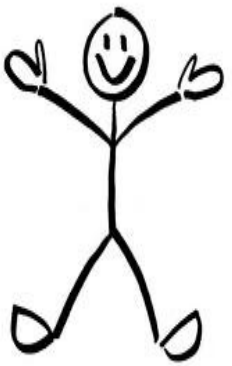
Reductions and Rice's Theorem

Announcements

- Welcome back! I hope you had a great break!
- Problem Set 8 out, due Friday, December 2 at 2:15 PM.
  - Stop by office hours with questions!
  - Email **cs103@cs.stanford.edu** with questions!
- Problem session tonight, 7-8PM, in 370-370.
  - Alternate time: 7-8PM tomorrow, Gates 100.

Reductions

A Simple Reduction



Thanks to David Dill for the idea

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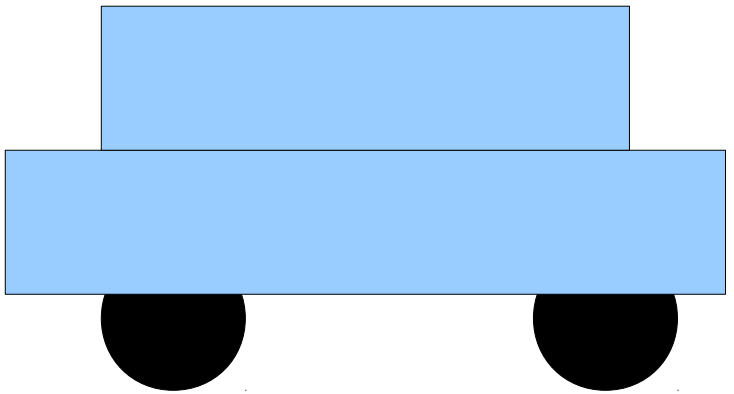

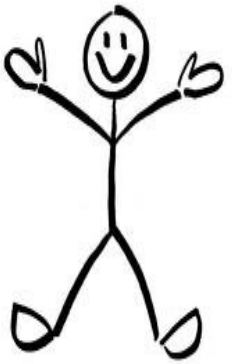

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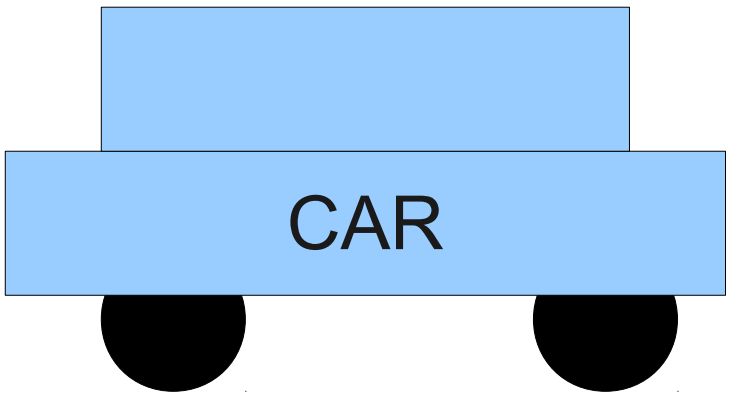

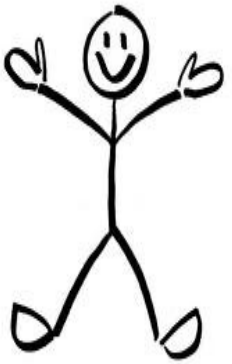

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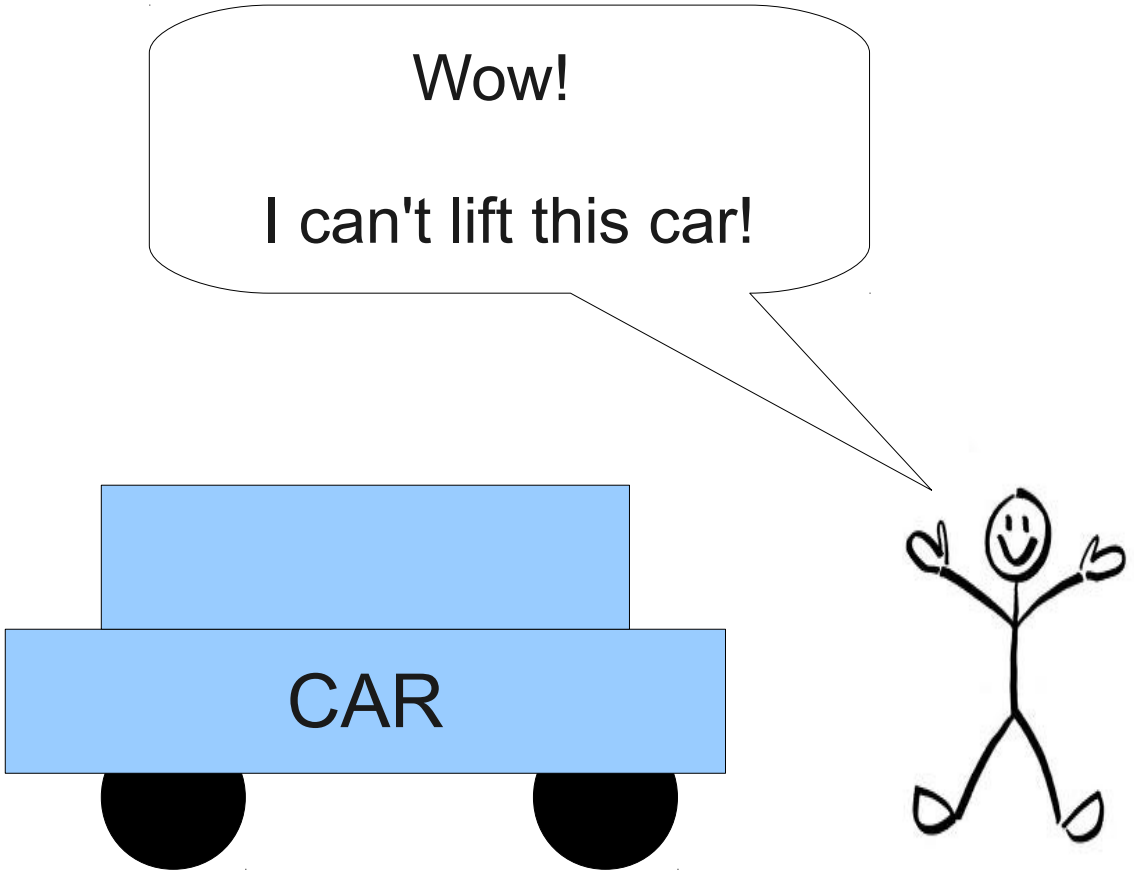

