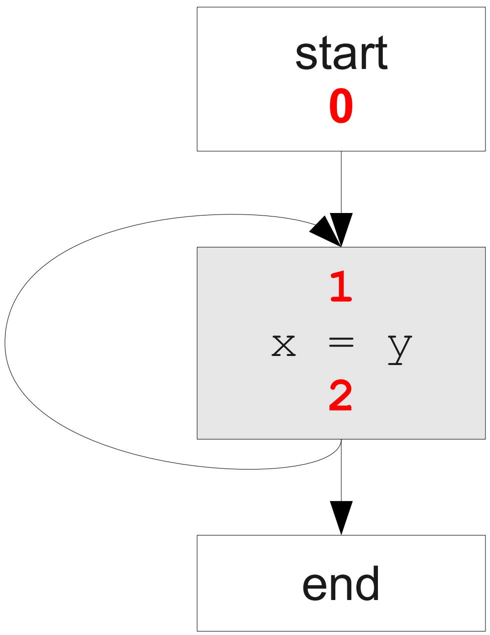

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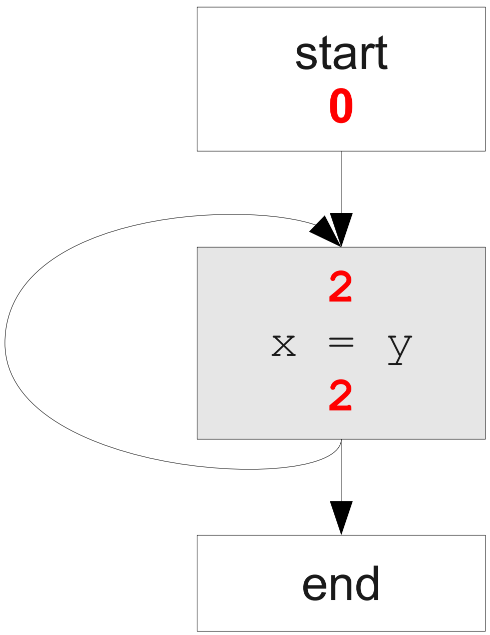

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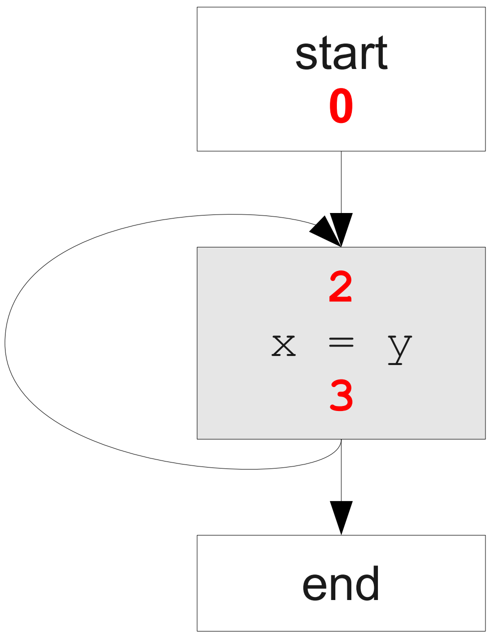

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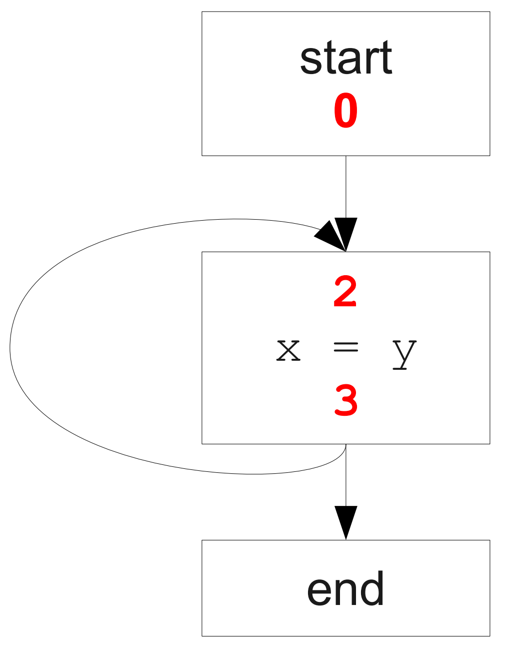
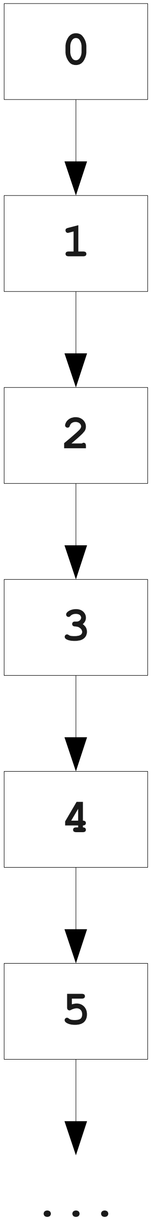


Why Doesn't This Terminate?

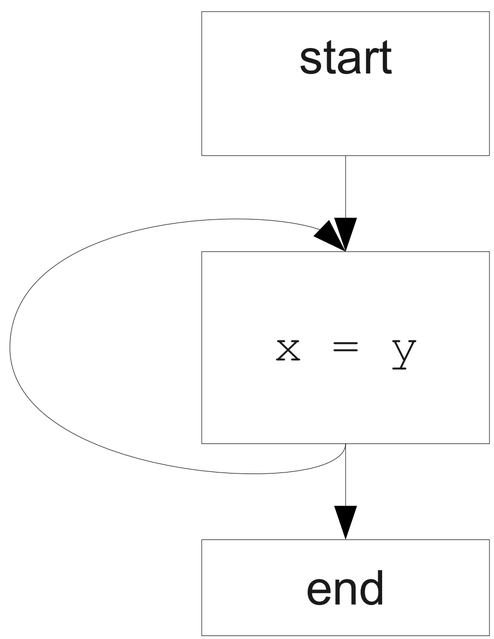
- **Values can decrease without bound.**
  - Note that “decrease” refers to the lattice ordering, not the ordering on the natural numbers.
- The **height** of a semilattice is the length of the longest decreasing sequence in that semilattice.
- The dataflow framework is not guaranteed to terminate for semilattices of infinite height.
- Note that a semilattice can be infinitely large but have finite height (e.g. constant propagation).



Another Nonterminating Analysis

- This analysis works on a finite-height semilattice, but will not terminate on certain CFGs:
- Direction: **Forward**
- Domain: **Boolean values `true` and `false`**
- Meet operator: **Logical AND**
- Transfer function: **Logical NOT**
- Initial value: **`true`**

Another Nonterminating Analysis



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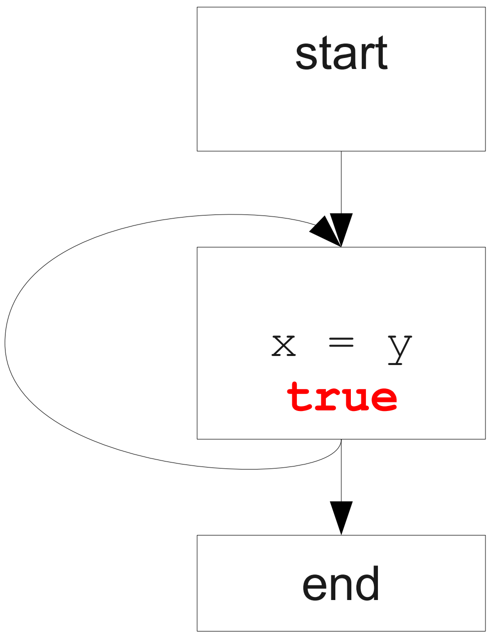

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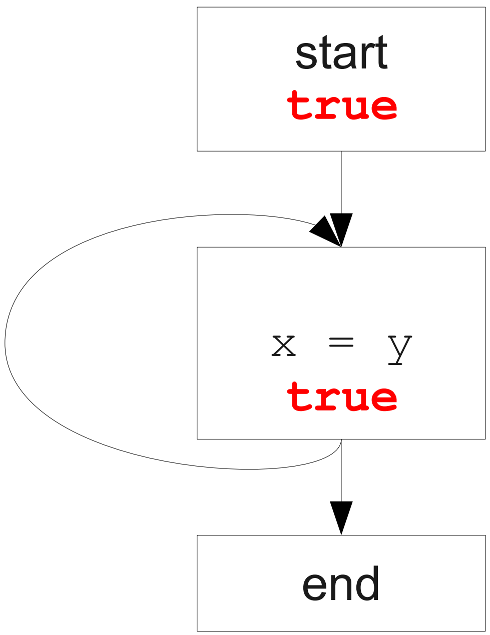

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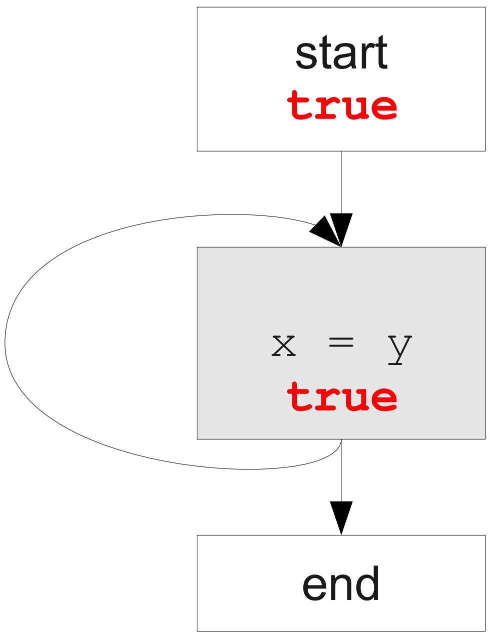

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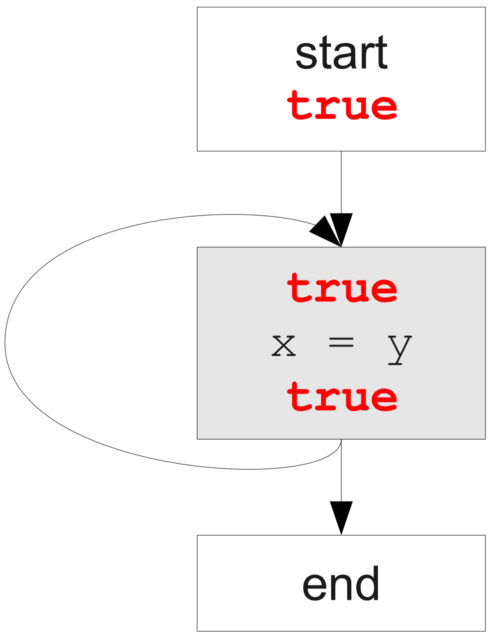

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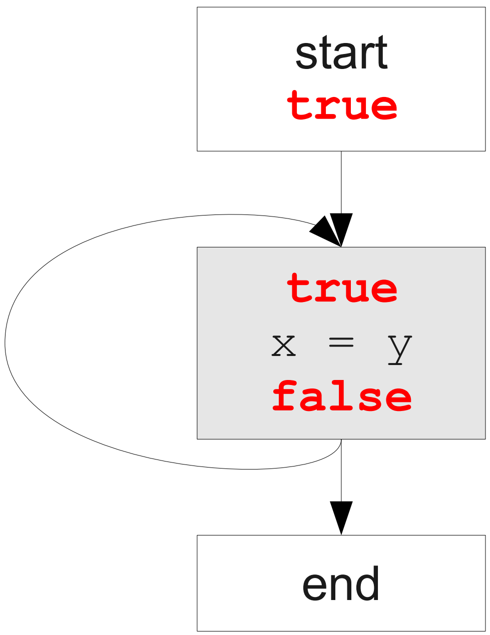

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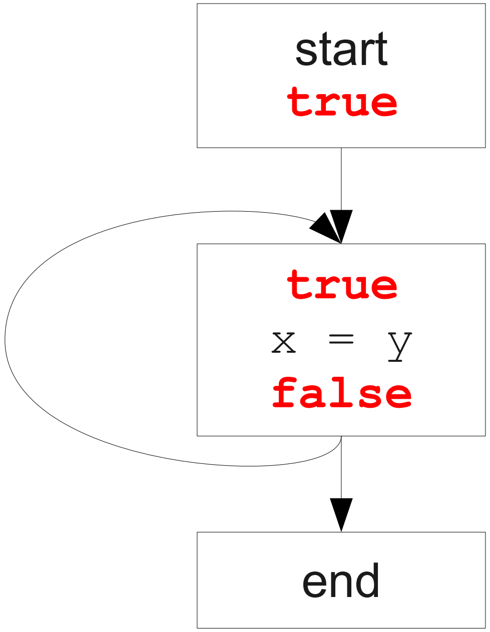

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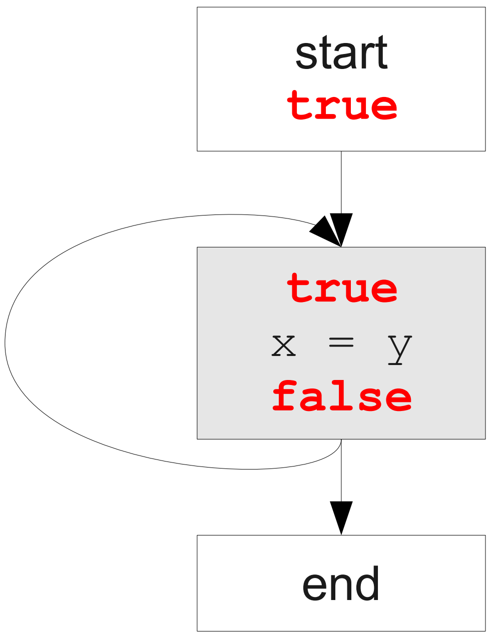

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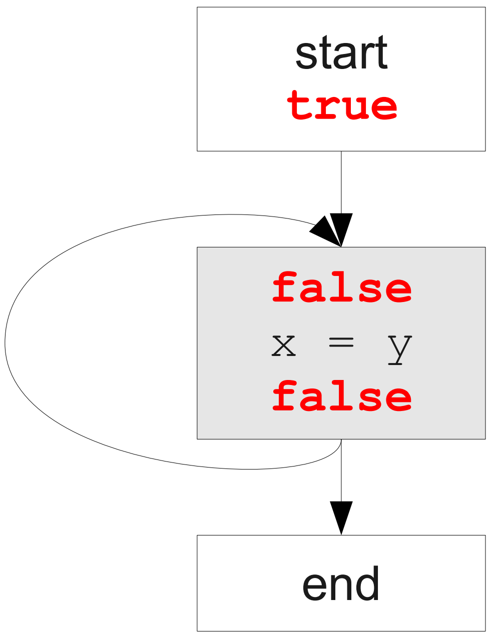

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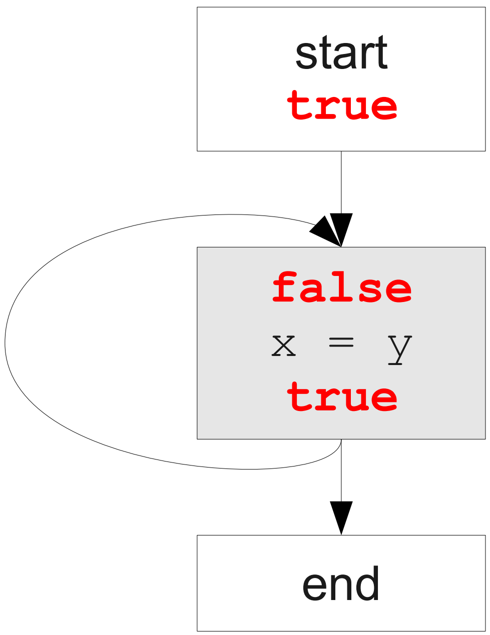

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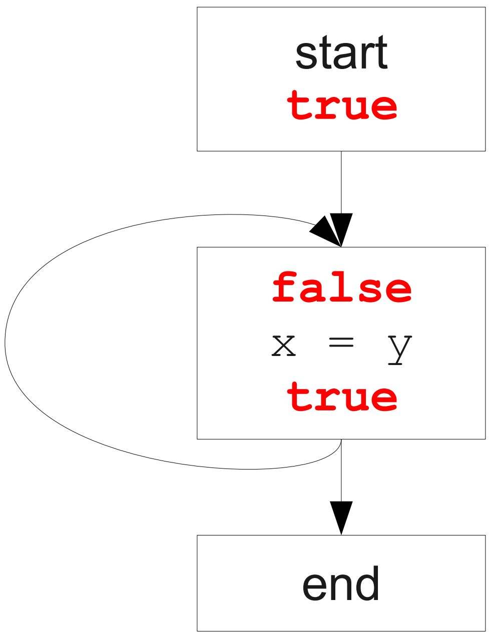

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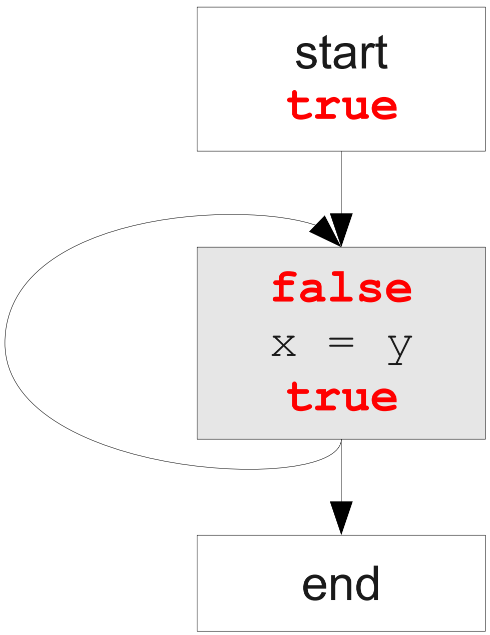

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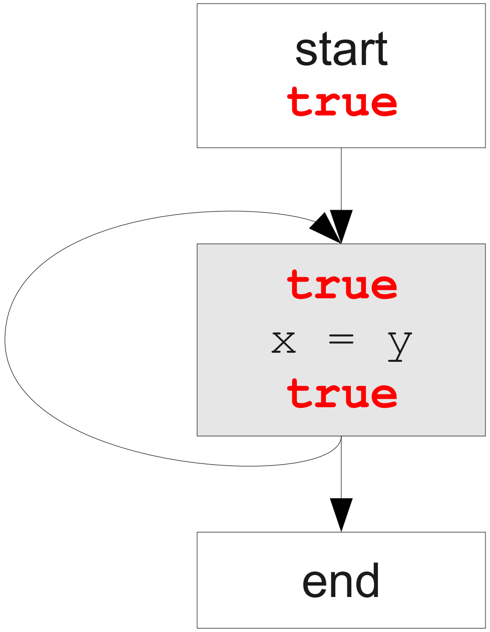

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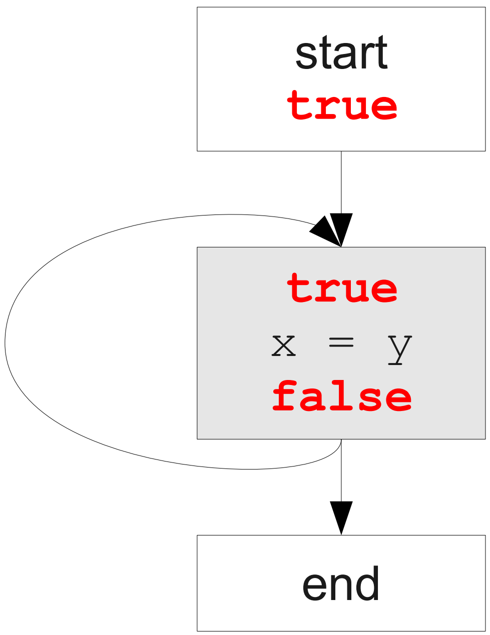

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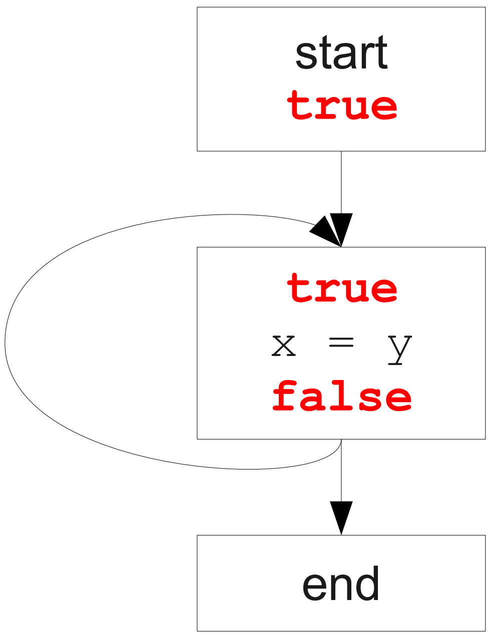

What Went Wrong (This Time)?

- **Values can loop indefinitely.**
- Intuitively, the meet operator keeps pulling values down.
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- If the transfer function can keep pushing values back up again, then the values might cycle forever.
- How can we fix this?

Monotone Transfer Functions

- A transfer function is **monotone** if for all x and y

  if $x \leq y$, $f(x) \leq f(y)$
- Intuitively, if you know less information about a program point, you can't “gain back” more information about that program point.
- Many transfer functions are monotone, including those for liveness and constant propagation.
- Note: Monotonicity does **not** mean that $f(x) \leq x$; we'll see an example.

Constant Propagation is Monotone

- A transfer function is **monotone** if for all x and y

  if $x \leq y$, $f(x) \leq f(y)$
- Recall our transfer functions are
  - $f_{x = k}(V) = k$
  - $f_{x = a + b}(V) =$ Not a Constant
  - $f_{y = a + b}(V) = V$
- The first two rules are monotone:
  - $f(x) = f(y)$ for all x and y, so $f(x) \leq f(y)$ for all x and y.
- The last rule is monotone:
  - if $x \leq y$, $f(x) = x \leq y = f(y)$

Liveness is Monotone

- A transfer function is **monotone** if for all x and y

  if $x \leq y$, $f(x) \leq f(y)$
- Recall our transfer function for **a** = **b** + **c** is
  - $f_{a = b + c}(V) = (V - \mathbf{a}) \cup \{\mathbf{b}, \mathbf{c}\}$
- Recall that our meet semilattice has set union as a transfer function and induces an ordering relationship $X \leq Y$ iff $X \supseteq Y$.
- Suppose $X \supseteq Y$.
- Then $(X - \mathbf{a}) \supseteq (Y - \mathbf{a})$.
- Then $(X - \mathbf{a}) \cup \{\mathbf{b}, \mathbf{c}\} \supseteq (Y - \mathbf{a}) \cup \{\mathbf{b}, \mathbf{c}\}$.
- So $f(X) \supseteq f(Y)$.

The Grand Result

- **Theorem:** A dataflow analysis with a finite-height semilattice and family of monotone transfer functions always terminates.
- Proof sketch:
  - Run the data-flow iteration once to get some initial values.
  - From this point forward:
    - The meet operator can only bring values down.
    - The transfer function can never raise values back up above where they were in the past (monotonicity)
    - Values cannot decrease indefinitely (finite height)

Partial-Redundancy Elimination

Code Size is Not Execution Time

- All of the analyses we've seen so far have worked by simplifying or eliminating IR code.
- However, much of optimization results from **moving** code from one basic block to another.

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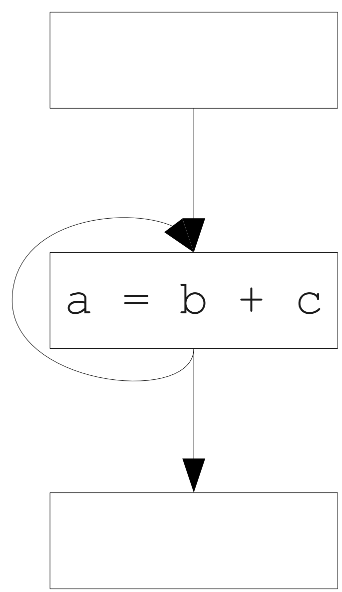

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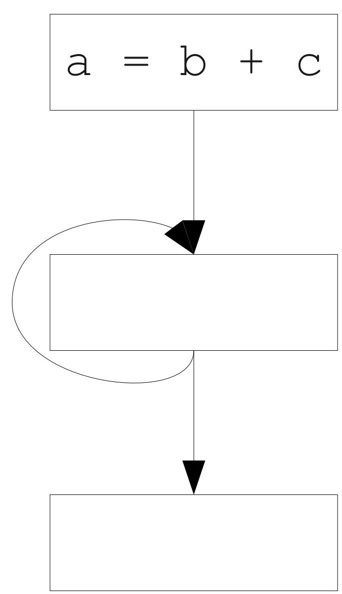

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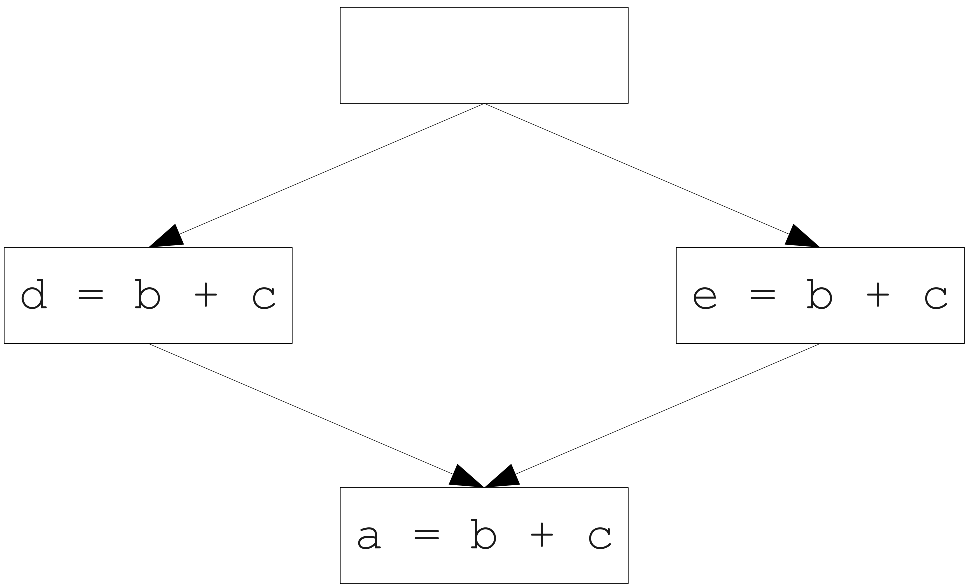

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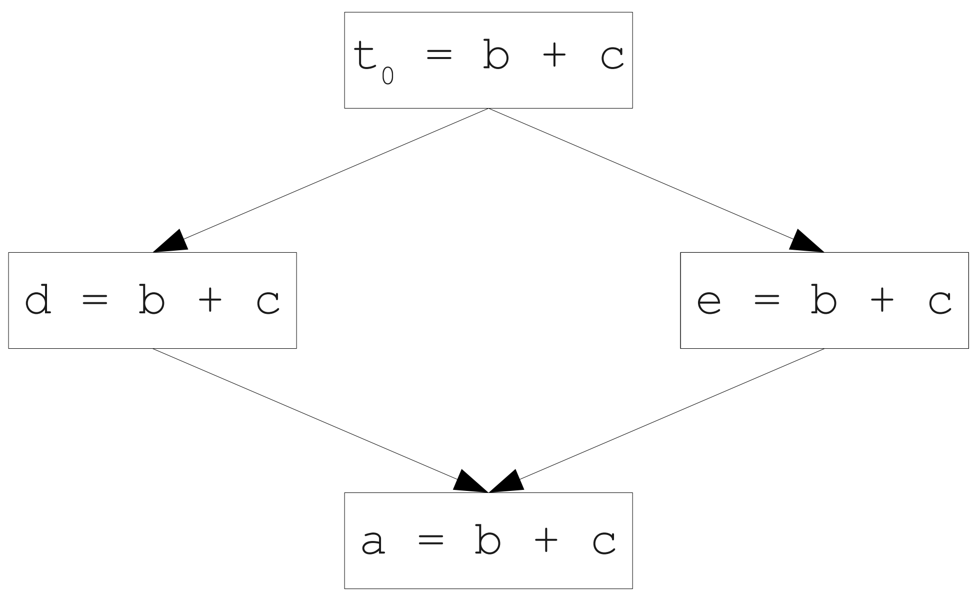

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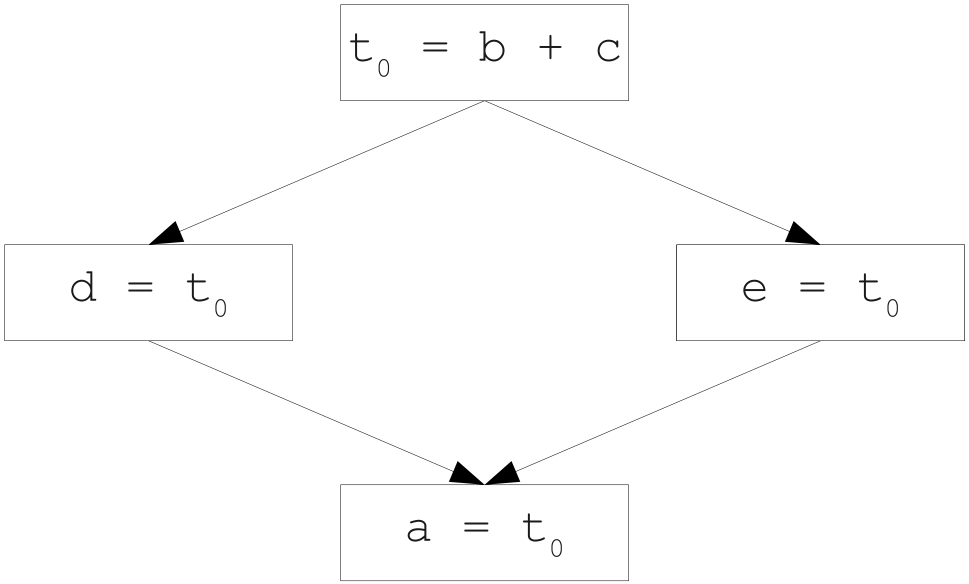
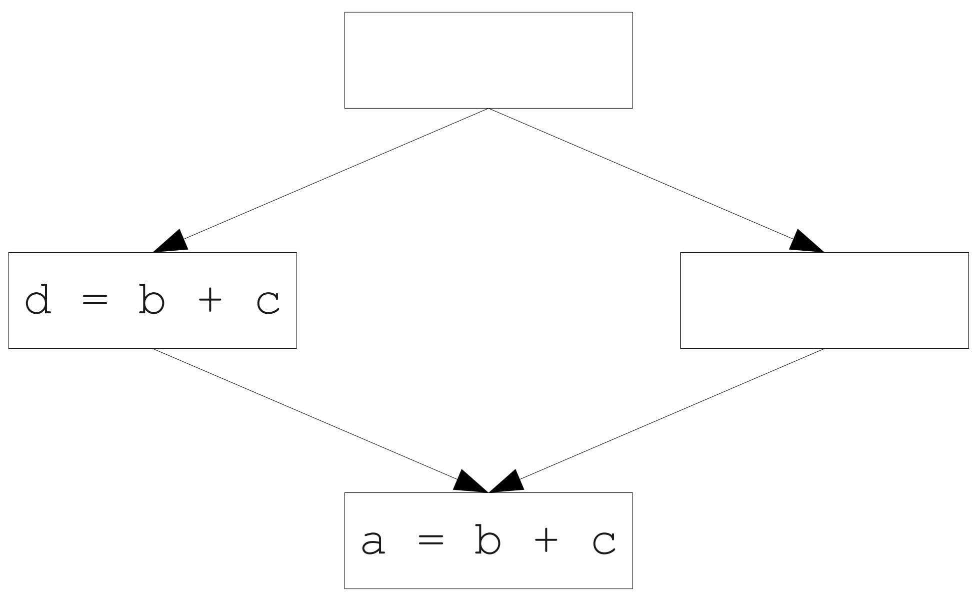


Eliminating Redundancy

- A computation in a program is said to be **redundant** if it computes a value that is already known.
- Common subexpressions are one example of redundancy.
- Loop-invariant code is another example.
- Virtually all optimizing compilers have some logic to try to eliminate redundancy.

Partial Redundancy

- One of the trickiest cases of redundancy to eliminate is **partial redundancy**.
- A computation is **partially redundant** if its value is known on only some of the paths that reach it.



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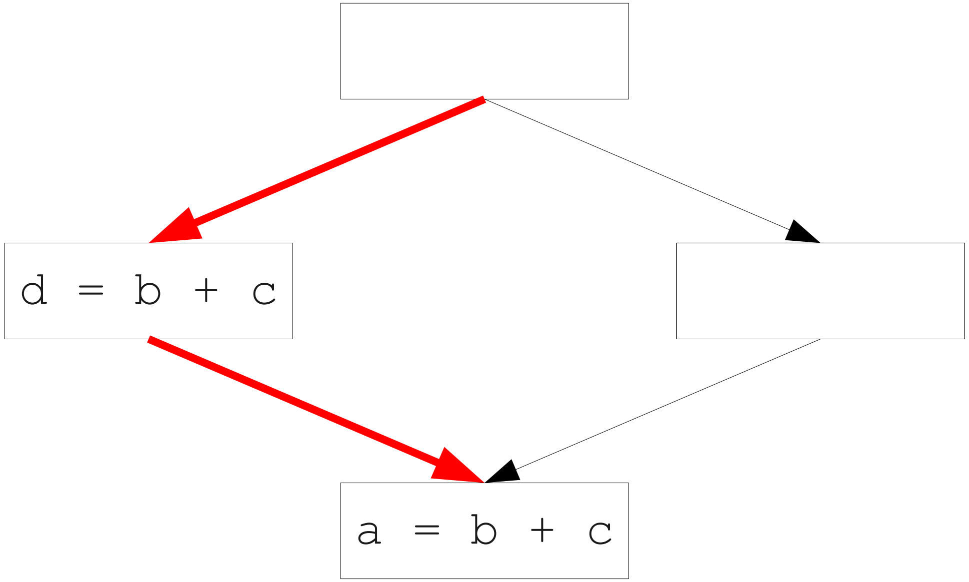

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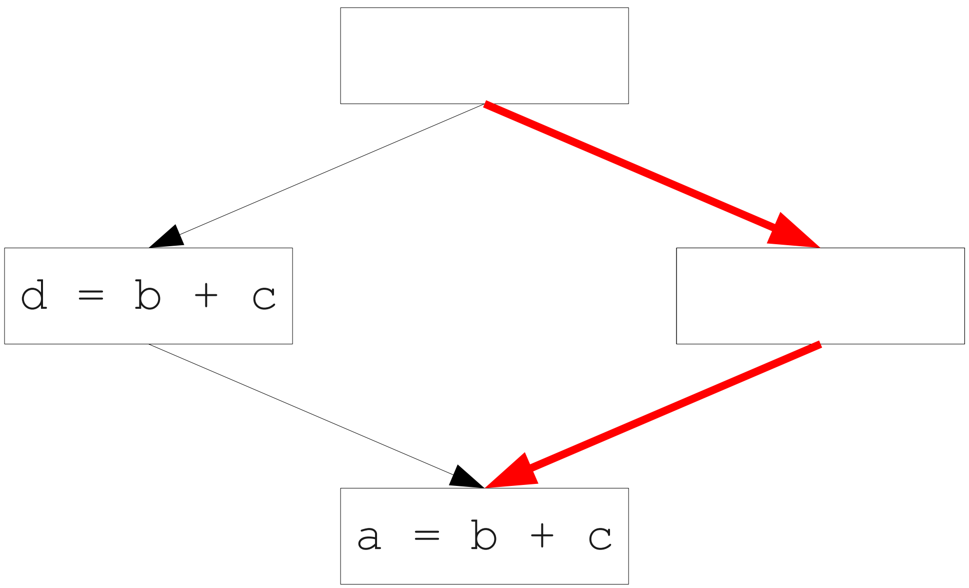
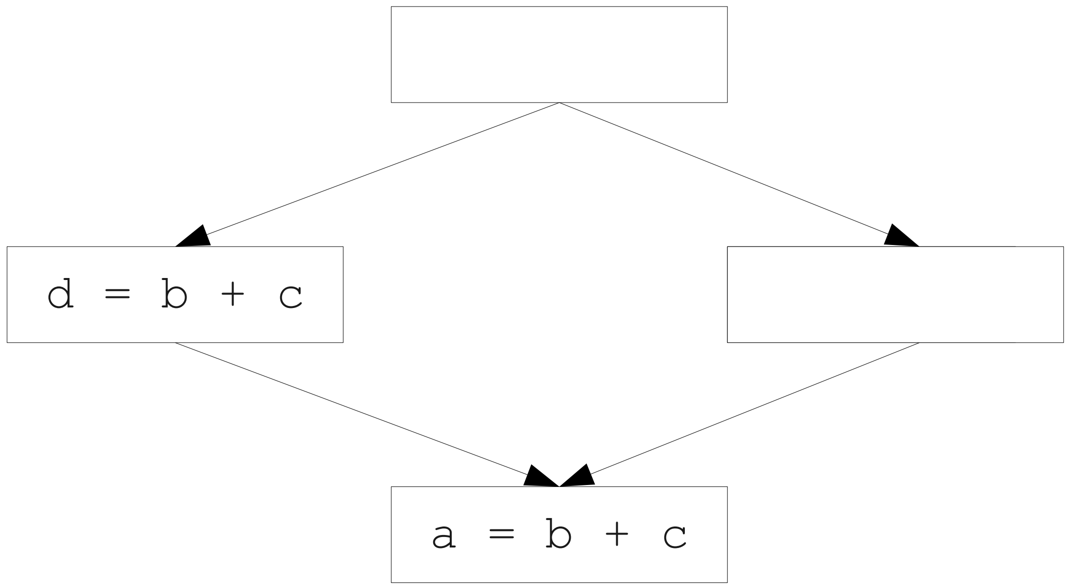


Eliminating Partial Redundancy

- Goal: Eliminate partial redundancy without making any execution of the program do more work than before.
- Optimized code should **always** be at least as good as the original.

The Key Observation



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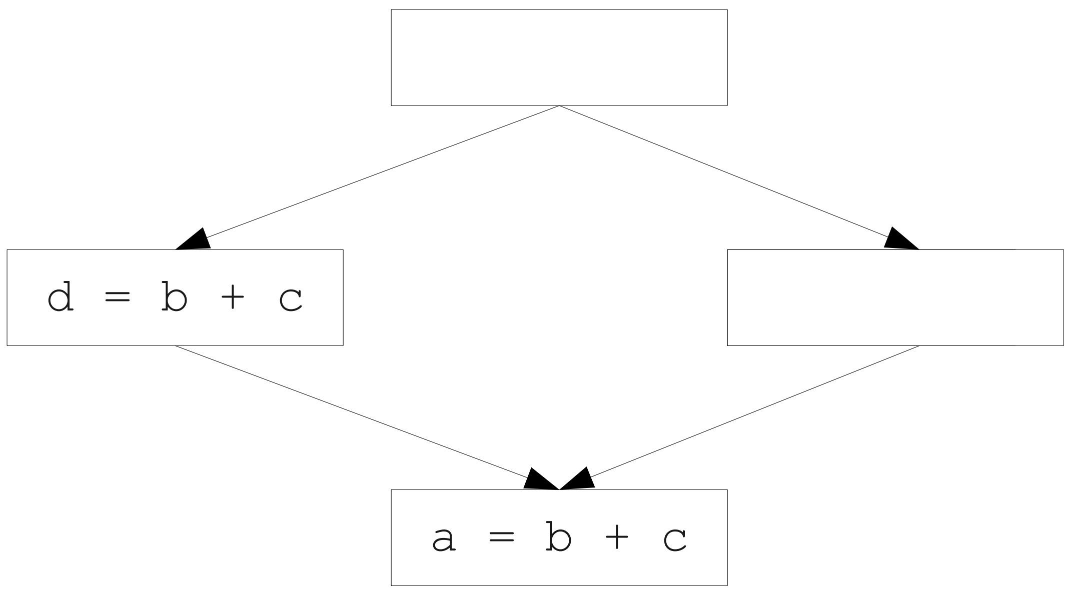

Where in the program is it <u>guaranteed</u> that we will need the value of **b + c**?