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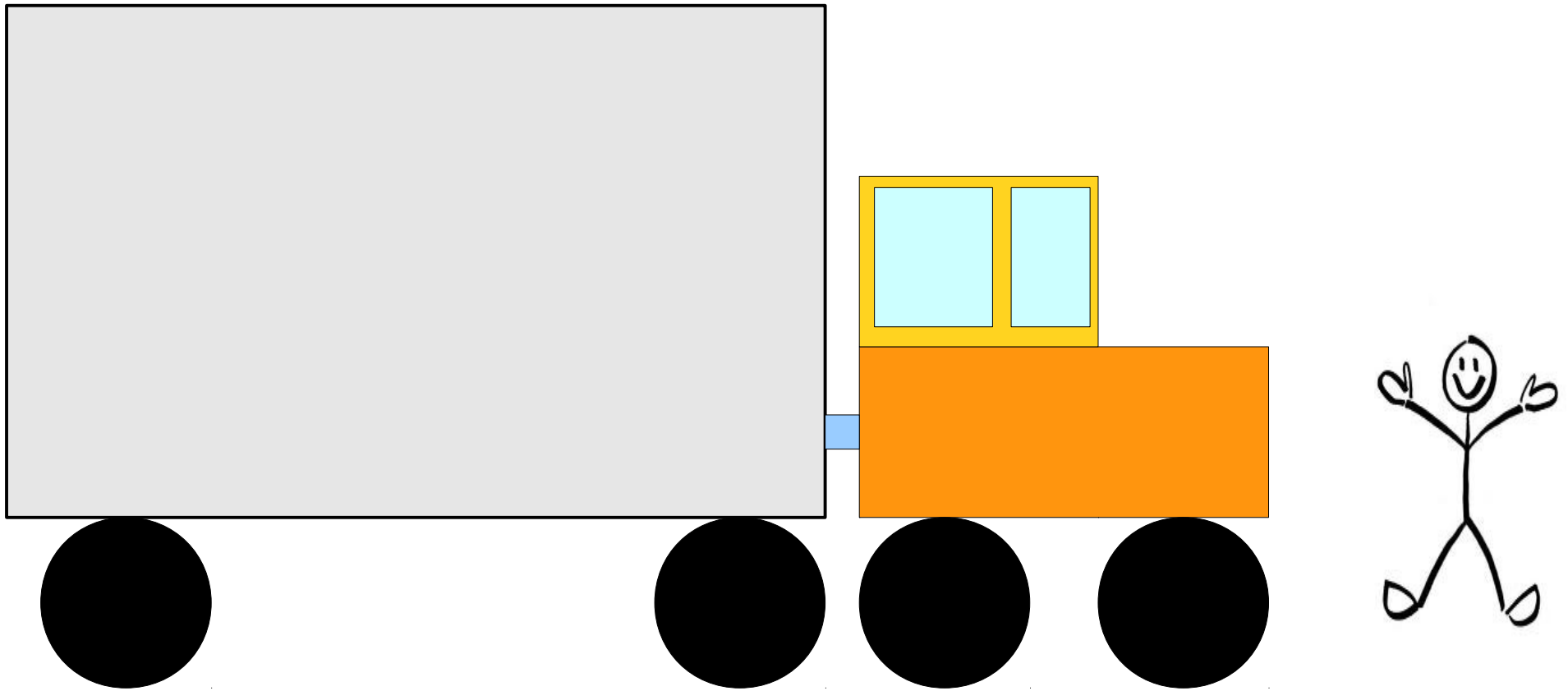

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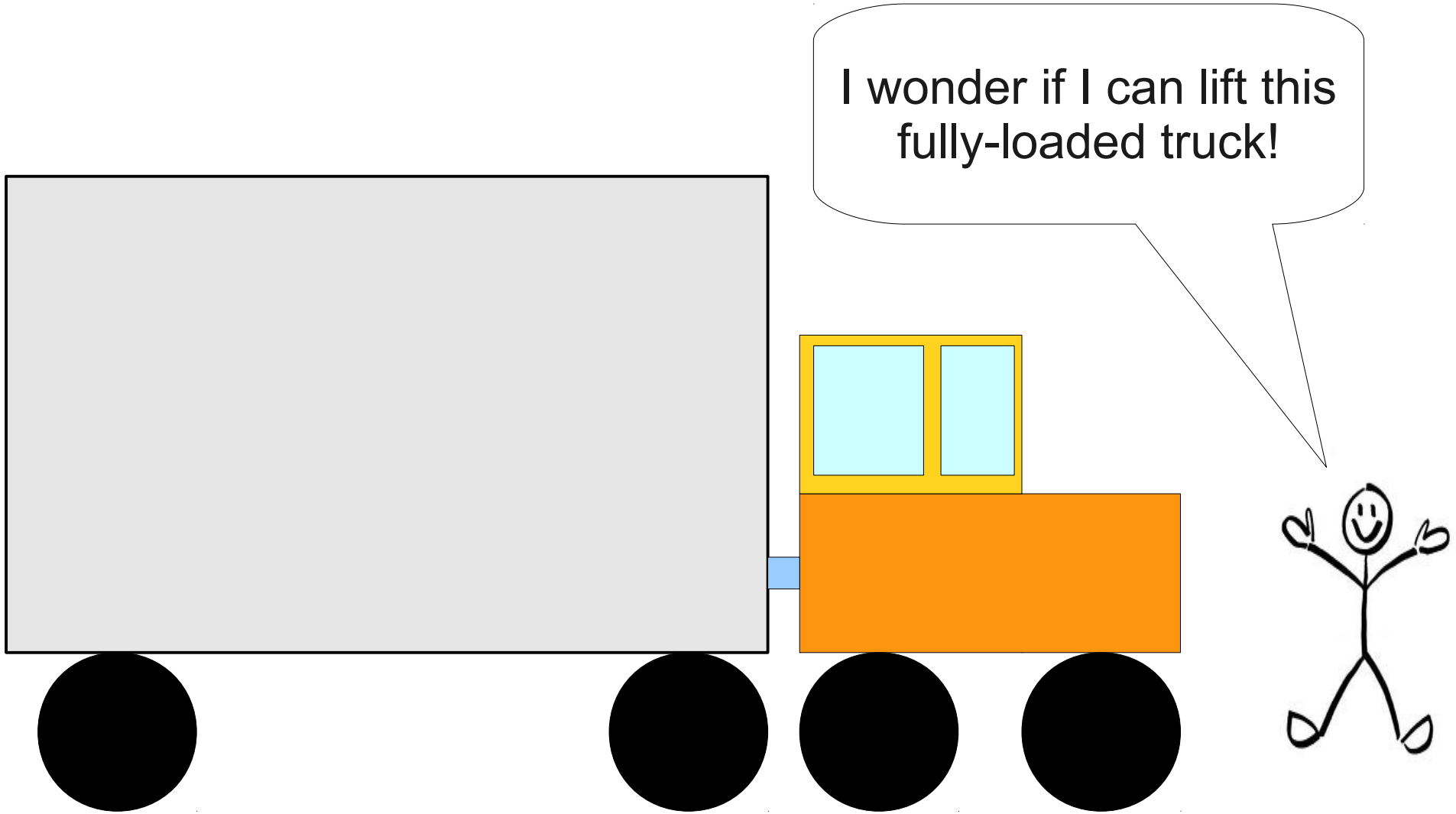