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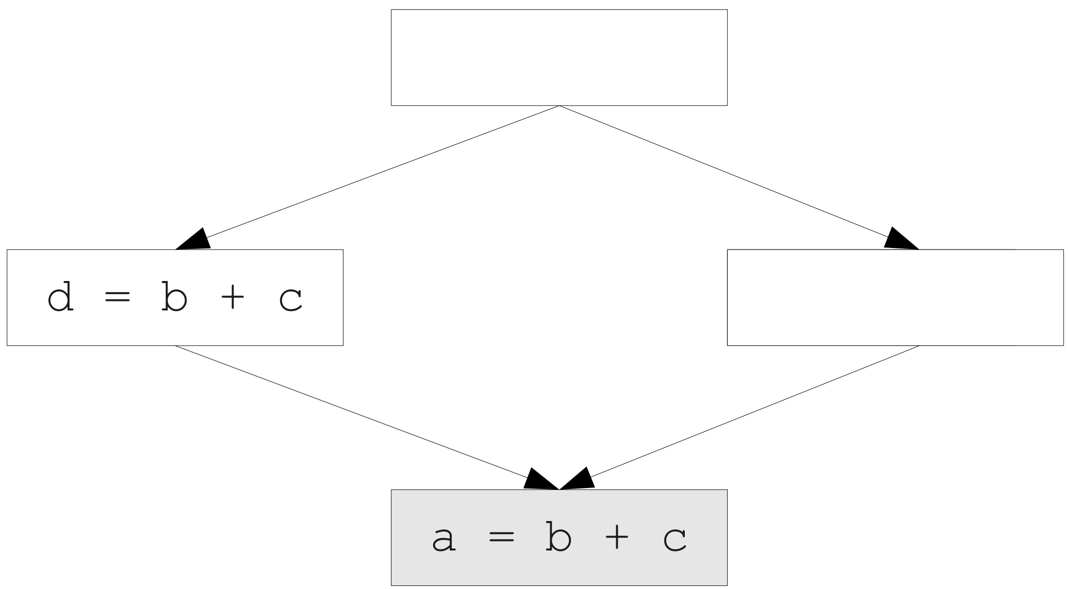

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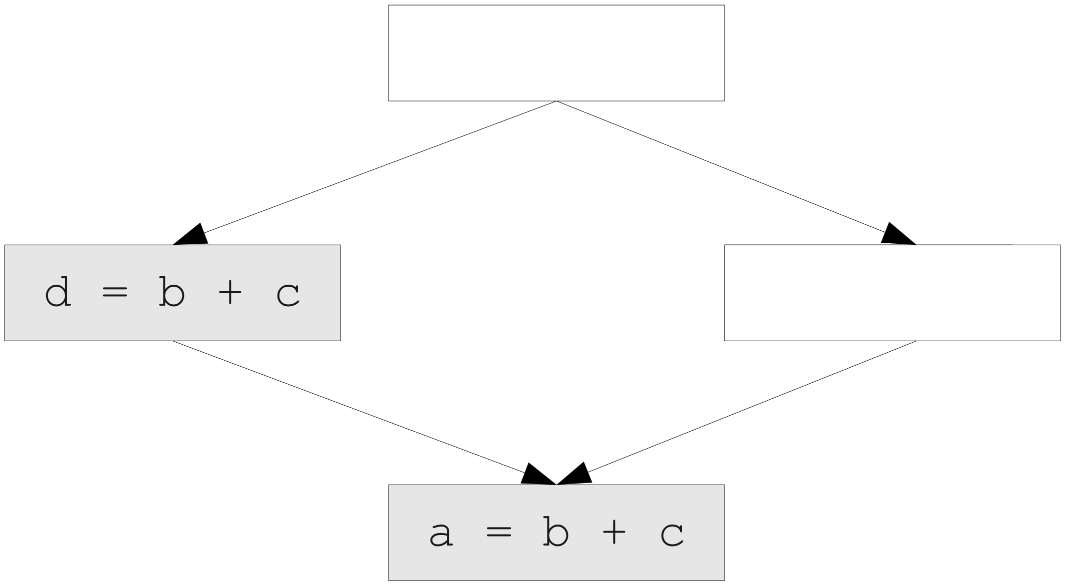

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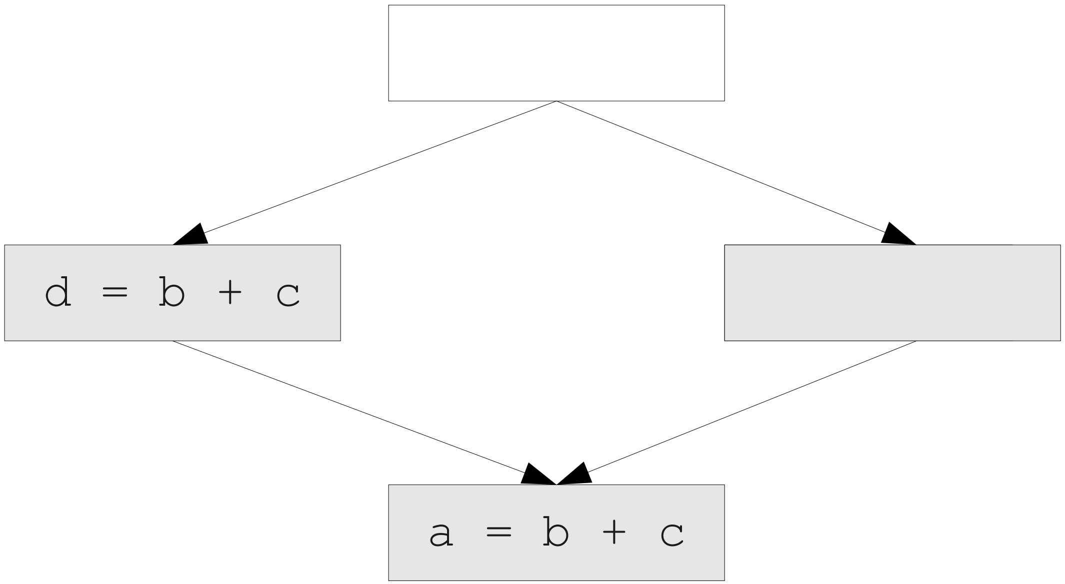

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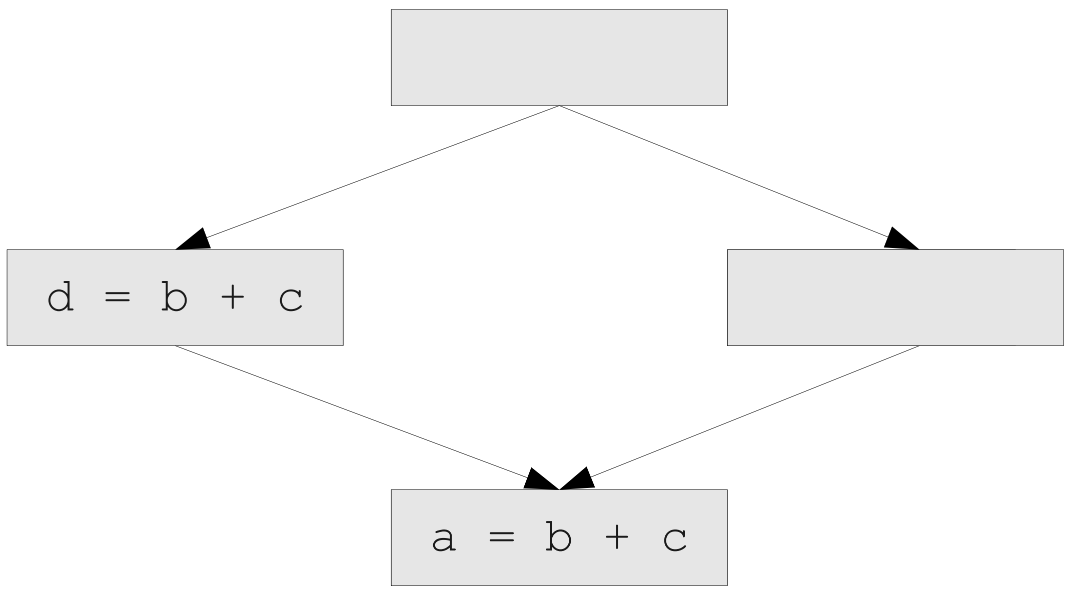
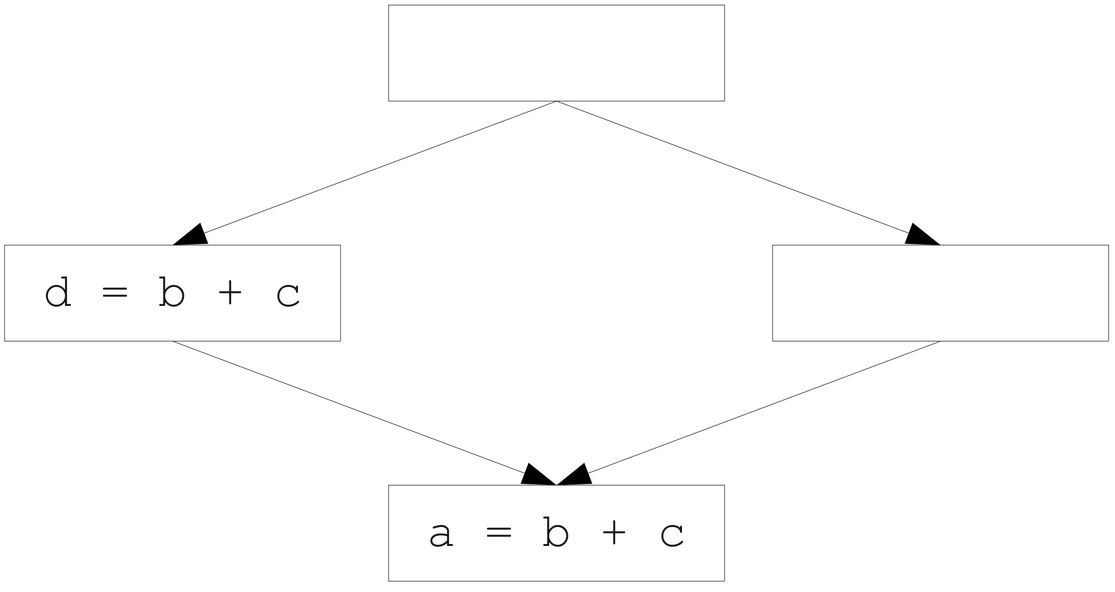


Where in the program is it guaranteed that we will need the value of **b + c**?

Although not all paths through the program might directly need an expression, they may **anticipate** the expression.

The Second Key Observation



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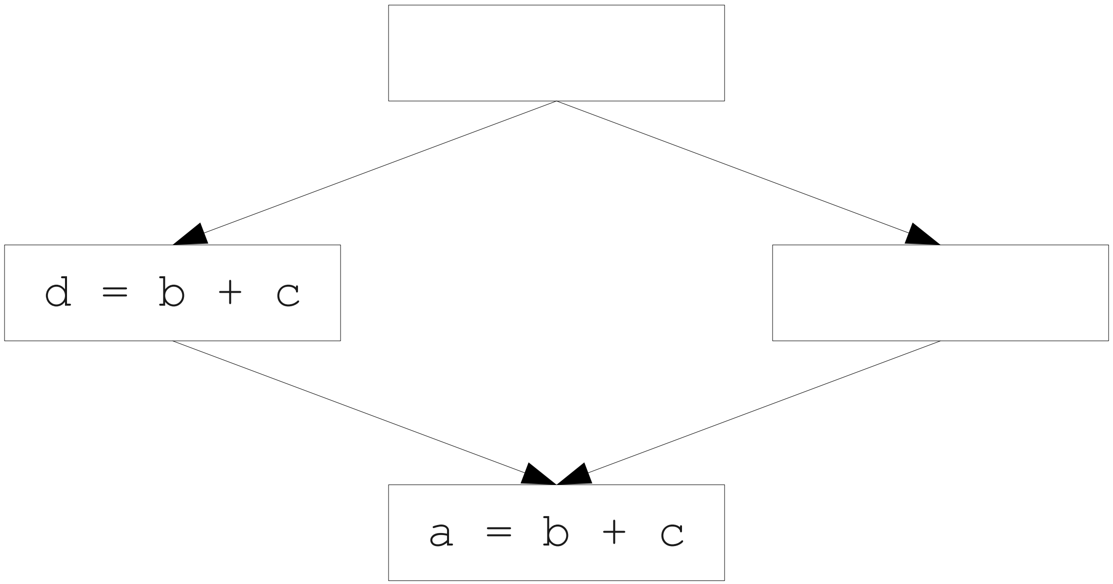

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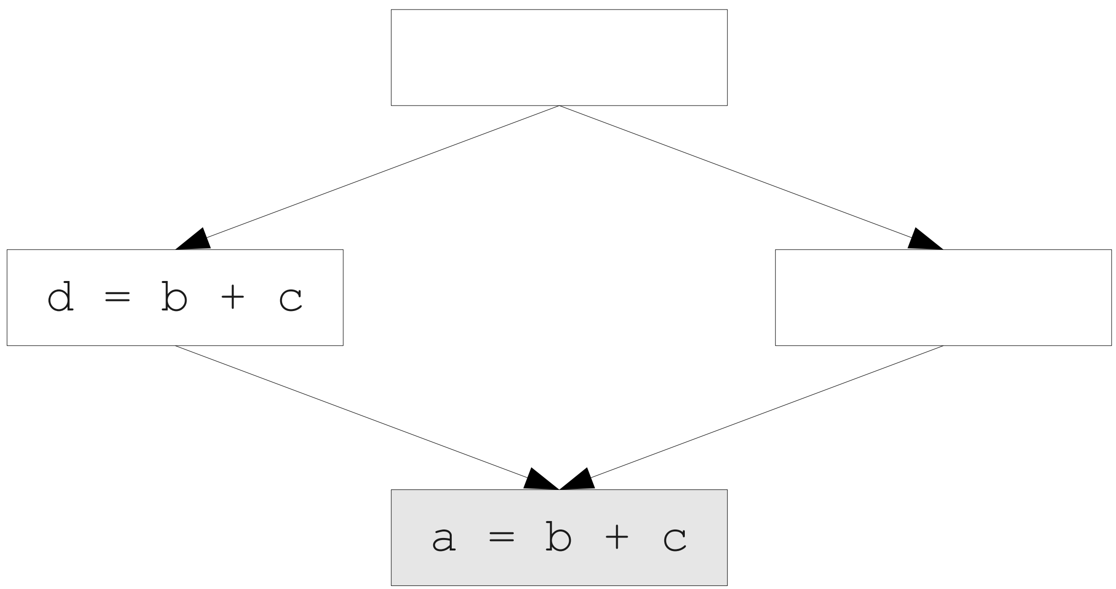

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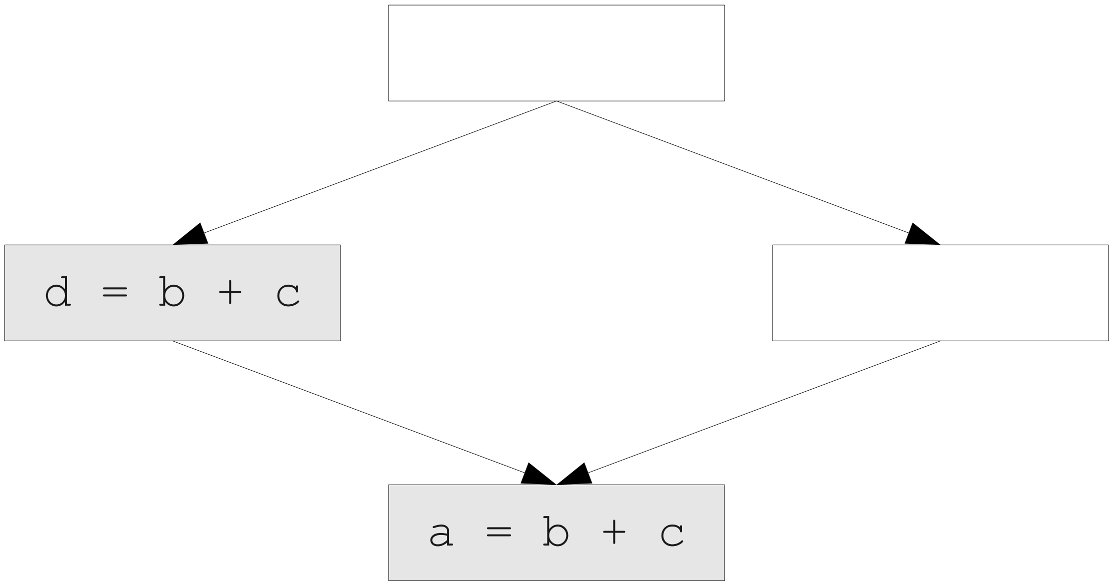
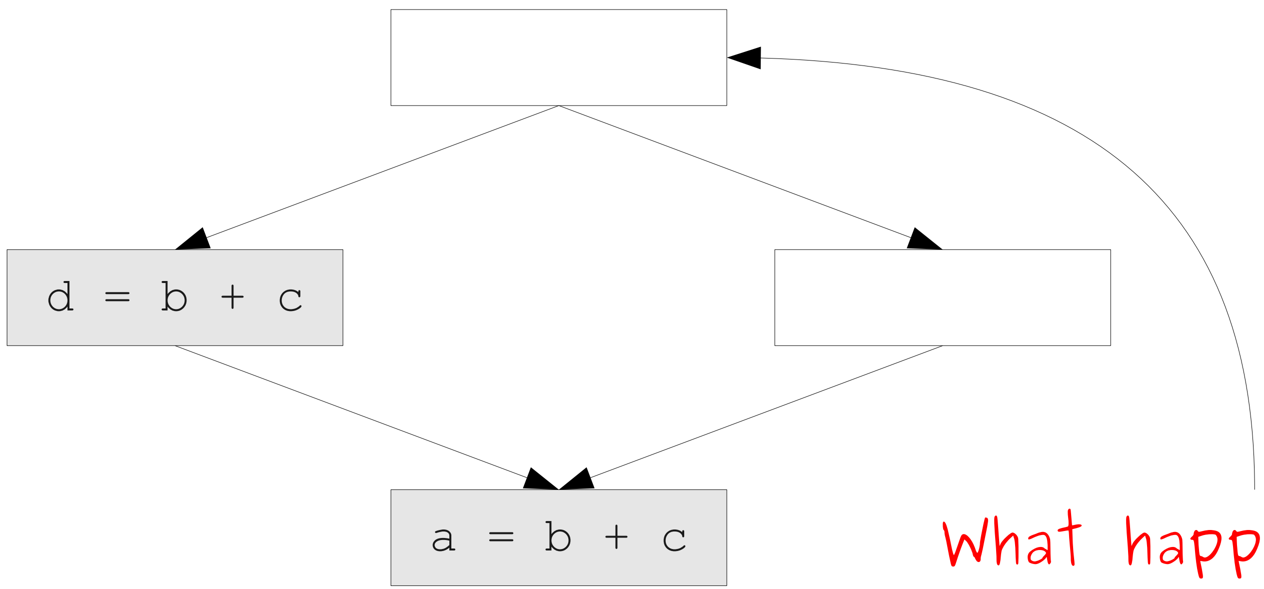
The Second Key Observation



Where in the program
is the value of **b + c**
already computed?

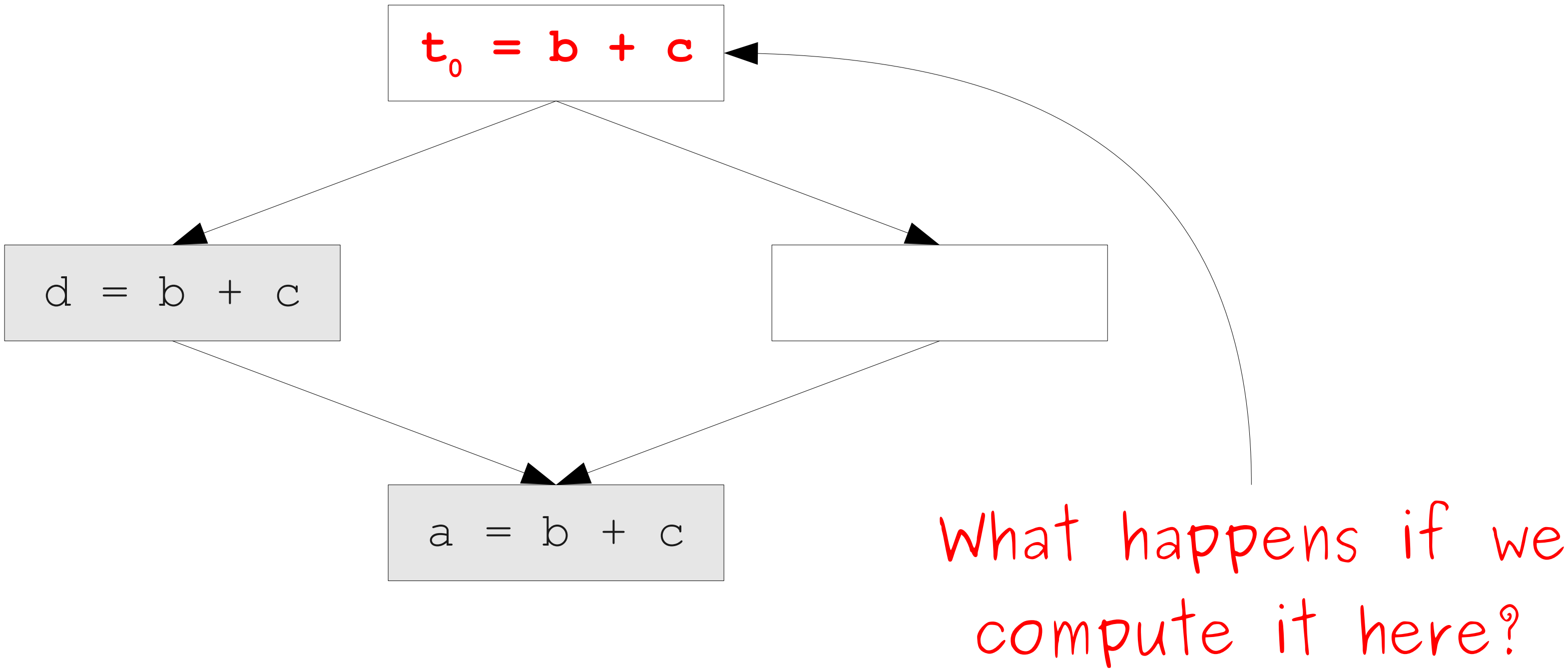
The Second Key Observation



What happens if we compute it here?

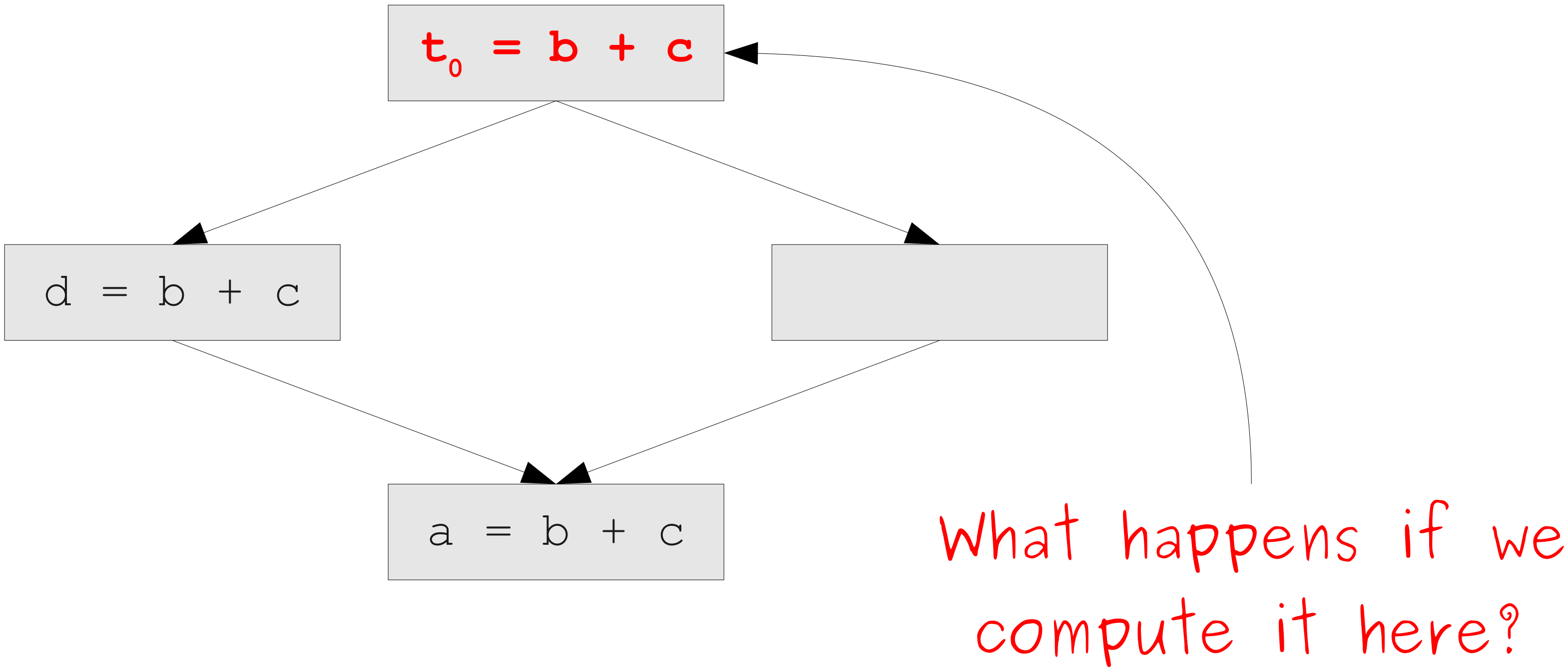
Where in the program is the value of **b + c** already computed?