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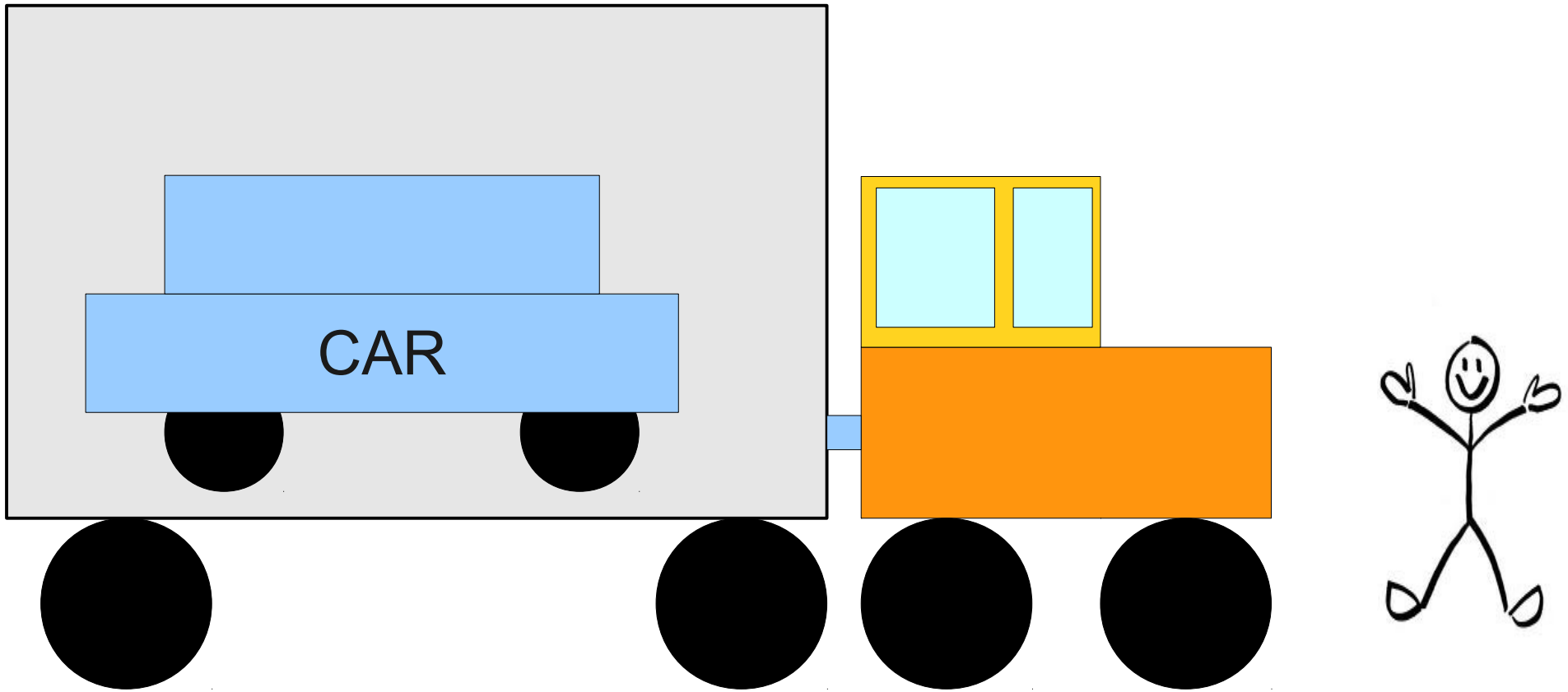