The Second Key Observation



Where in the program is the value of **b + c** already computed?

The Second Key Observation



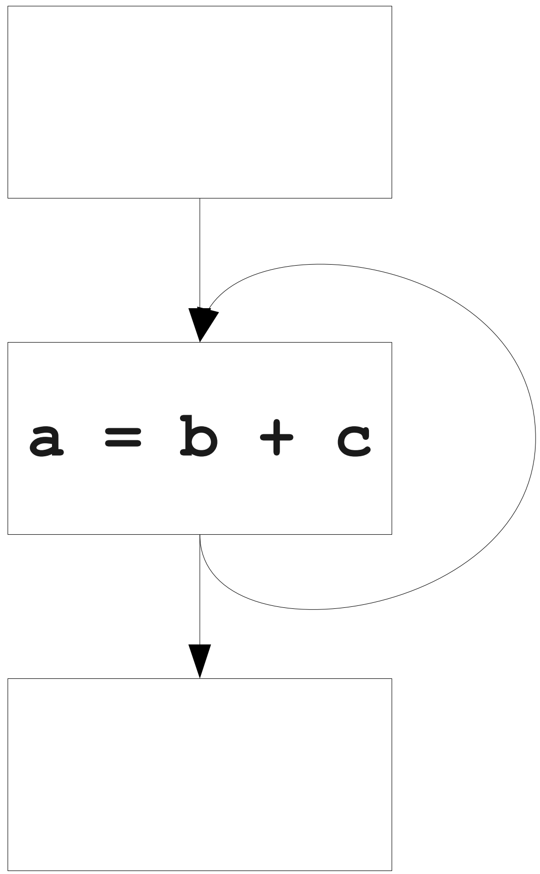
Where in the program is the value of **b + c** already computed?

Partial-Redundancy Elimination

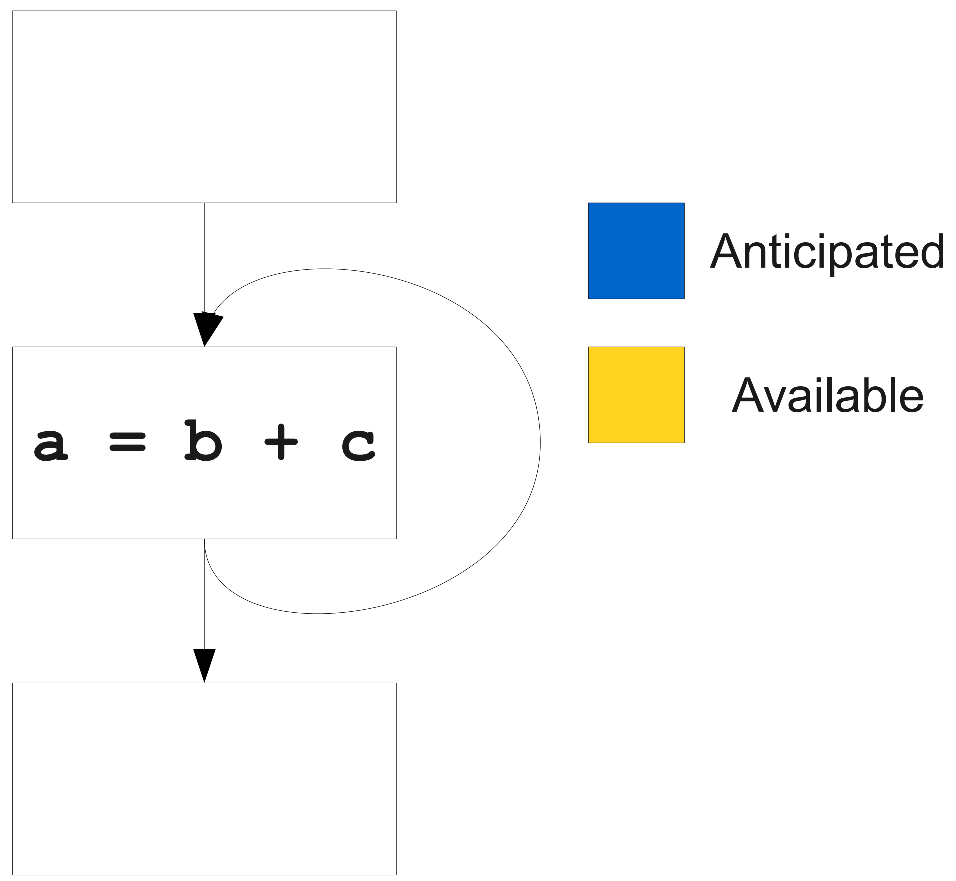
- **Idea:** Make the expression available everywhere that it's anticipated.
- Run an analysis to locate where the expression is anticipated.
- Run a second analysis to locate where the expression is available.
- Place the expression at the earliest locations where the expression is **anticipated** but not **available**.

Eliminating Redundancy

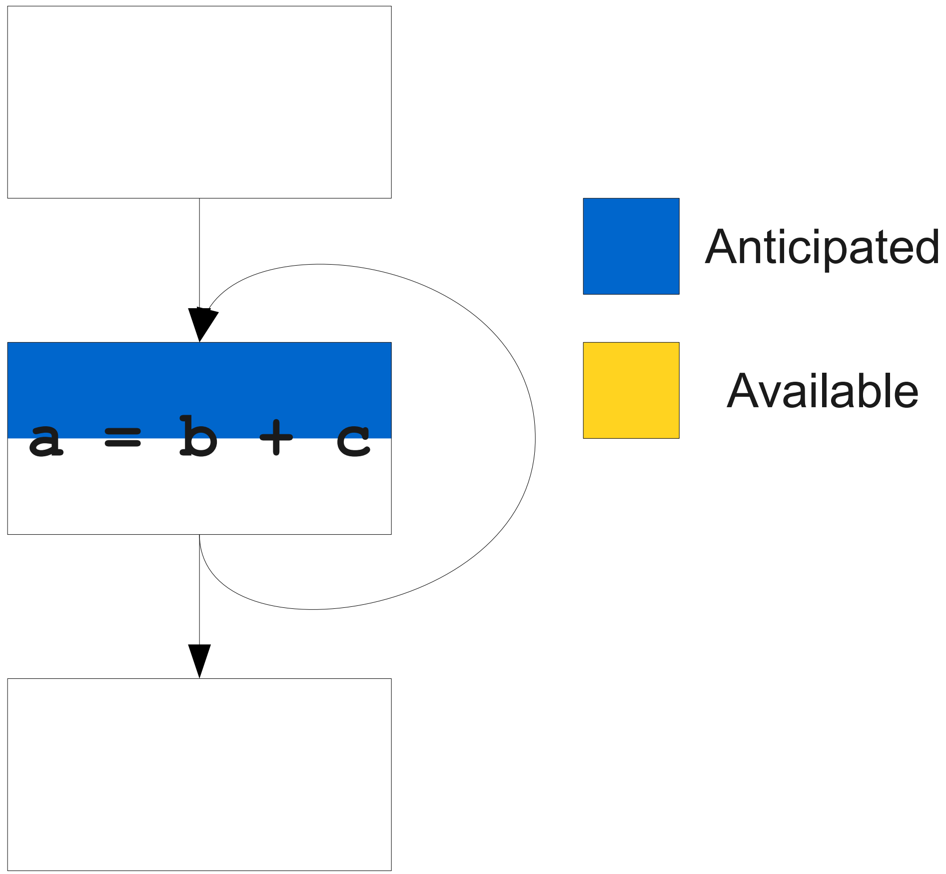
Eliminating Redundancy



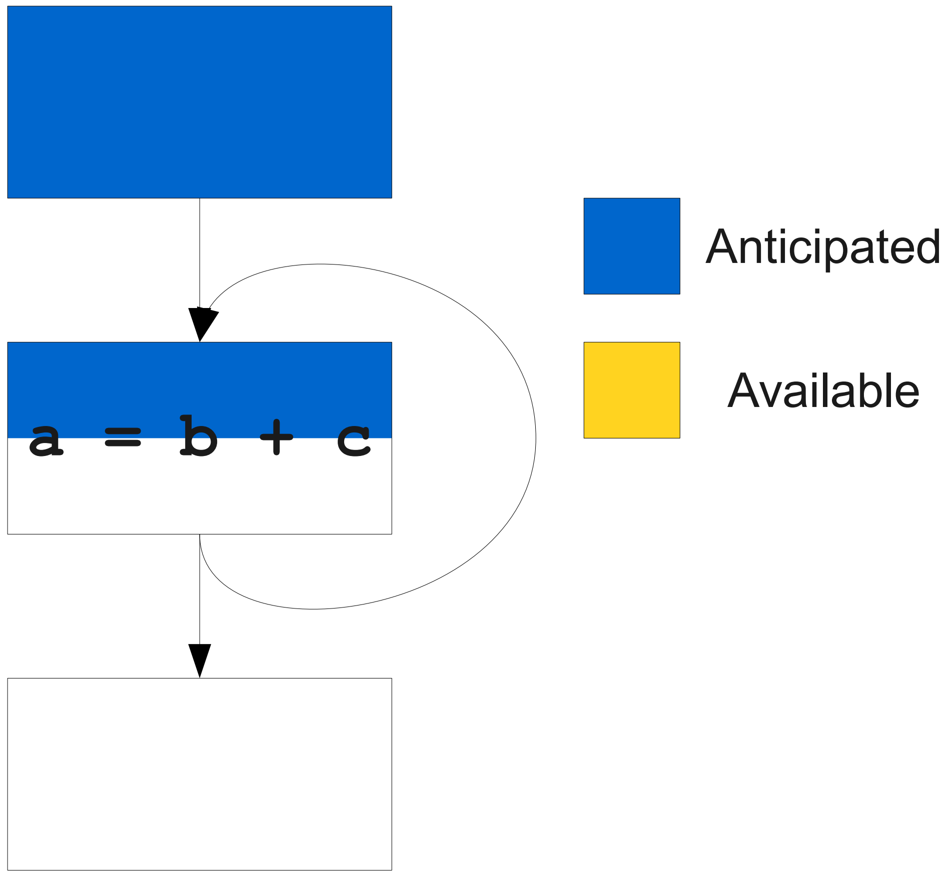
Eliminating Redundancy



Eliminating Redundancy



Eliminating Redundancy



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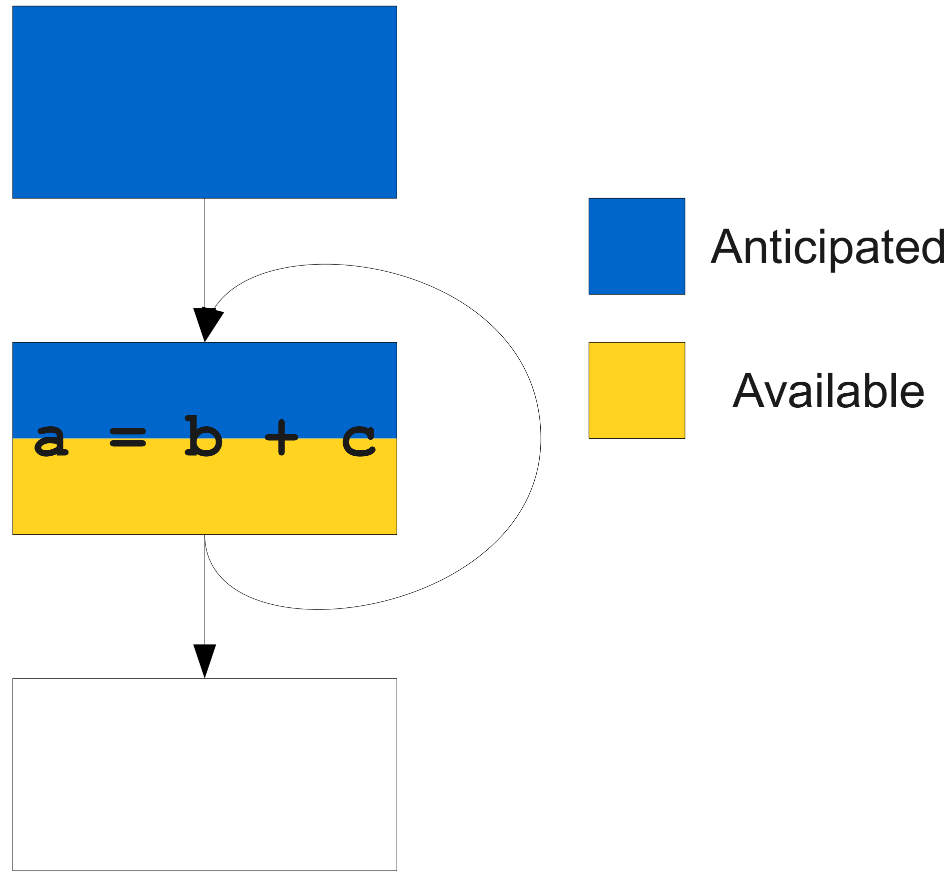

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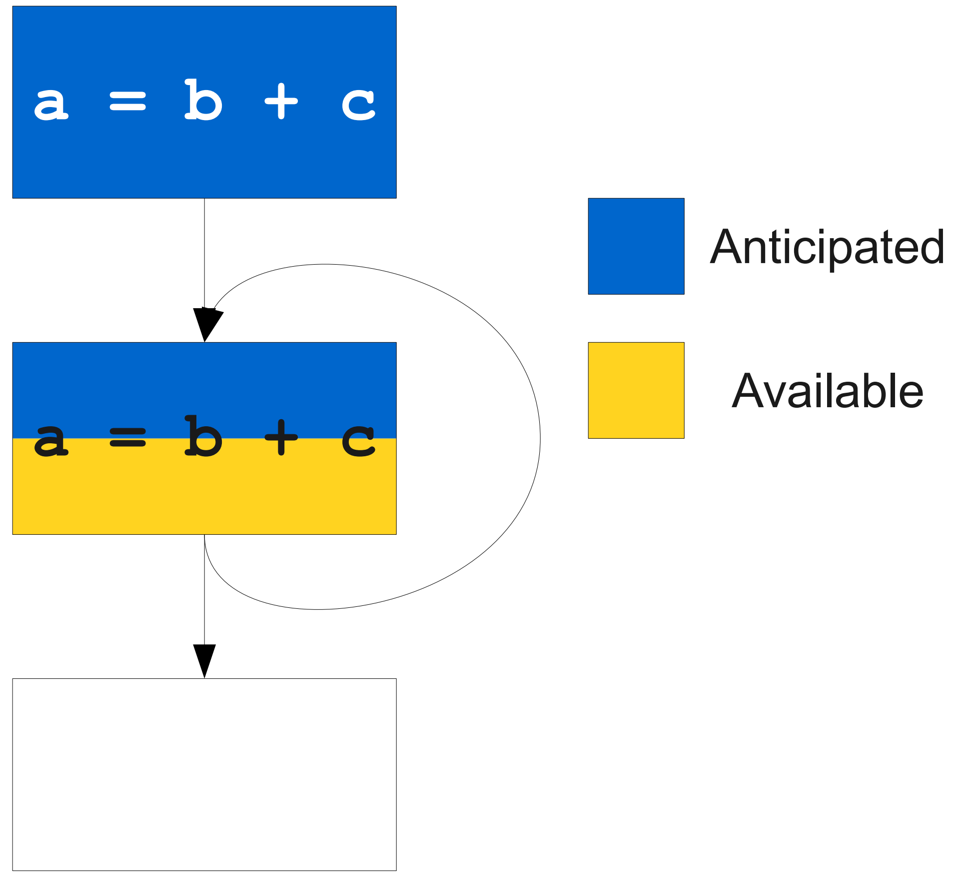

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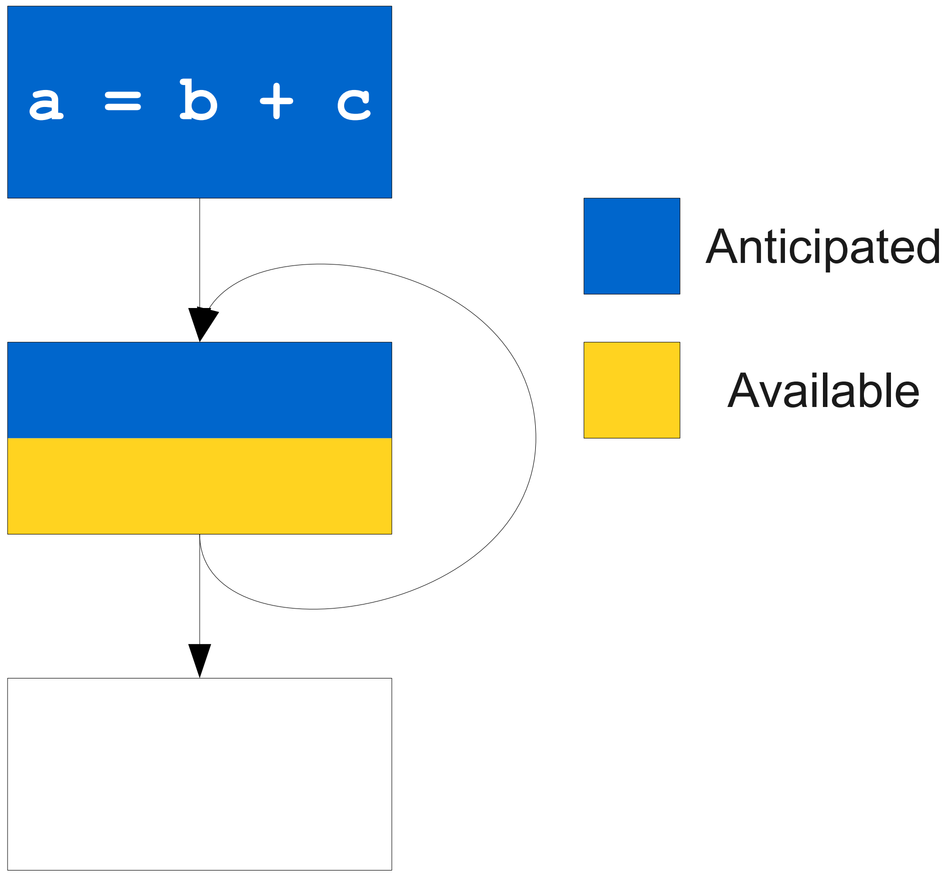
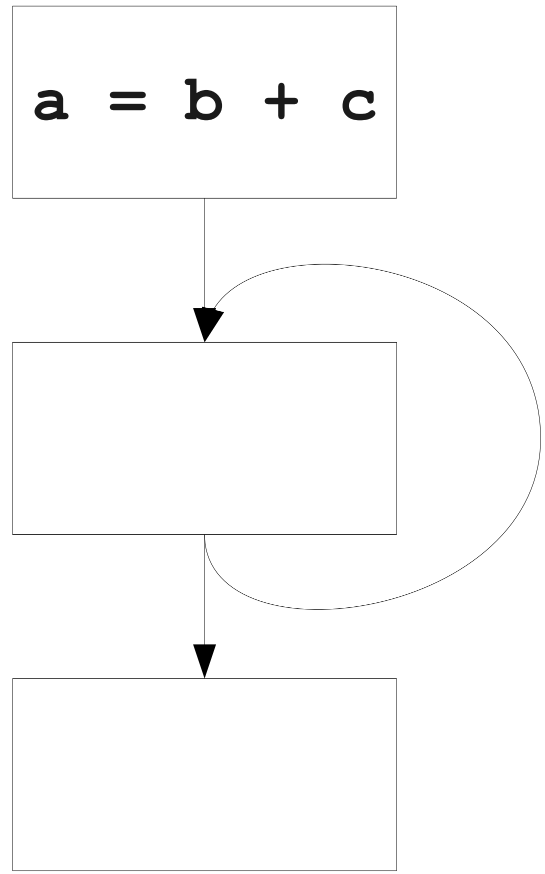
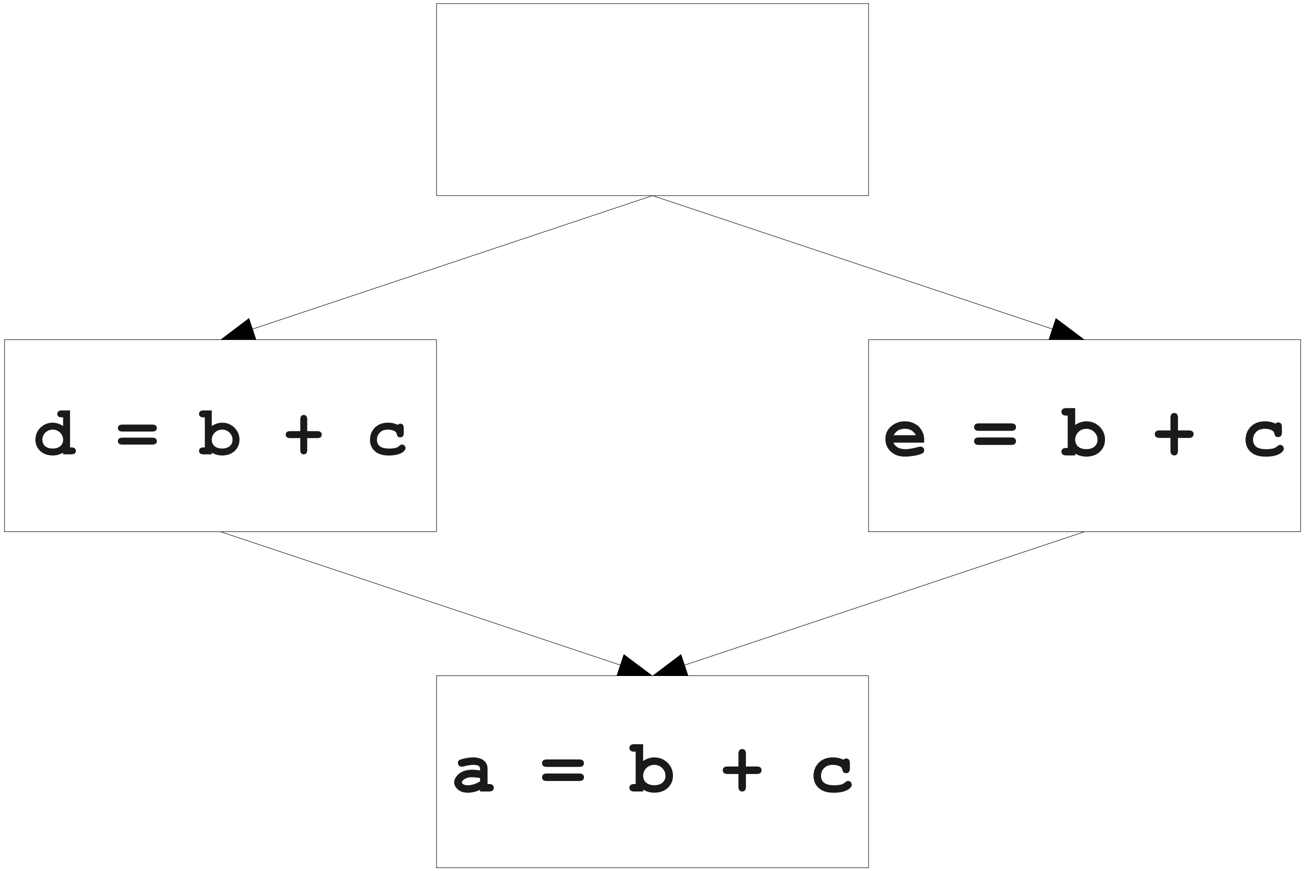