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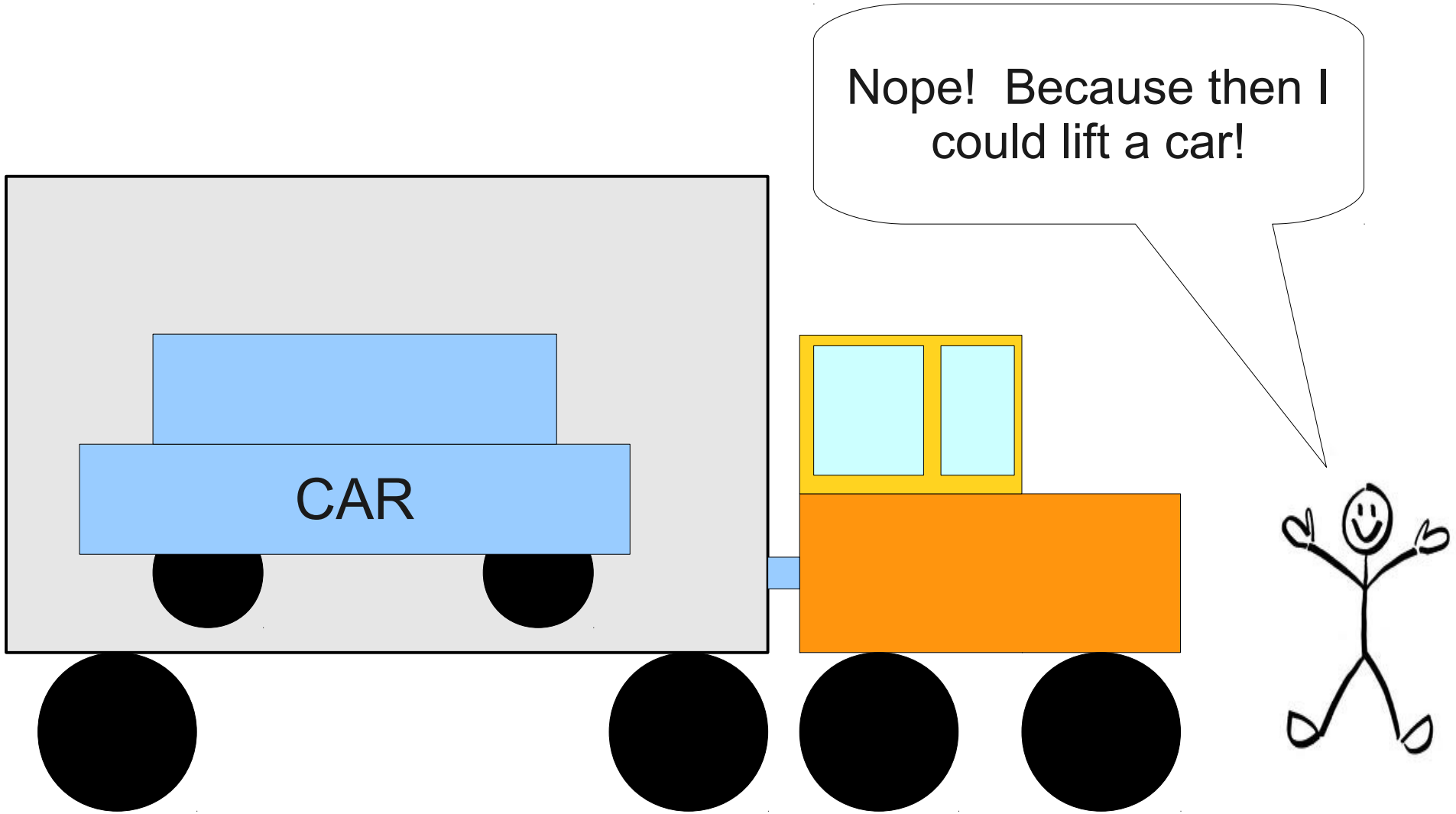

Reductions

- A **reduction** is a way of solving a problem EASY with a problem HARDER.
- In this example, we could solve the problem of "lift the car" by *reducing* it to the harder problem of "lift the truck."

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- A **reduction** works by turning an instance of EASY into an instance of HARDER.
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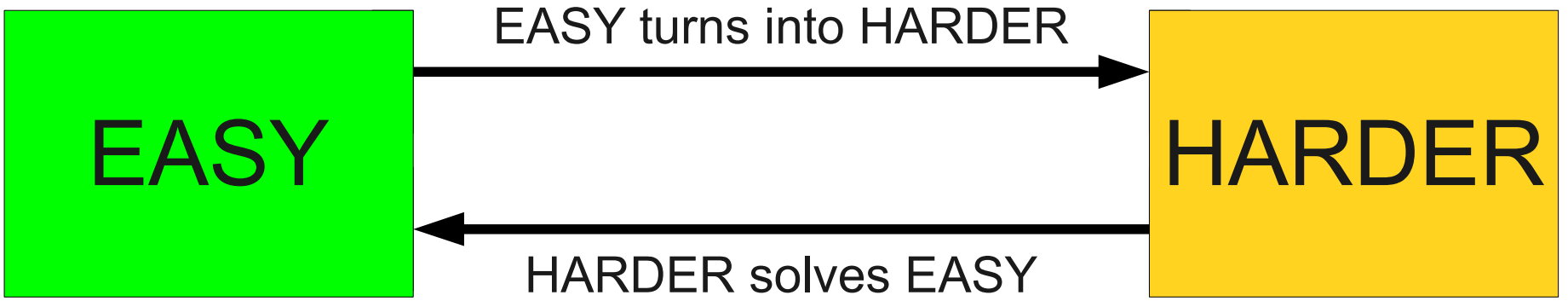
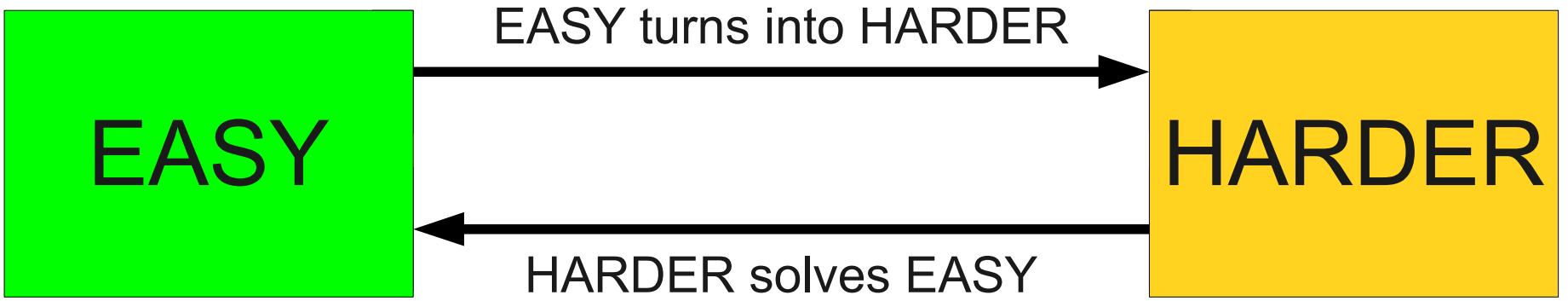


Reductions

- Suppose that we cannot solve EASY.
- If we can **reduce** EASY to HARDER, we cannot solve HARDER either.