Eliminating Redundancy

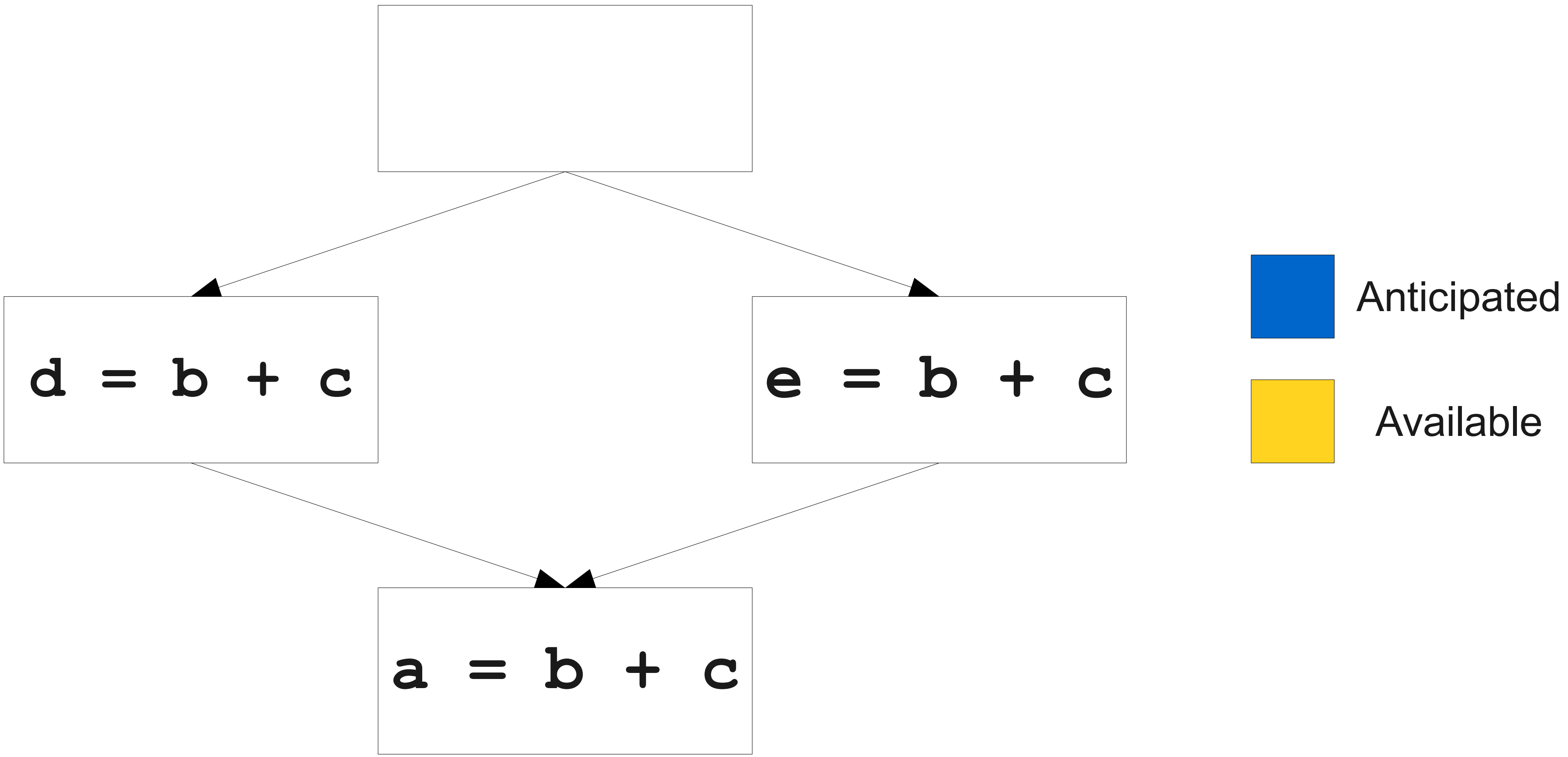


Eliminating Redundancy II

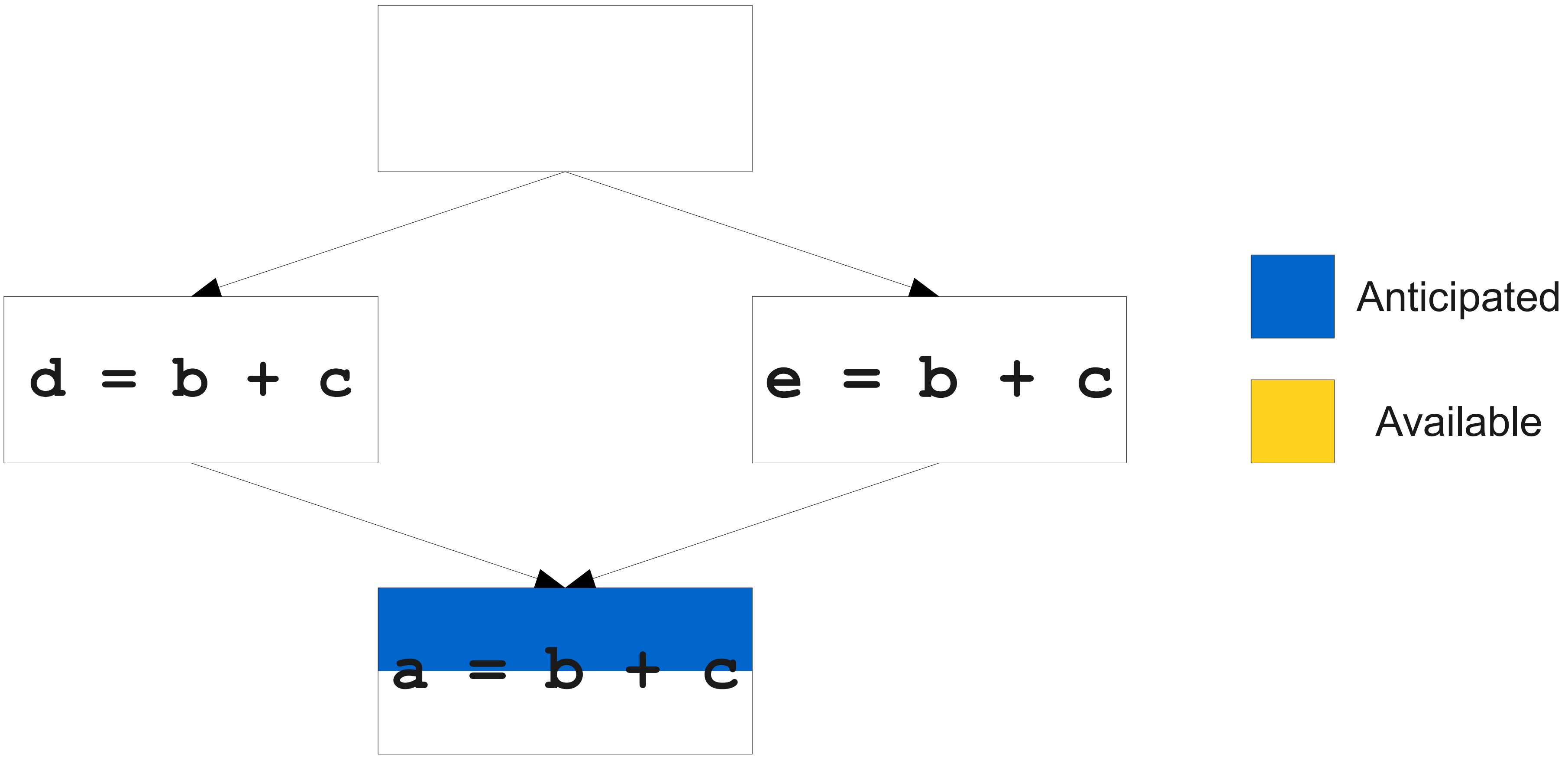
Eliminating Redundancy II



Eliminating Redundancy II



Eliminating Redundancy II



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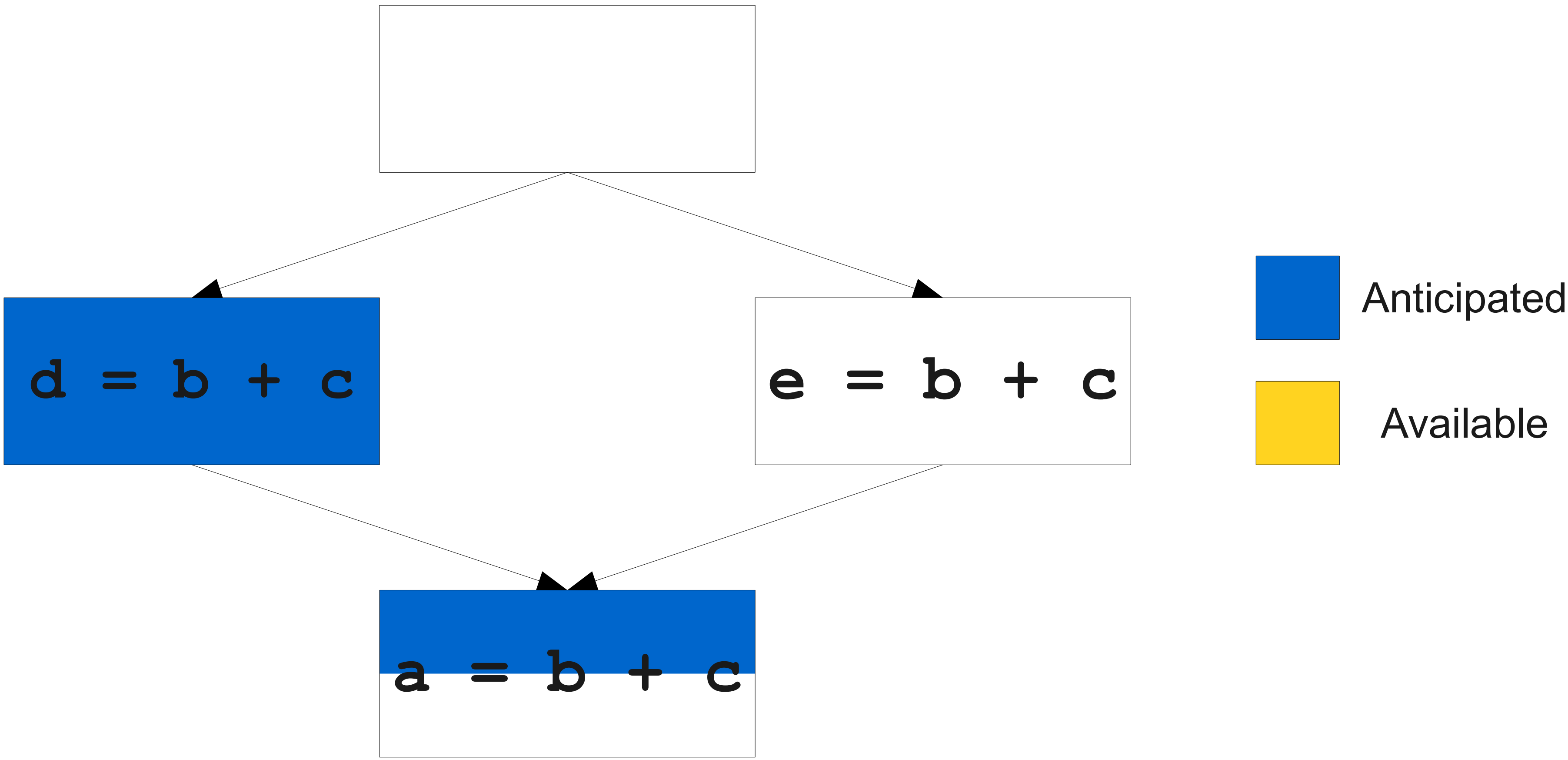

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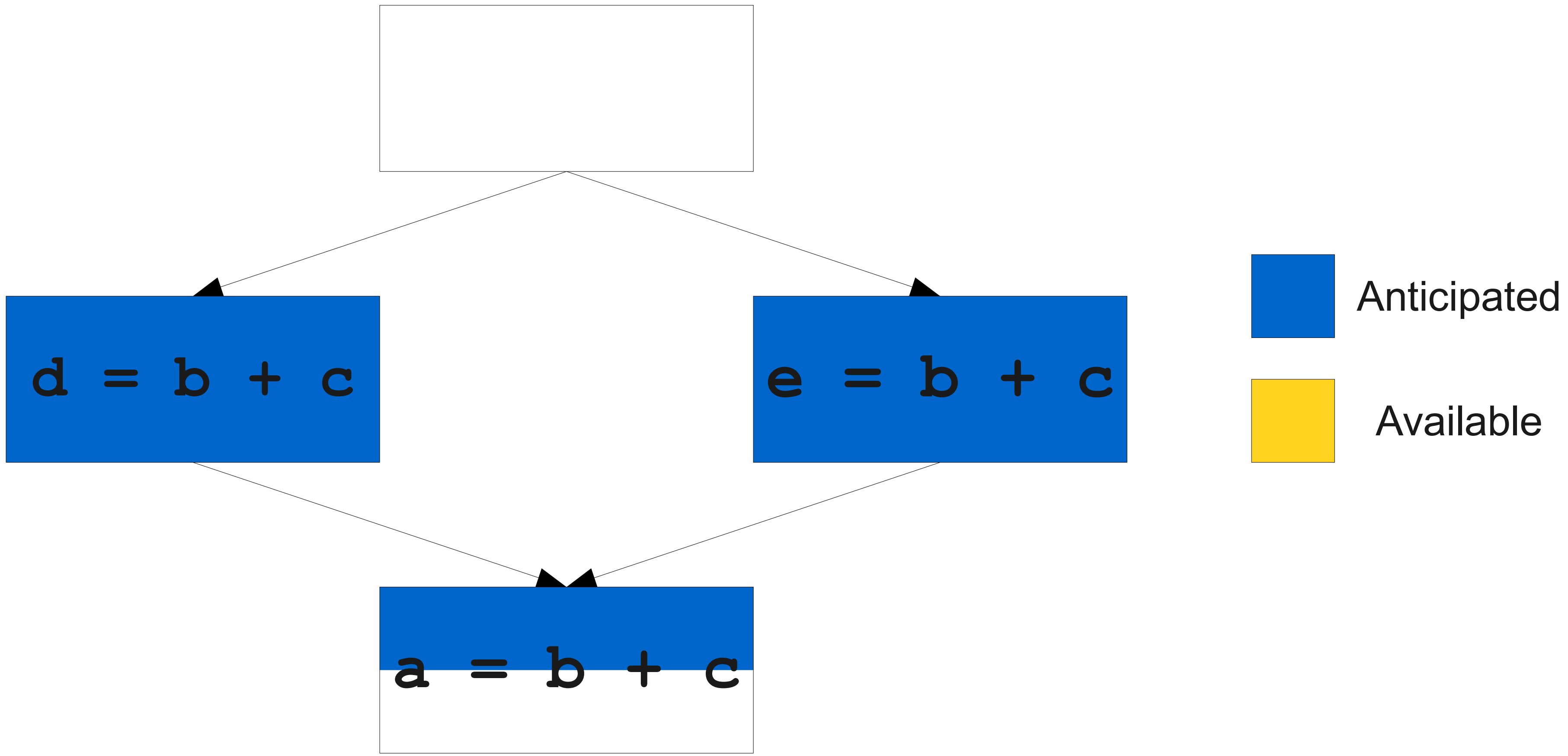

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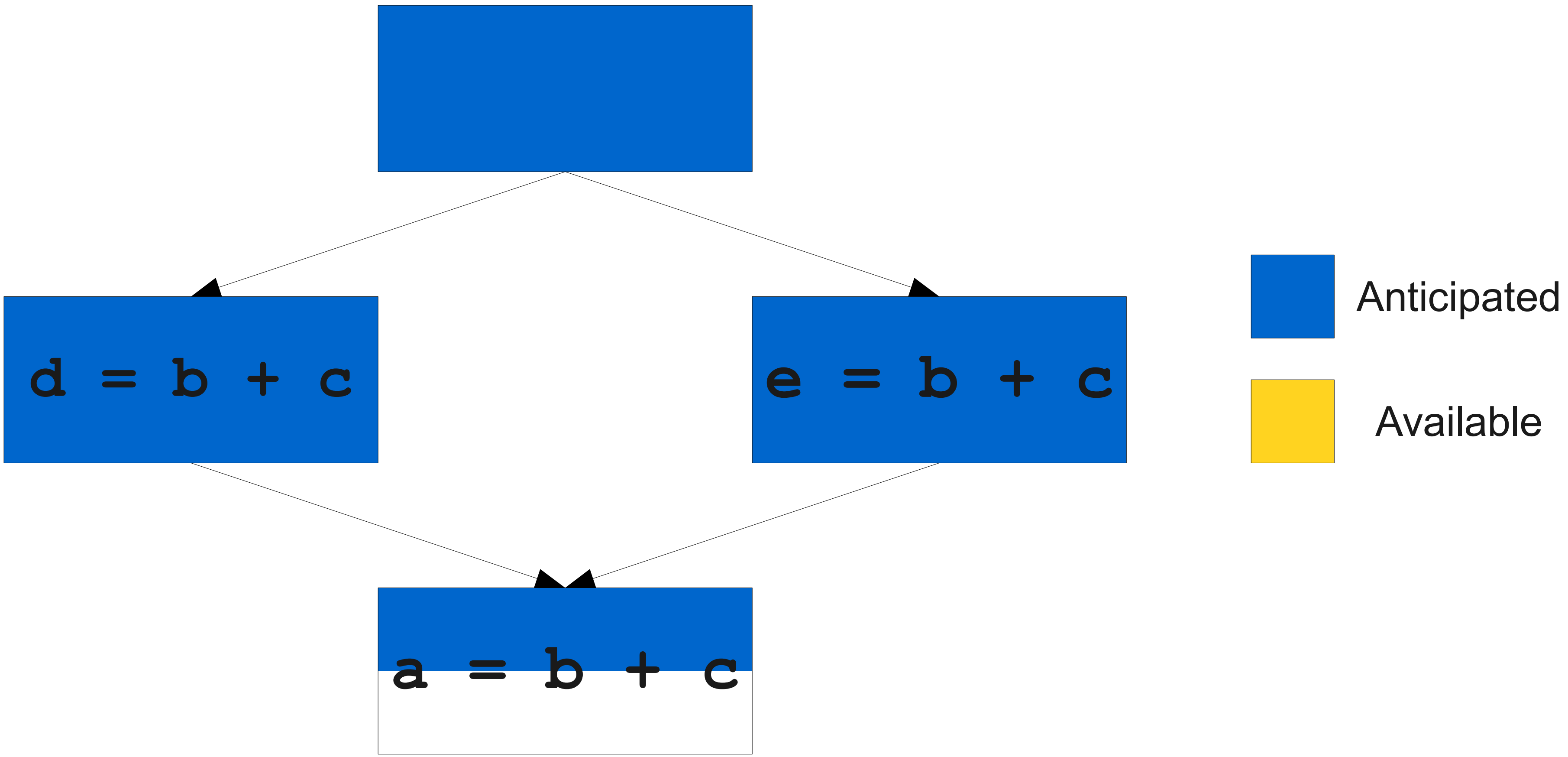

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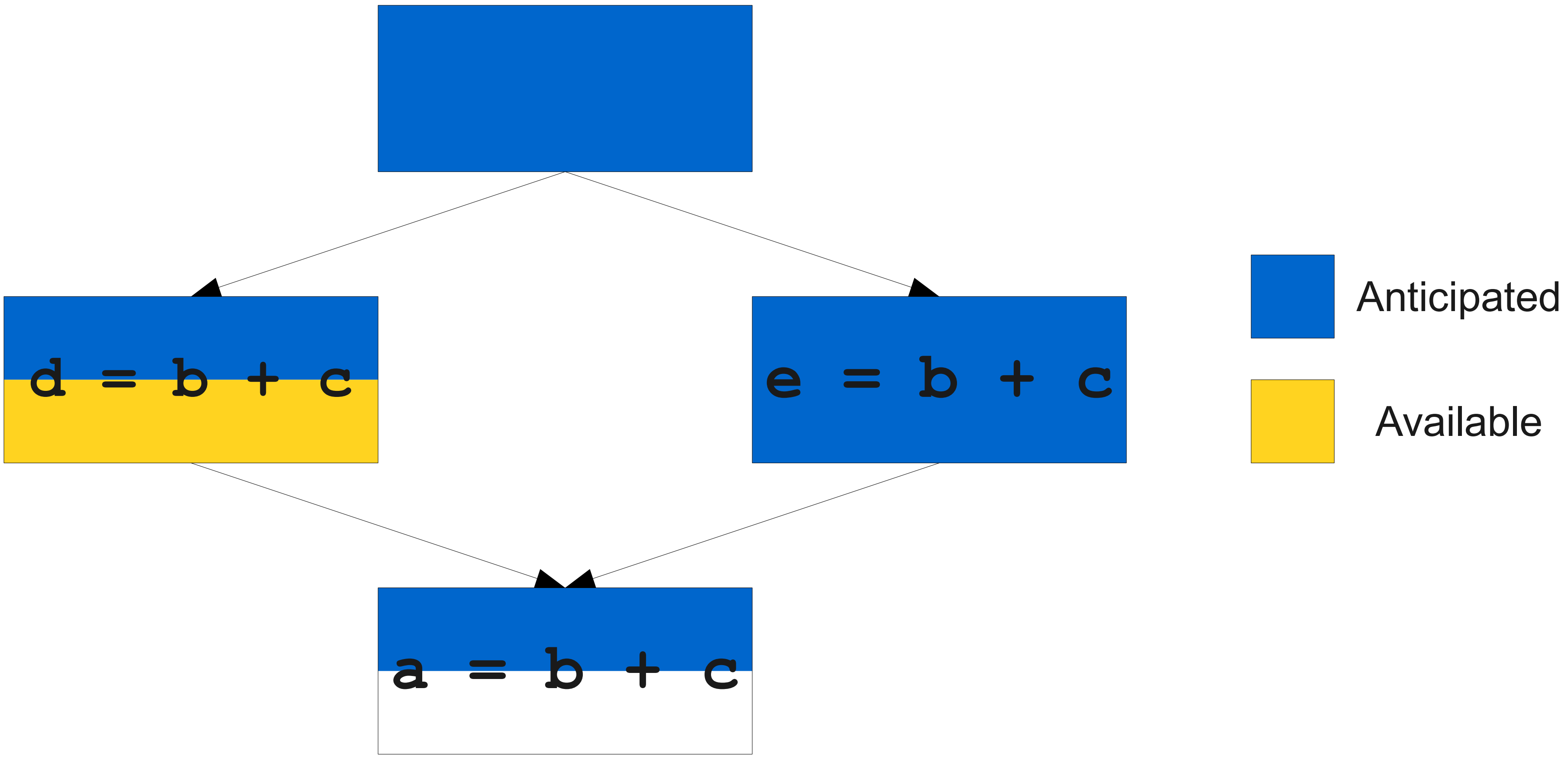

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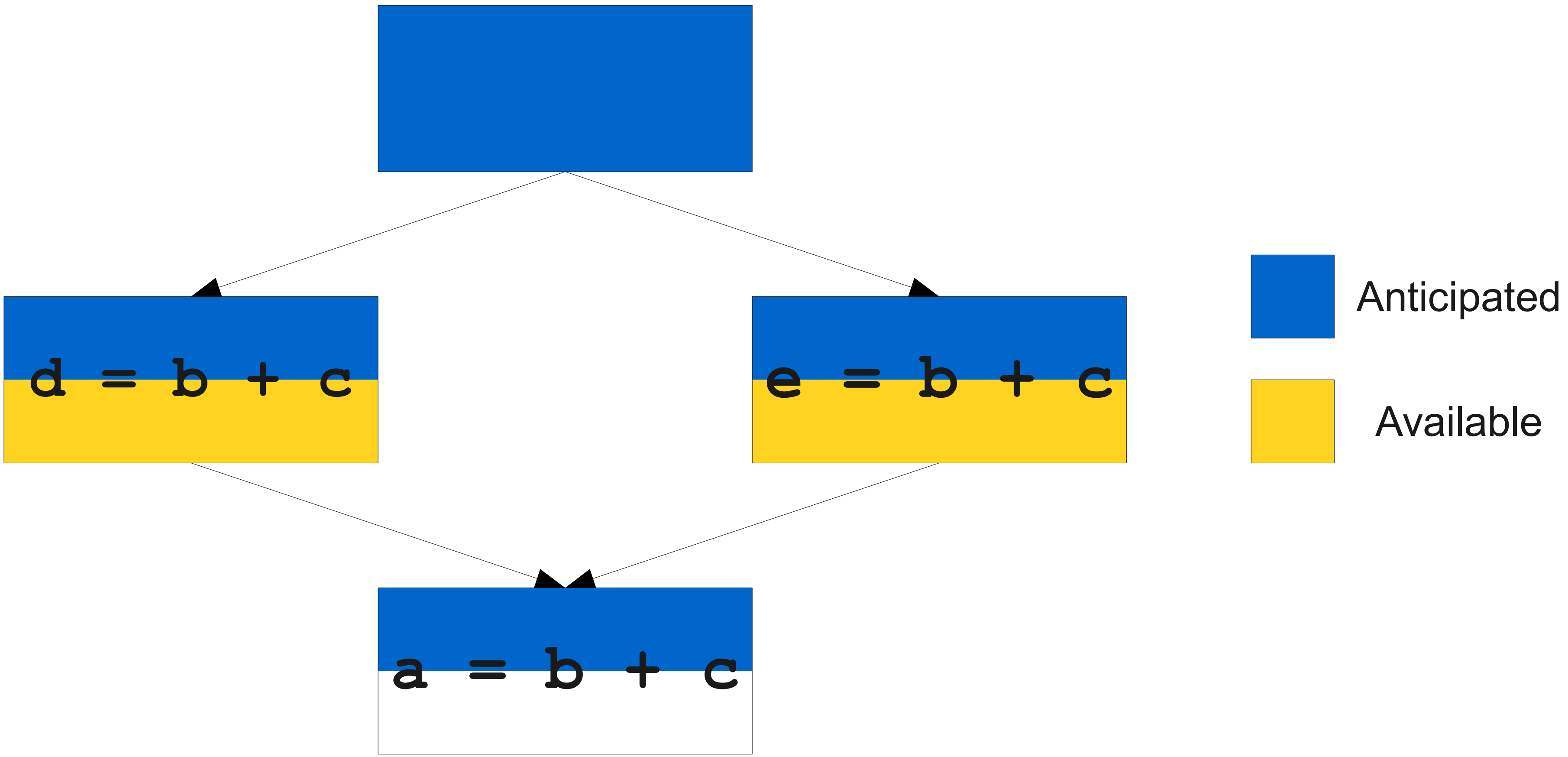

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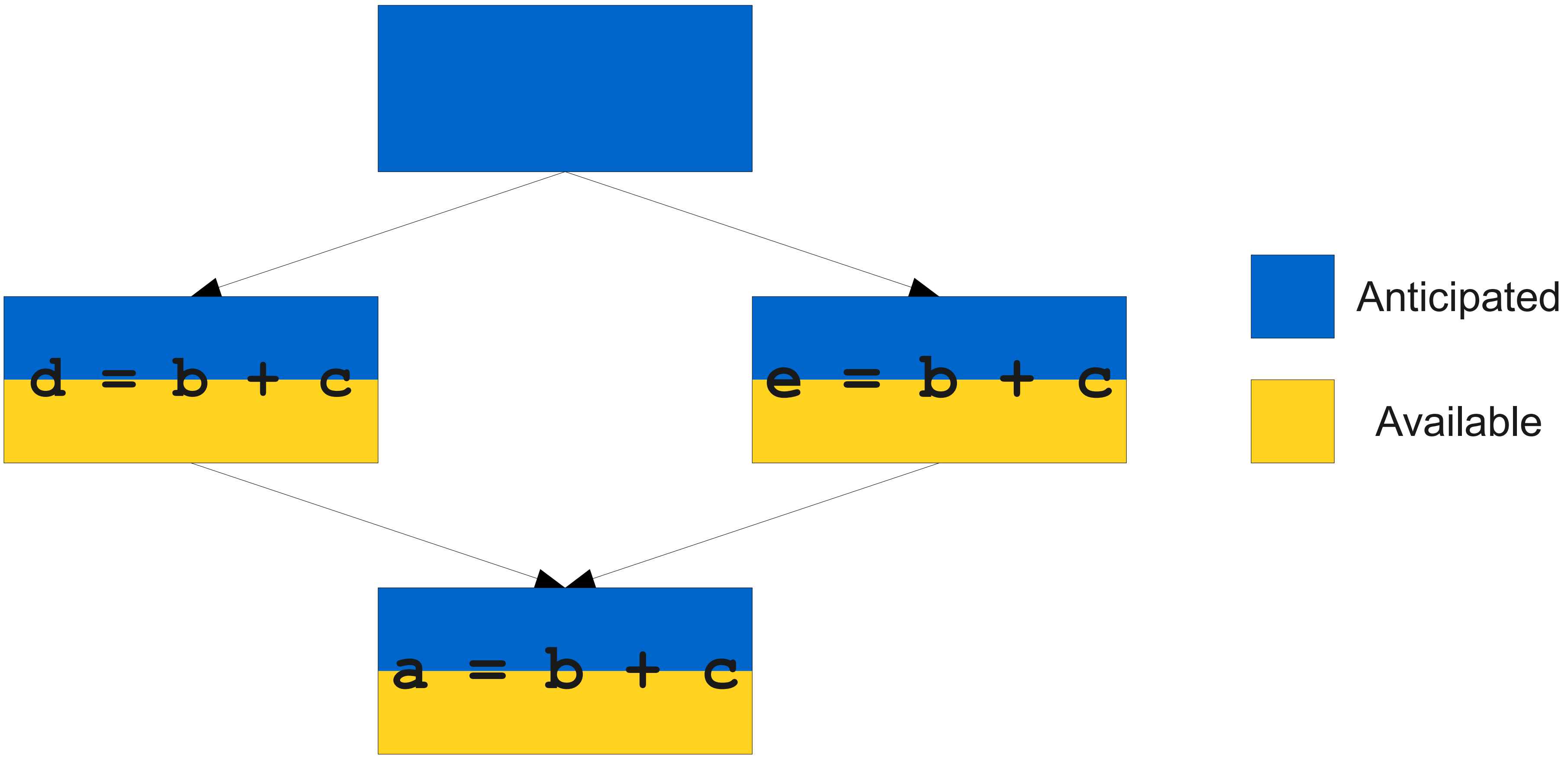

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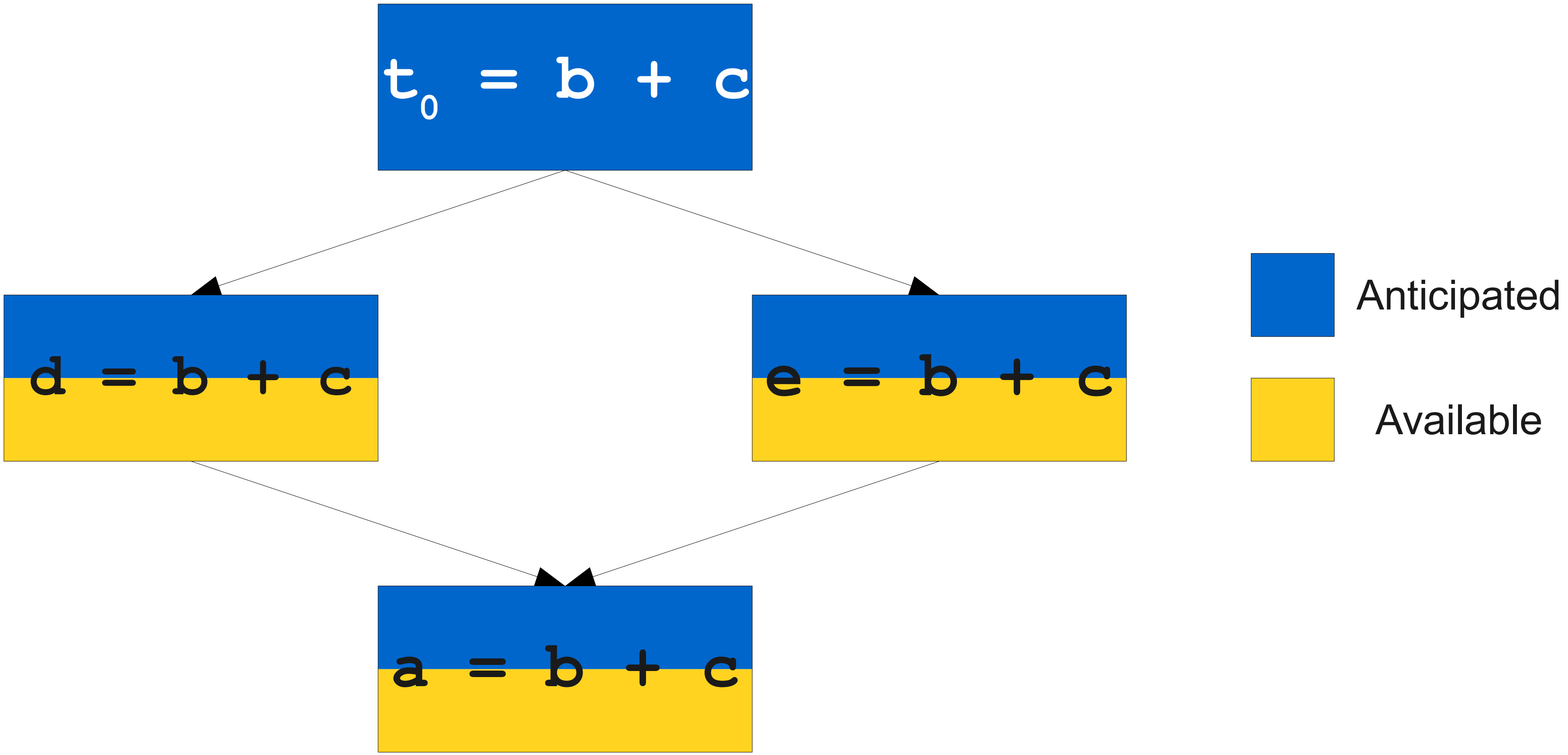

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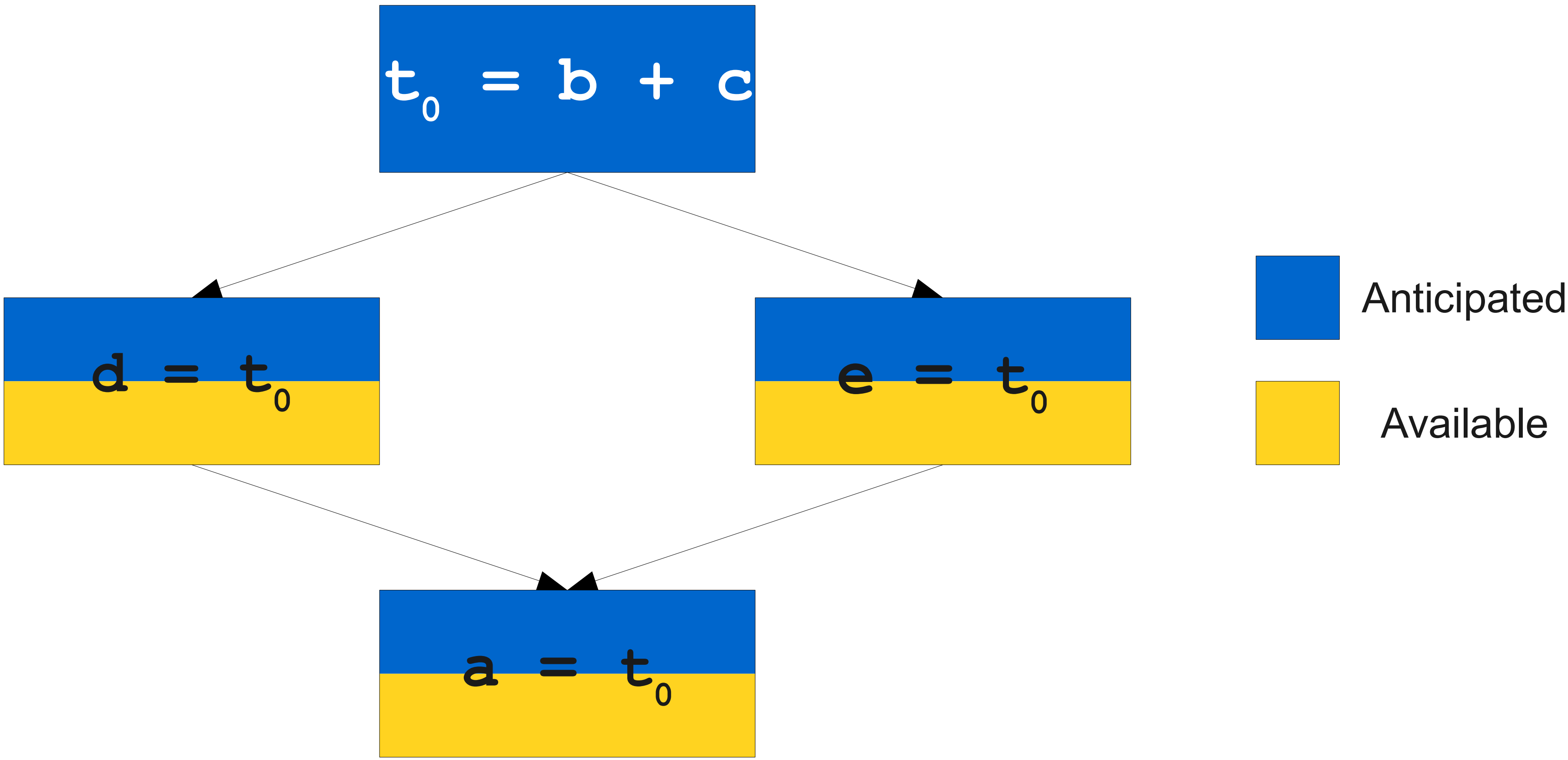

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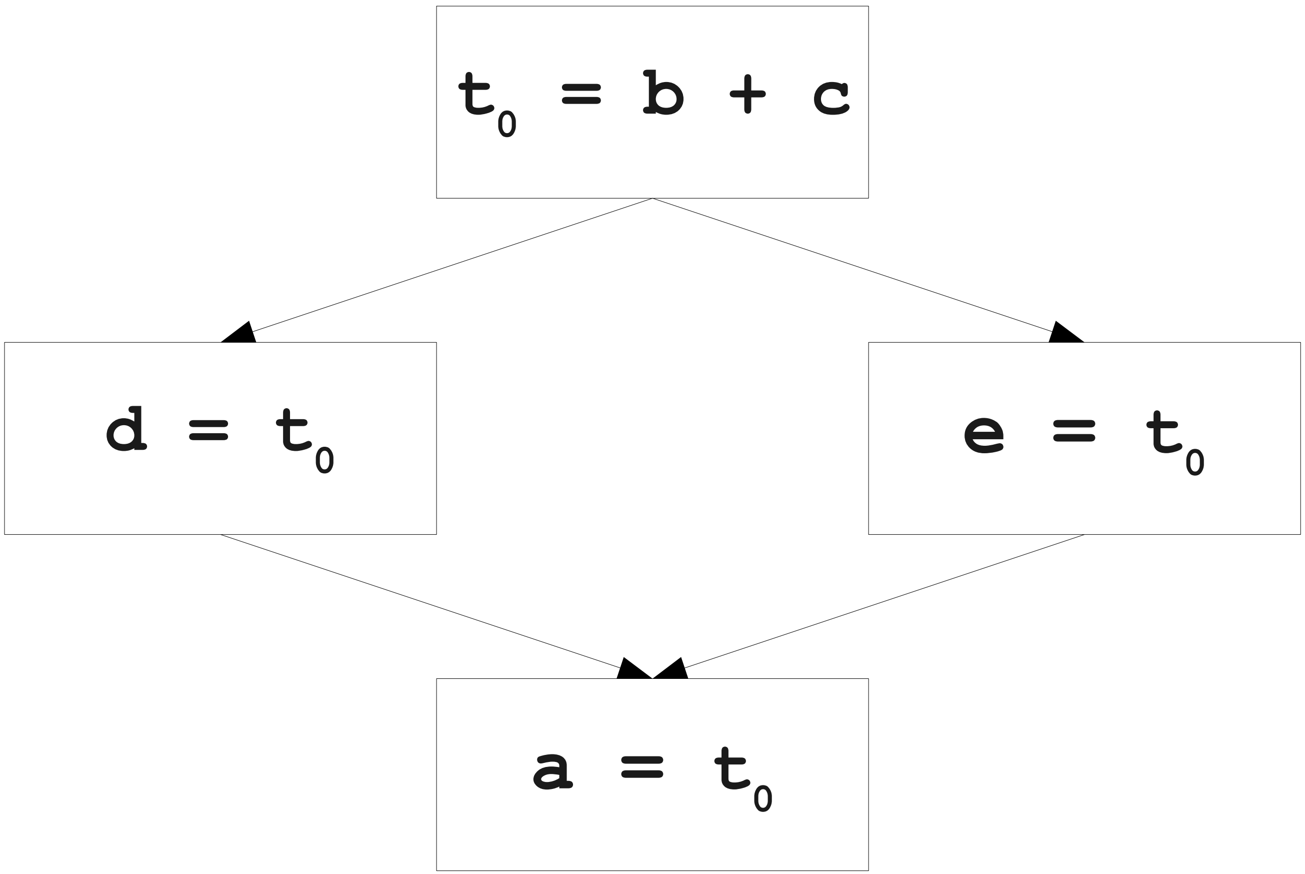