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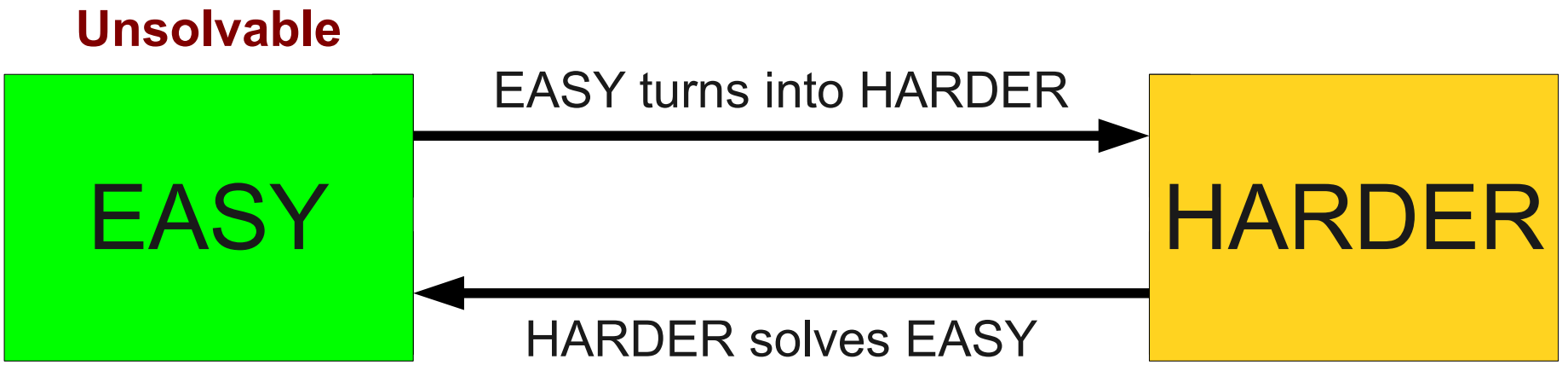

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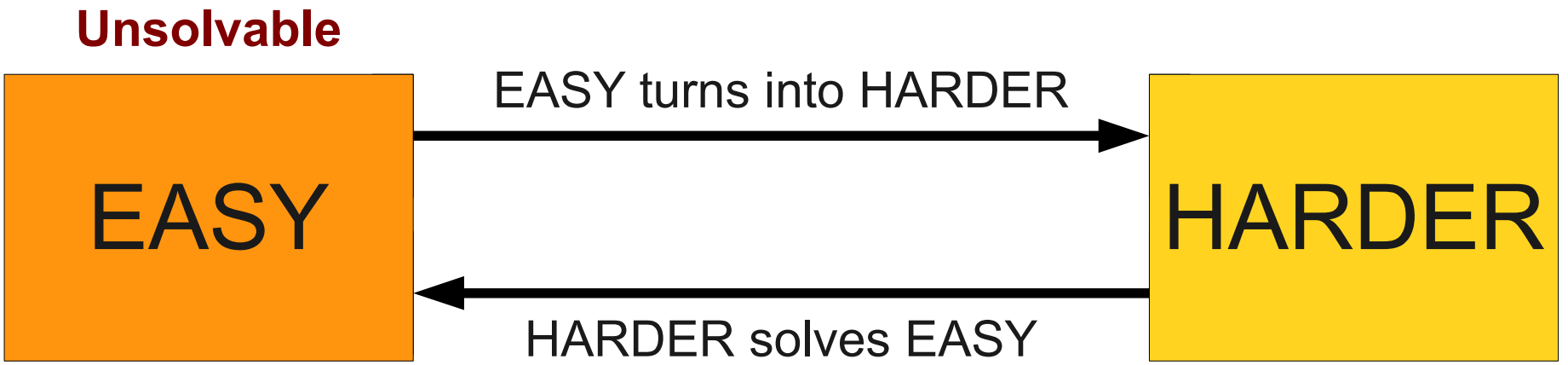

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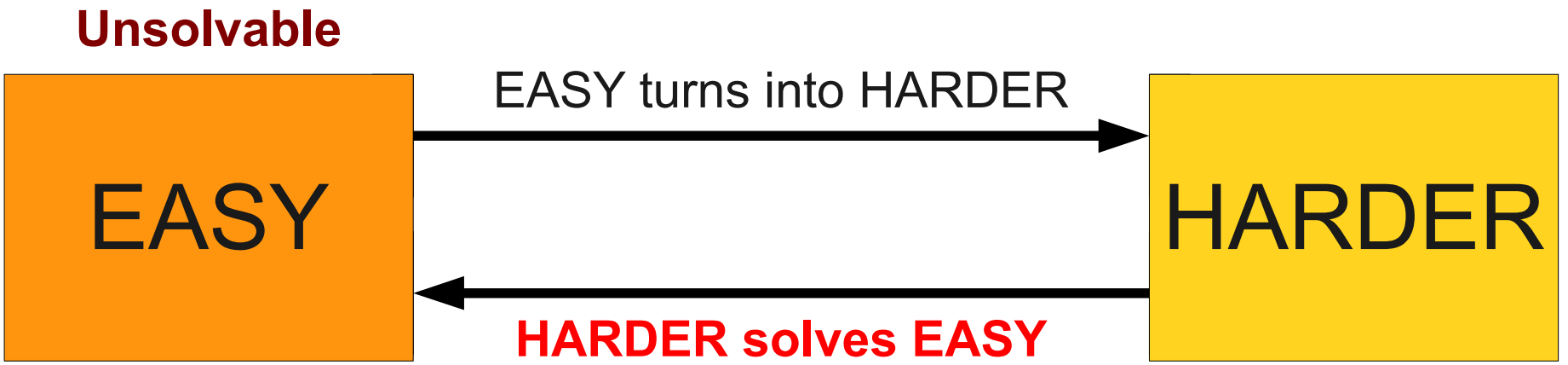