Eliminating Redundancy III

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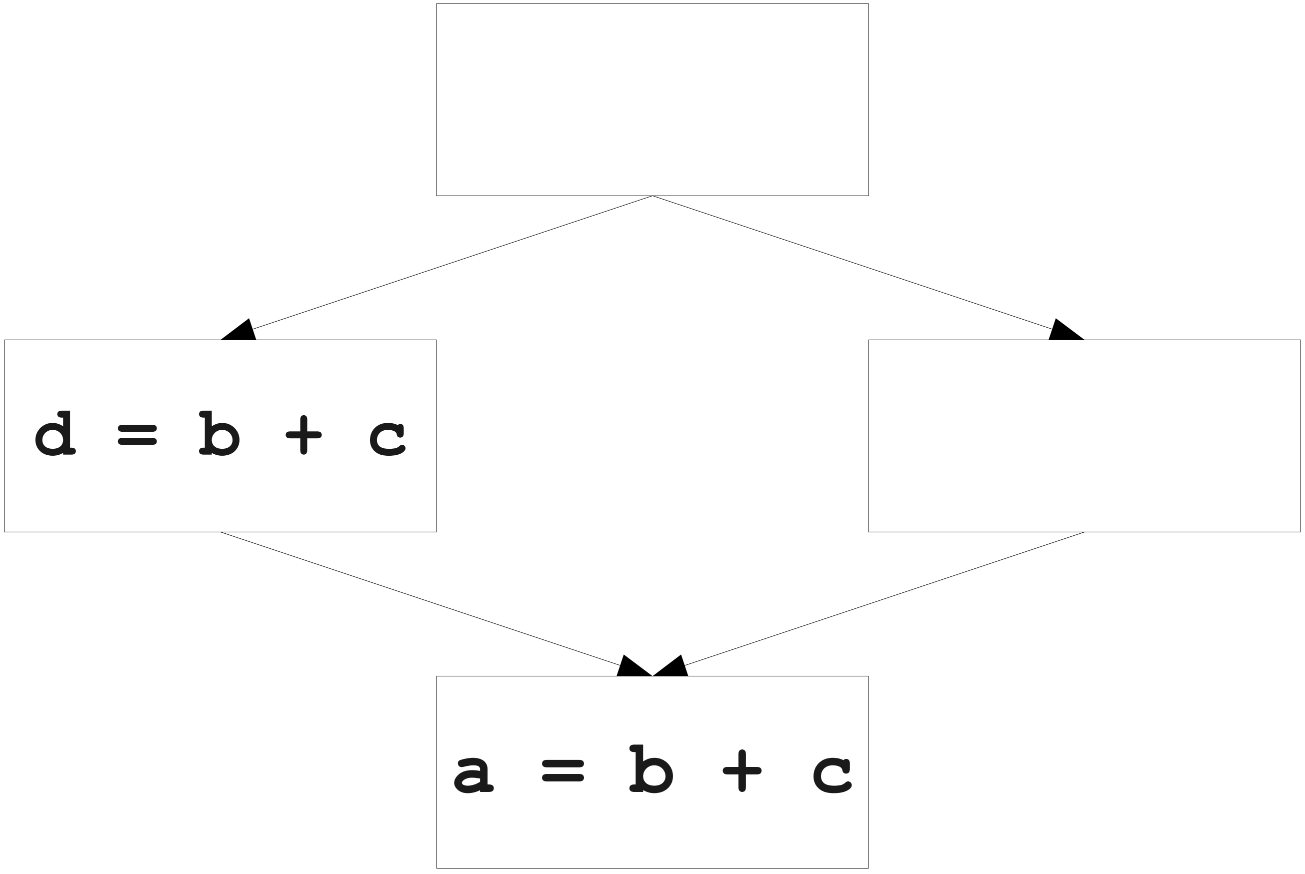
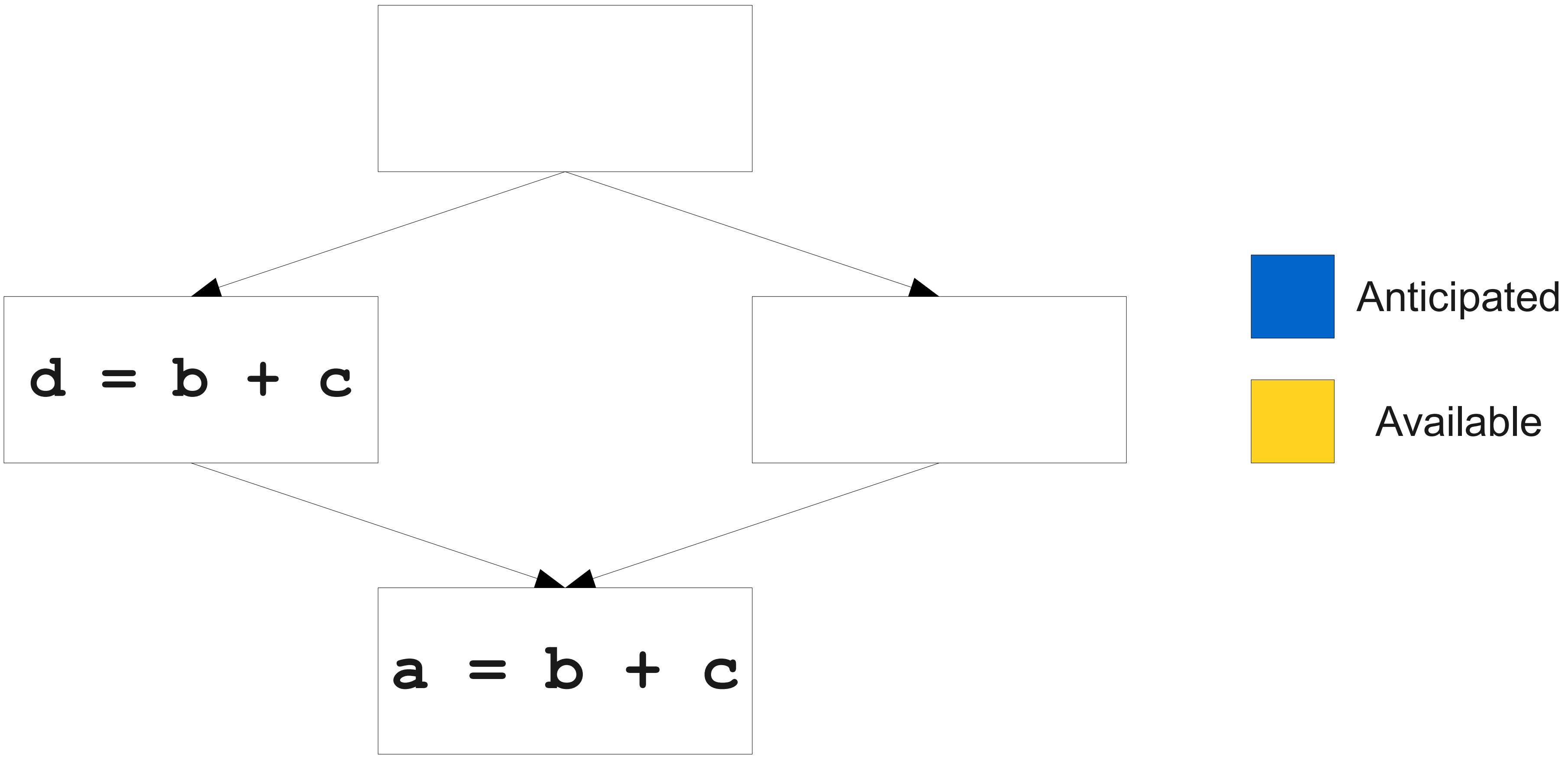


Eliminating Redundancy III



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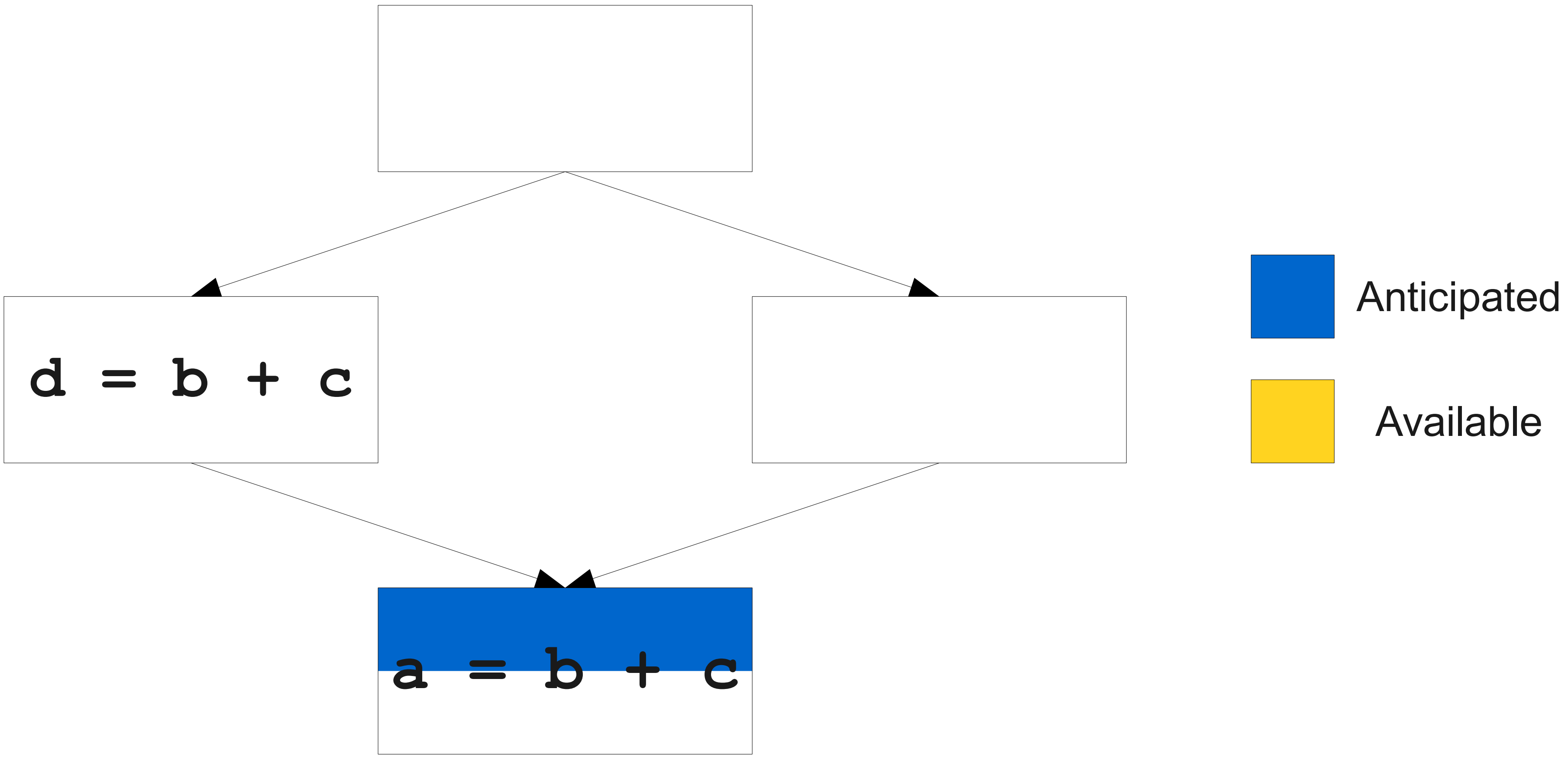

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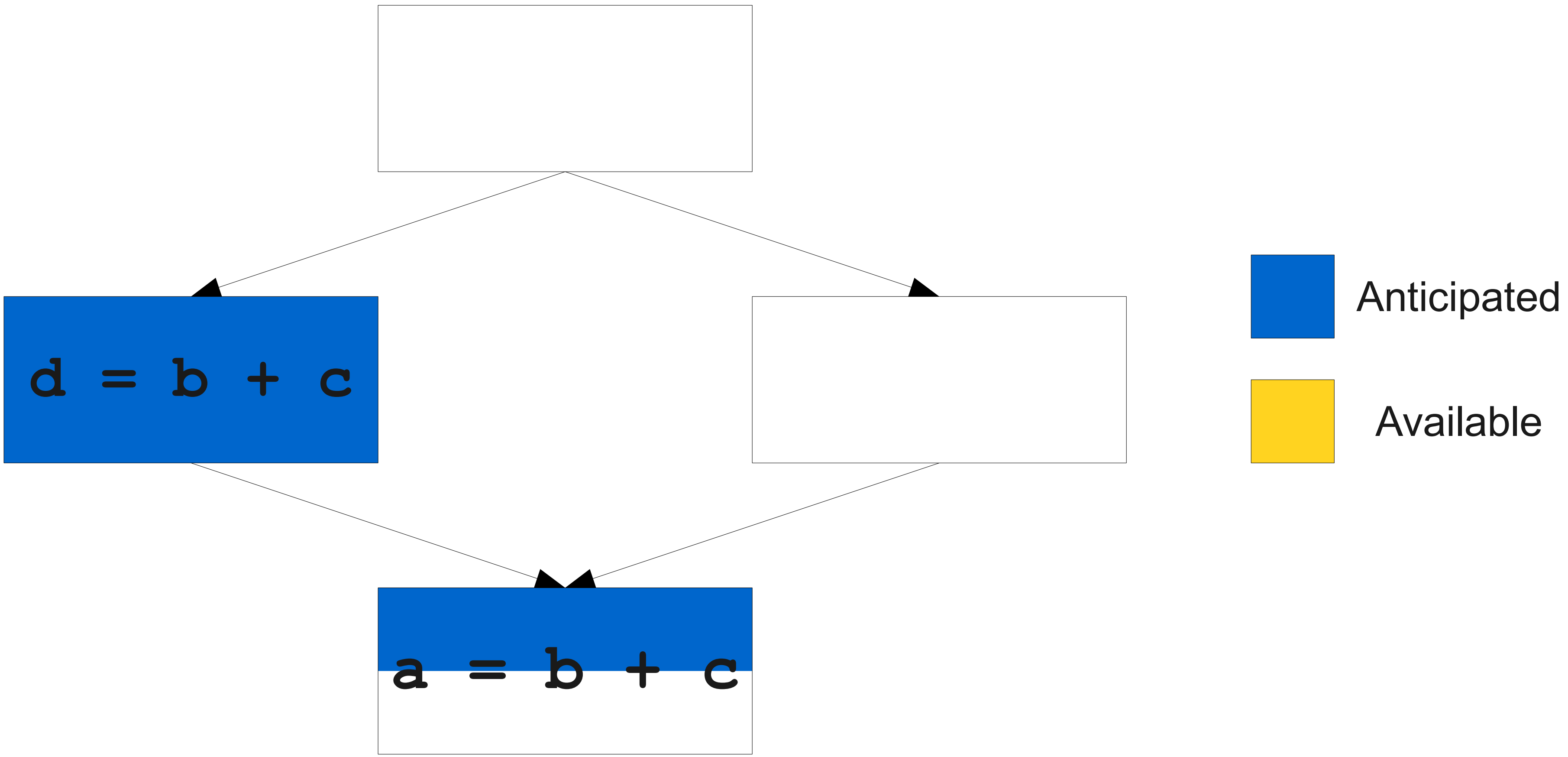

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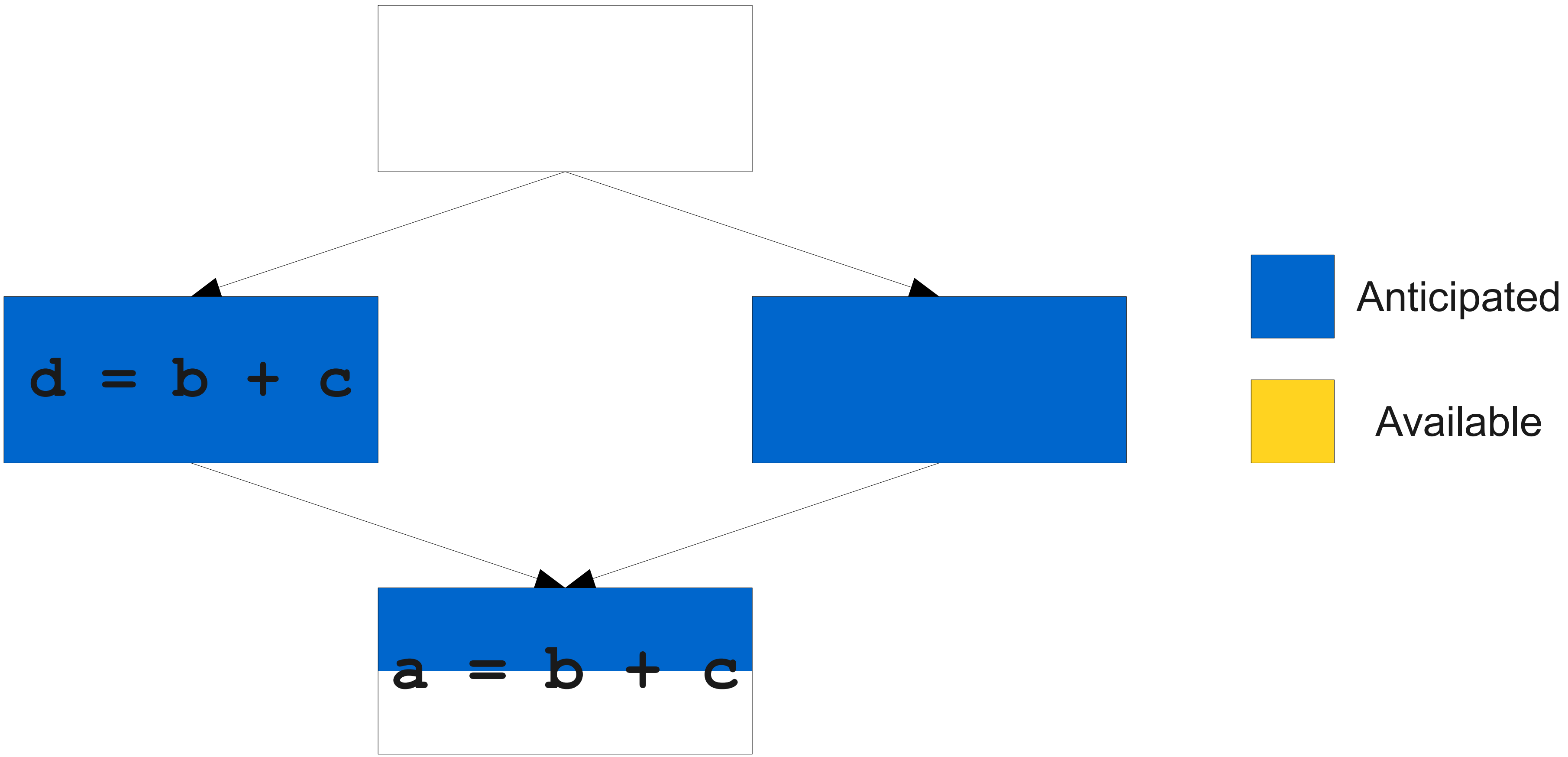