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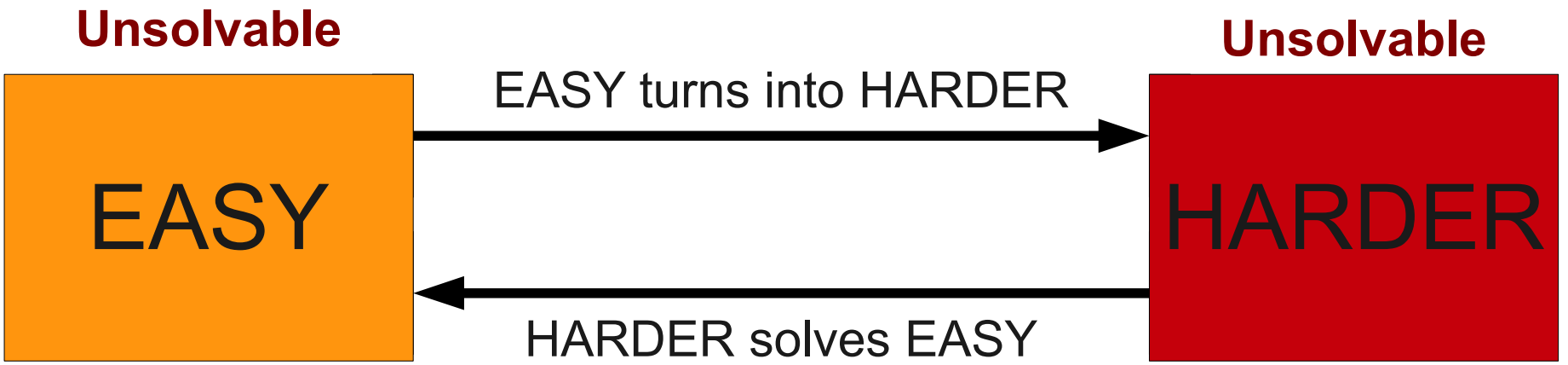

Reductions and Decidability

- Suppose we want to prove that some problem HARDER is undecidable.
- If we can **reduce** an undecidable problem HARD to HARDER, then we know that we cannot decide HARDER.

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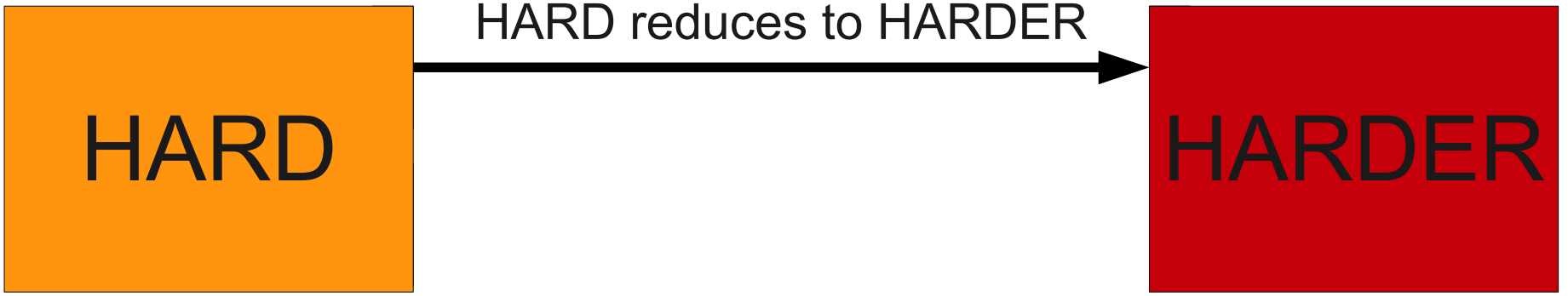

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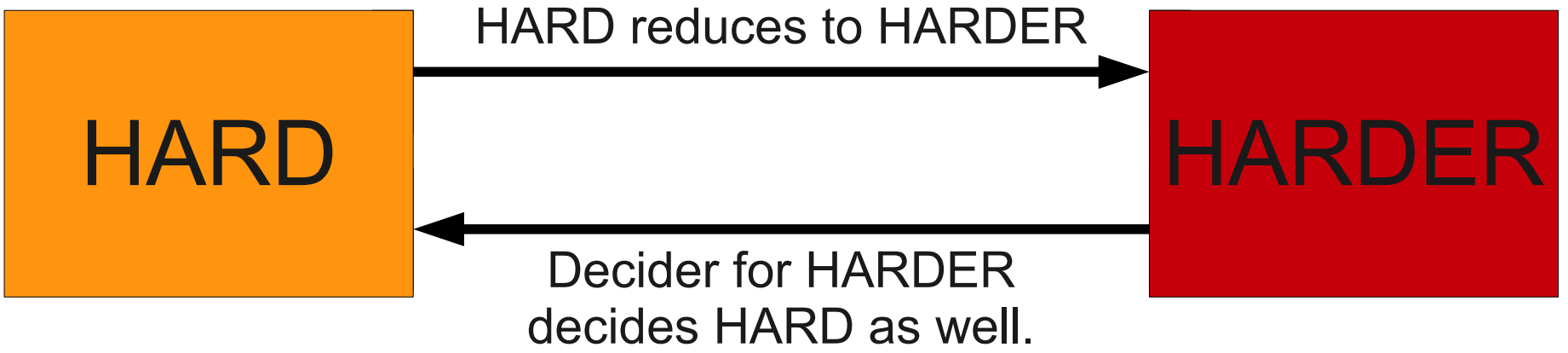

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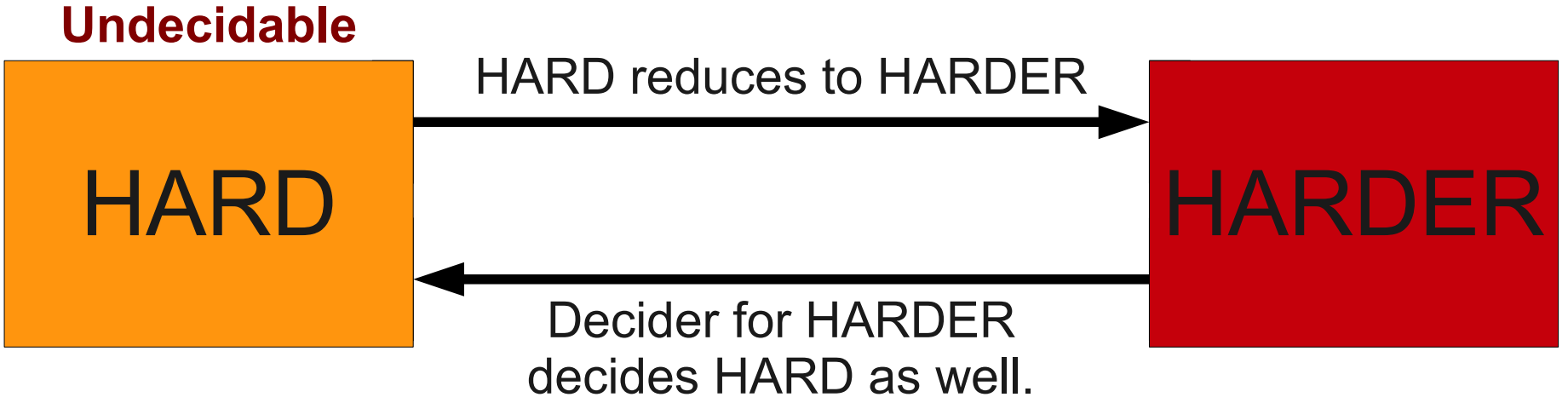

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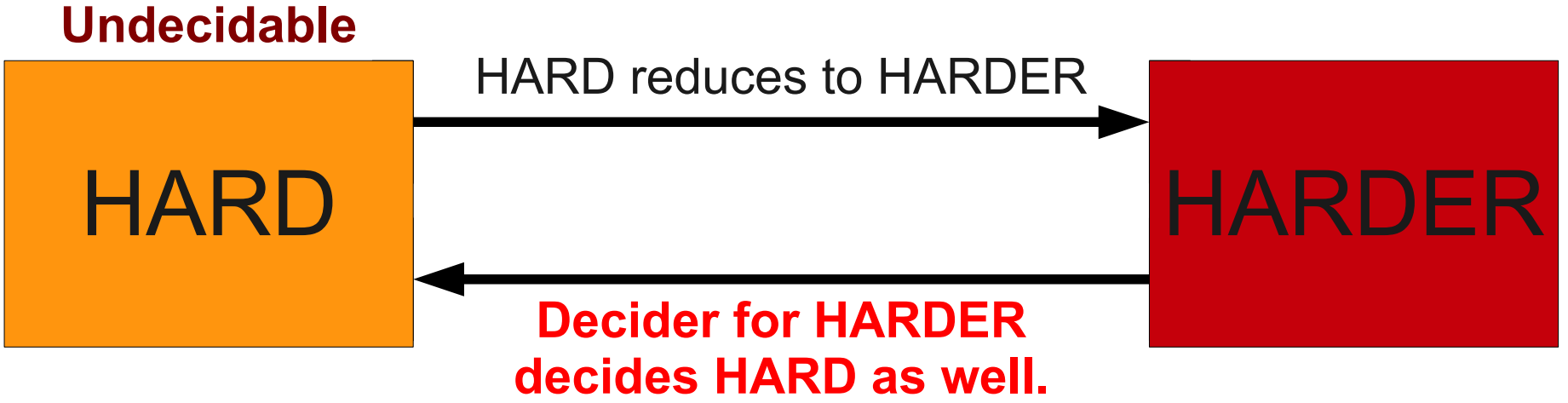

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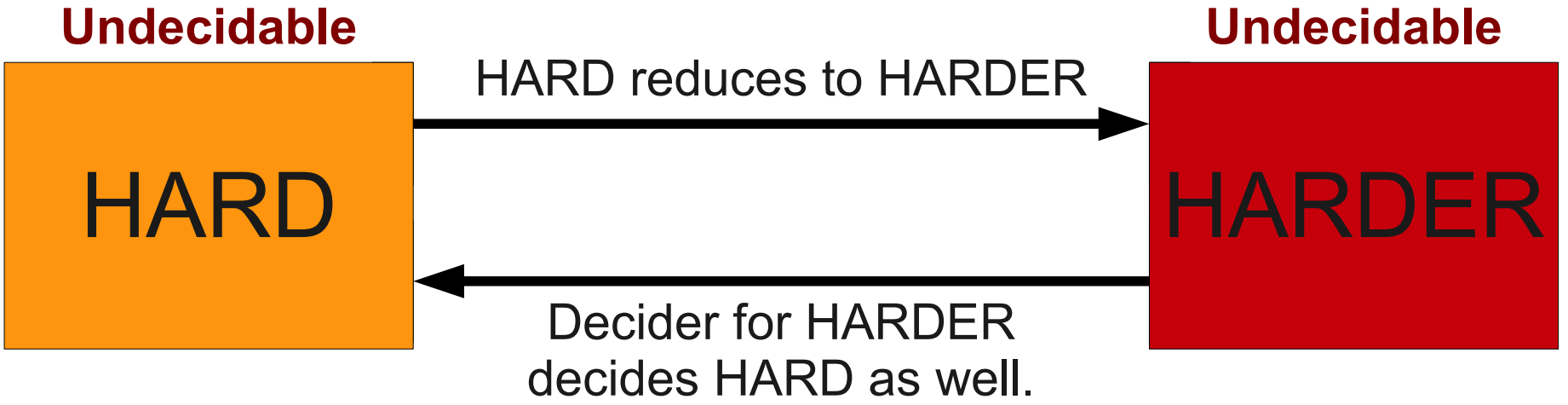

Reduction from the Halting Problem

- The language *HALT* is RE but not recursive:

$$HALT = \{ \langle M, w \rangle \mid M \text{ halts on } w \}$$

- If we can **reduce** *HALT* to some problem P, then that new problem cannot be decidable.

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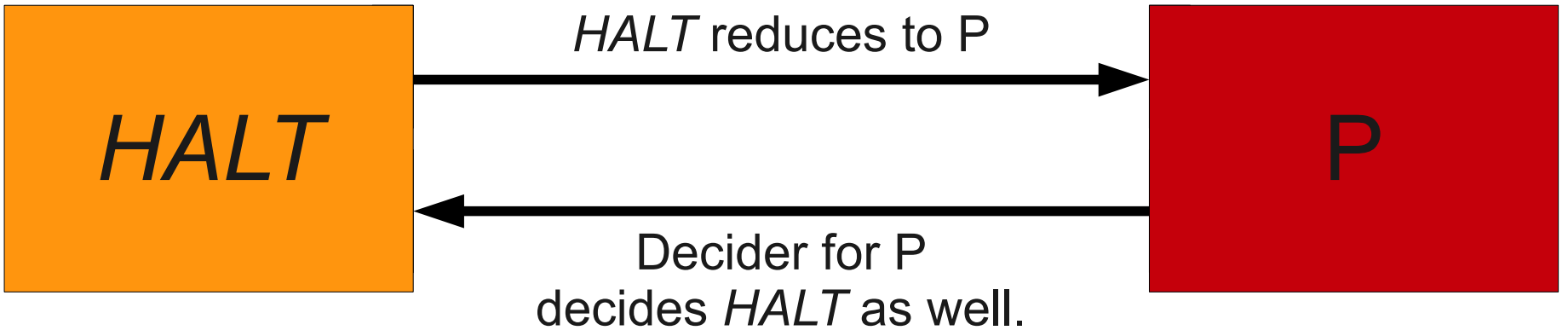