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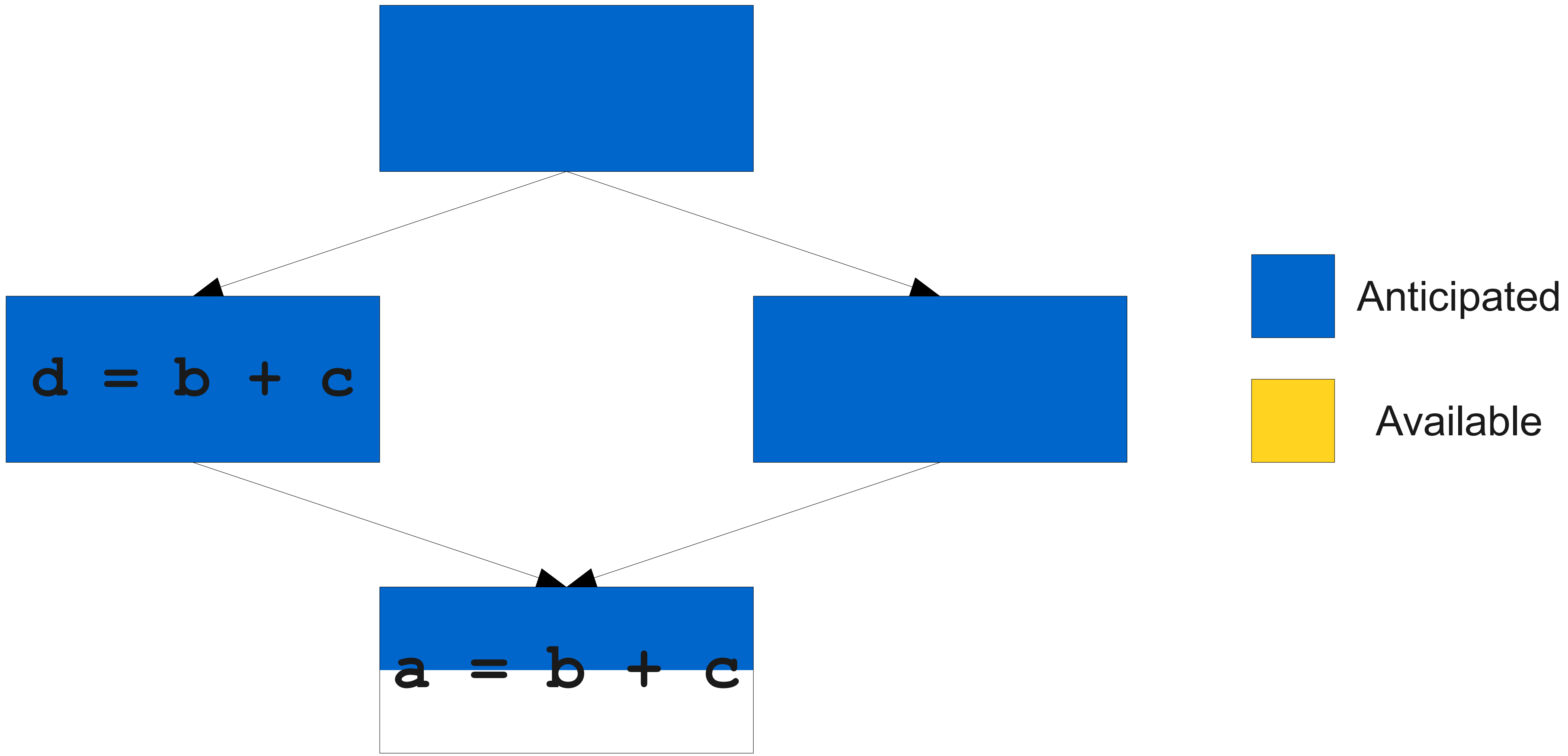

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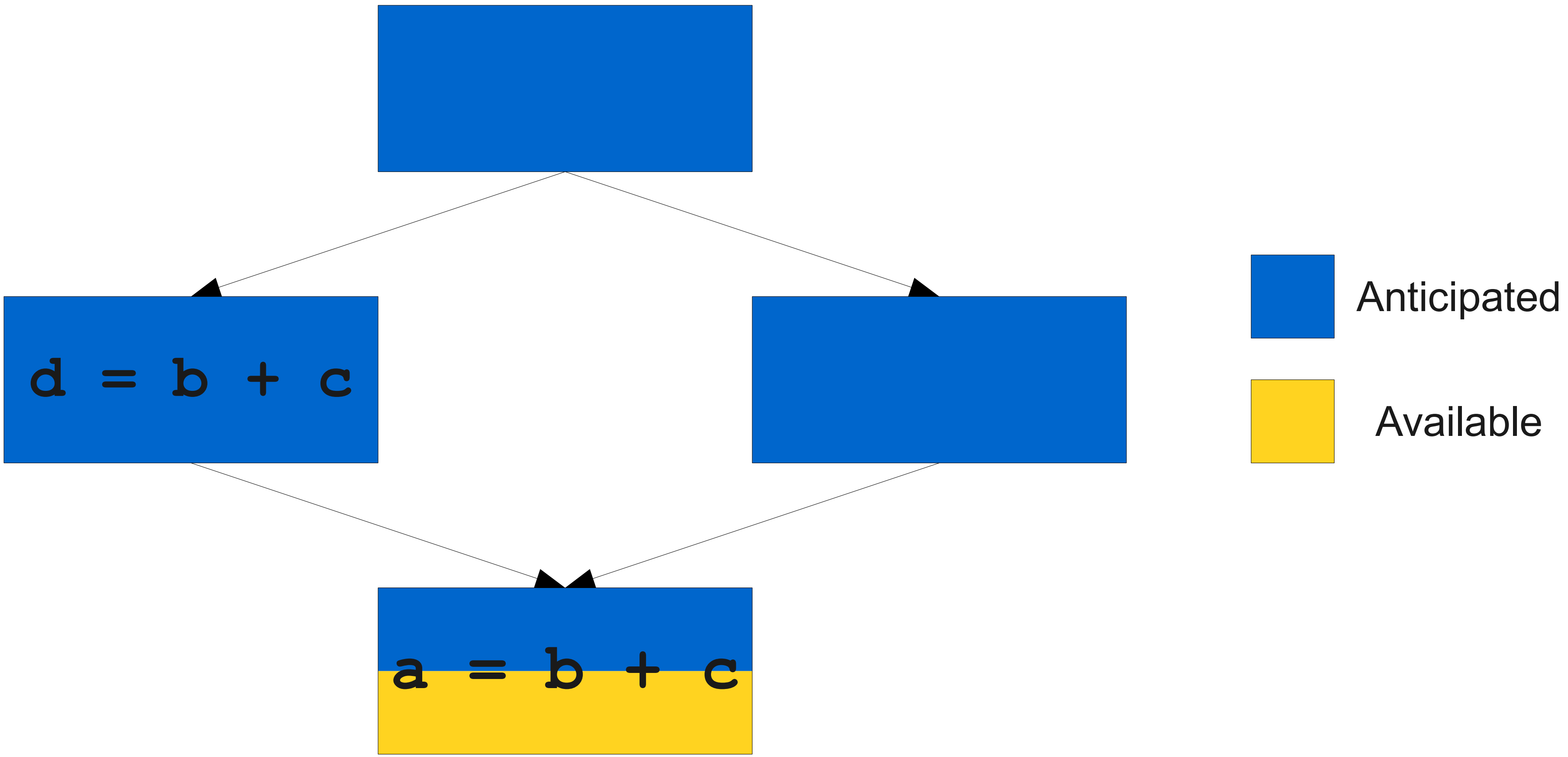

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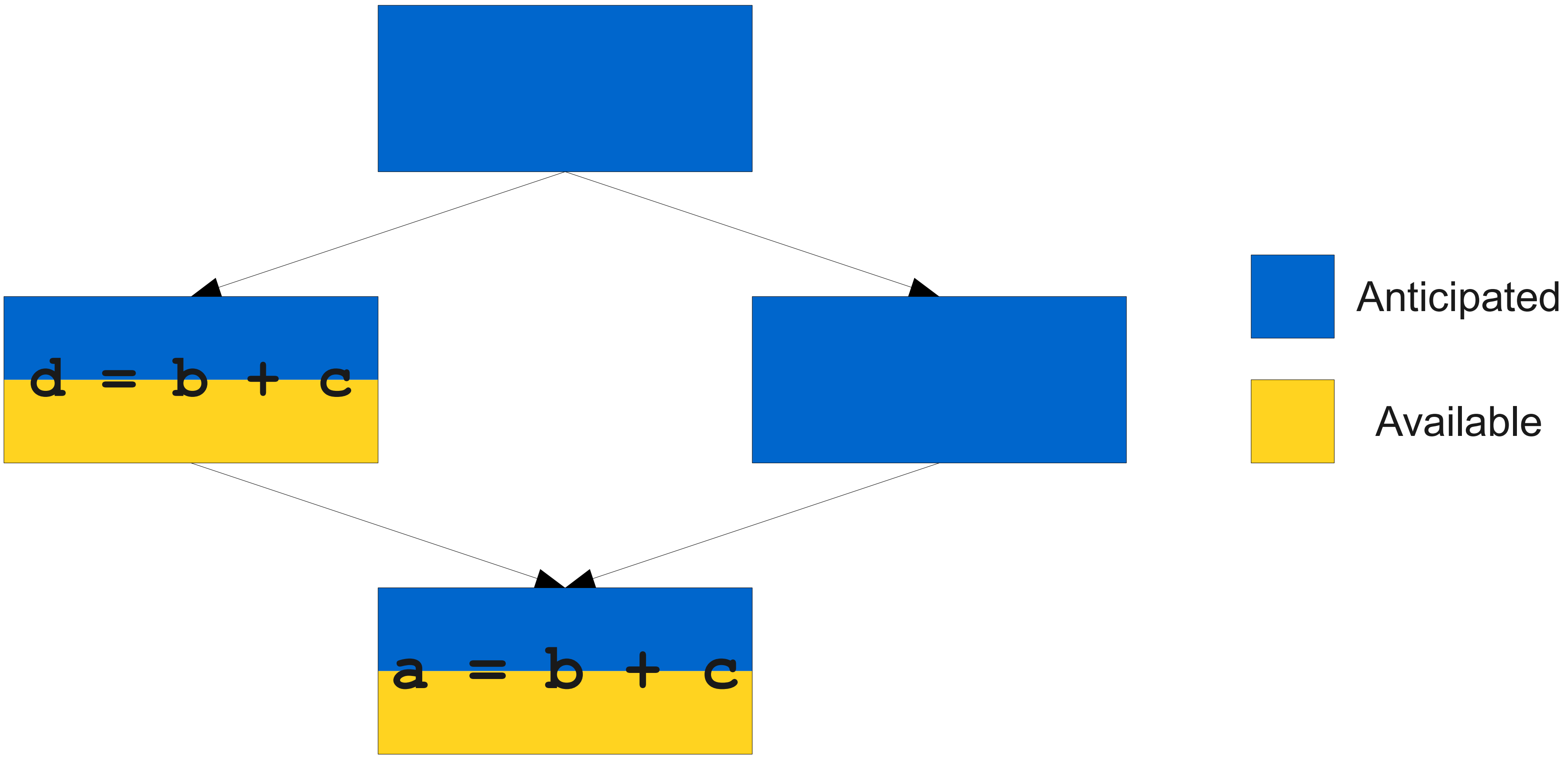

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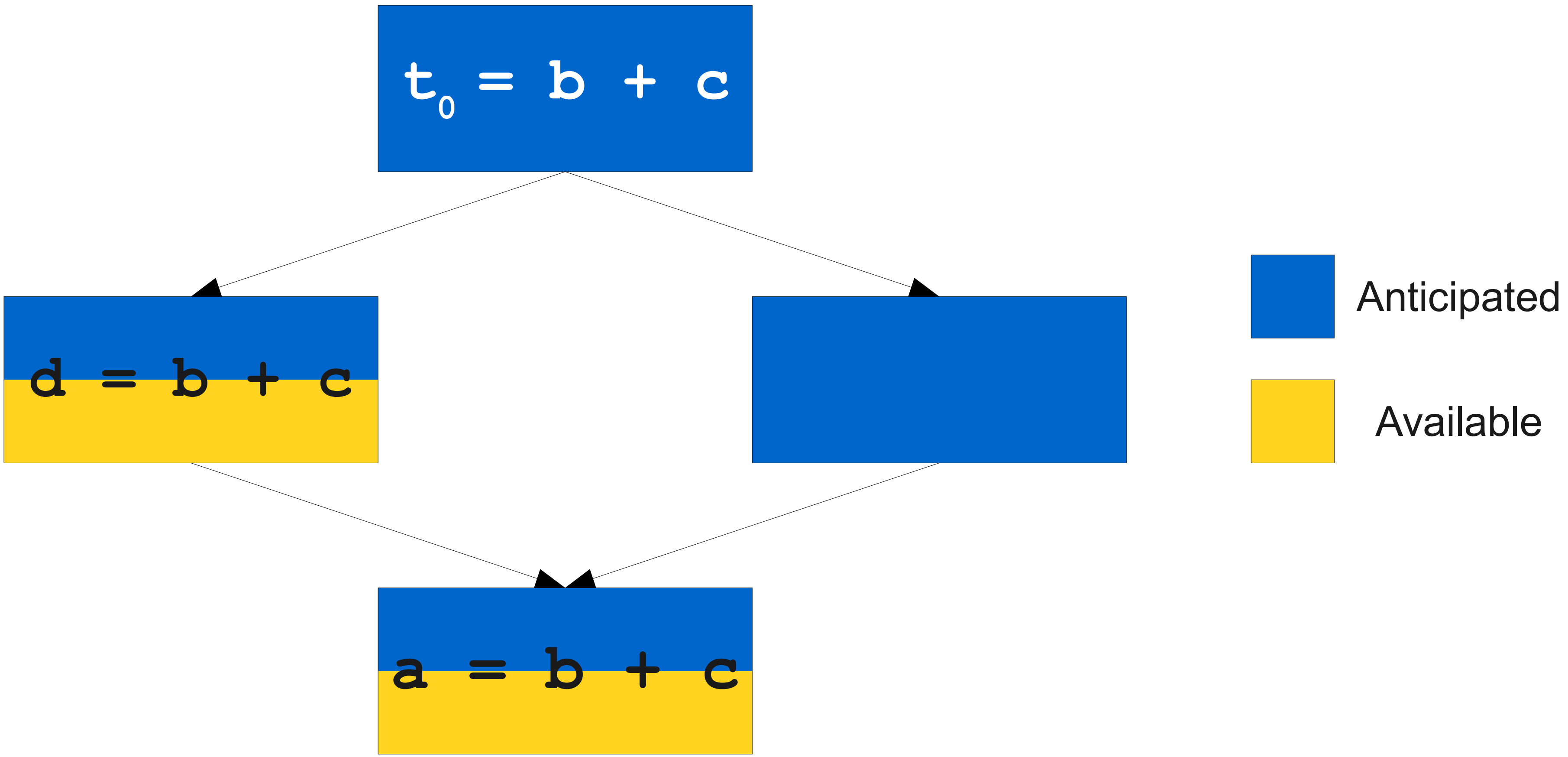

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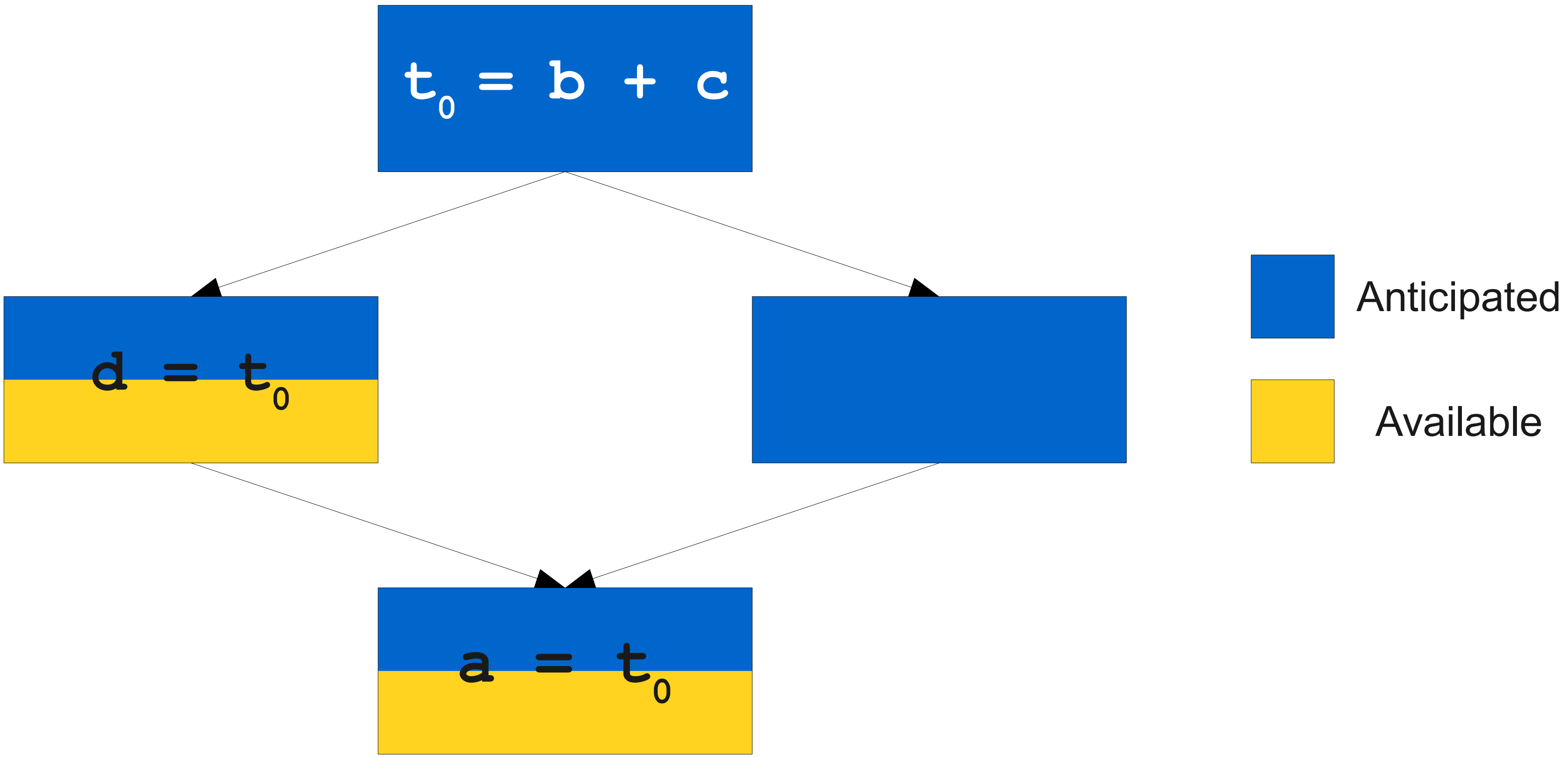

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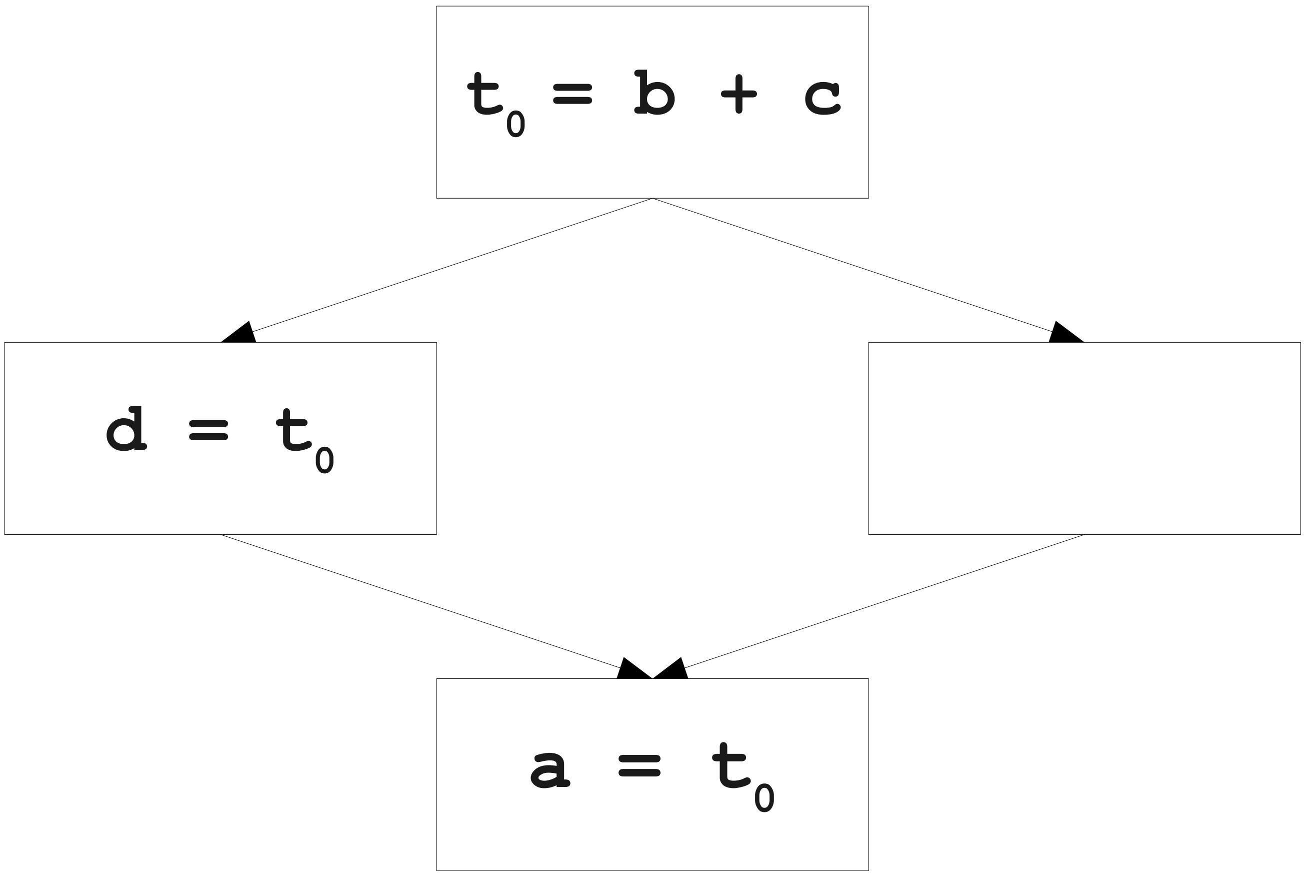

In Practice

- Partial-redundancy elimination is typically implemented using **four** dataflow analyses.
- A bit more complex than what we covered:
  - How to avoid keeping expressions around too long?
  - How to avoid introducing unnecessary temporaries?
- See Dragon Book, Ch. 9.5 for more precise details.

Next Time

- **Code Generation**
  - The machine at a glance.
  - Register allocation.