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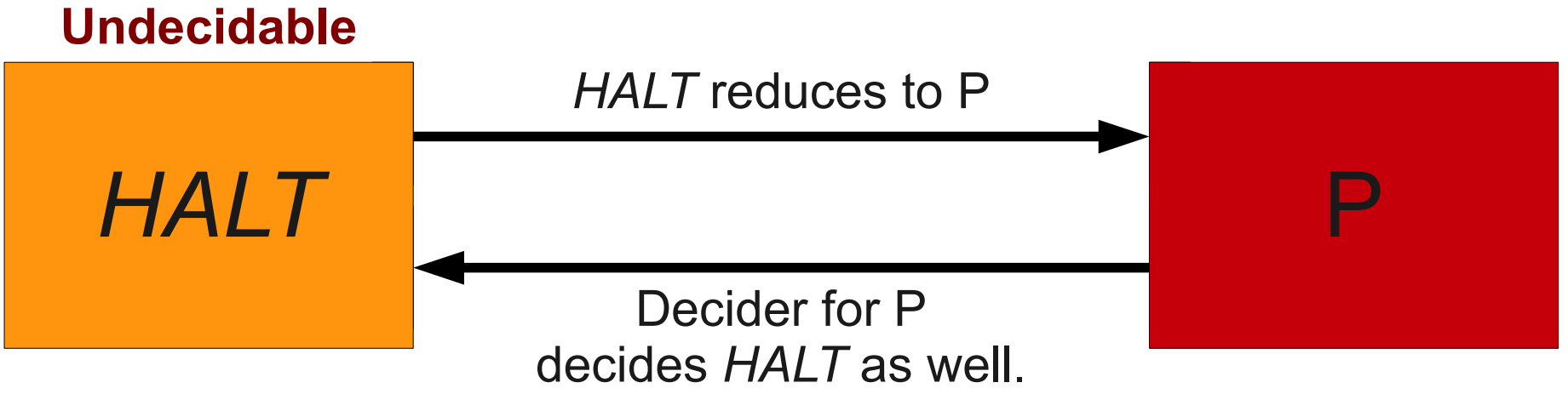

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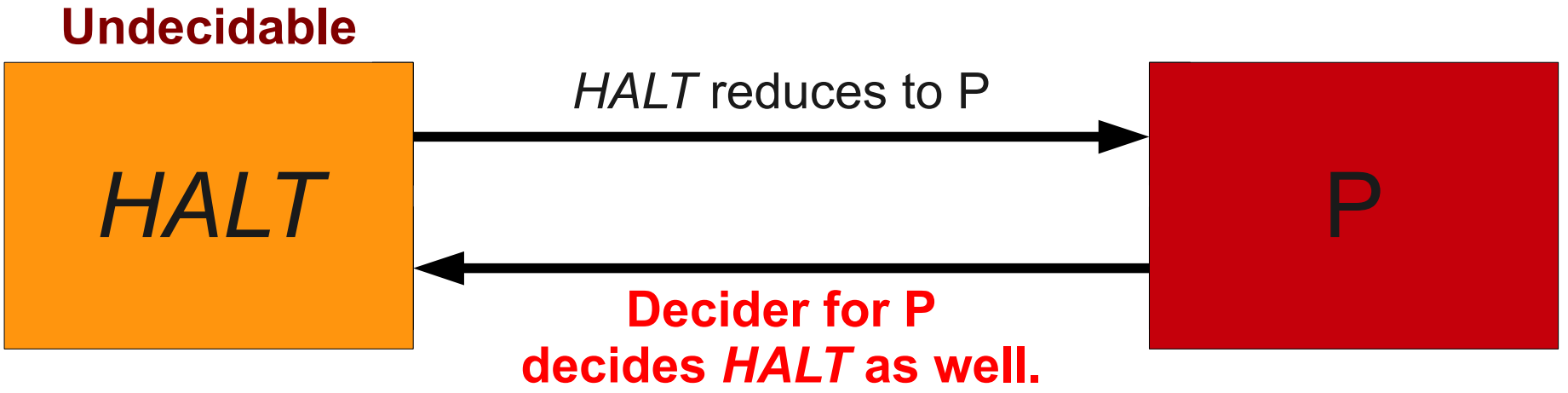

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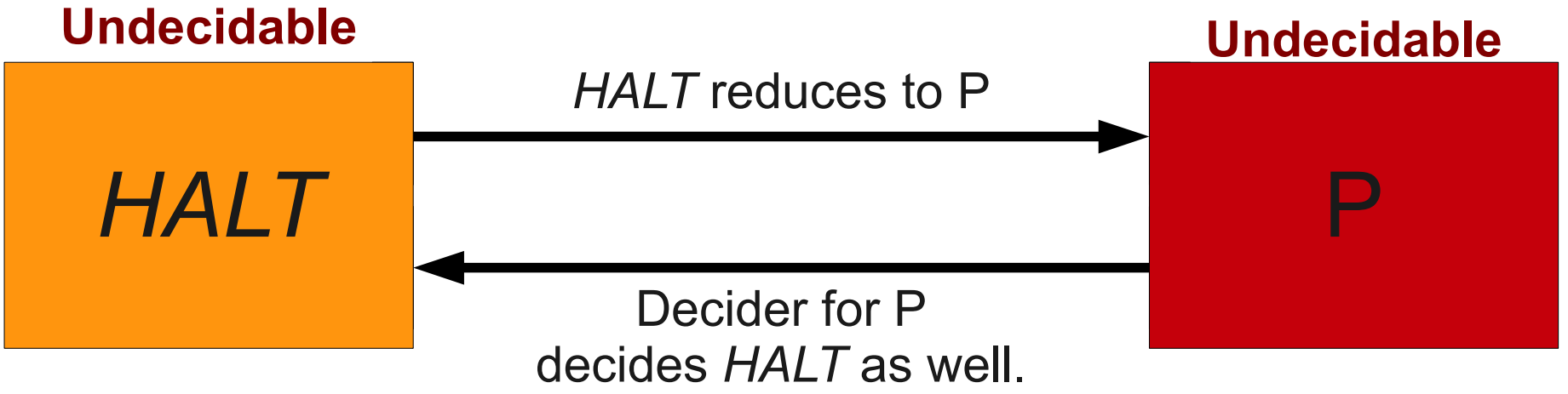

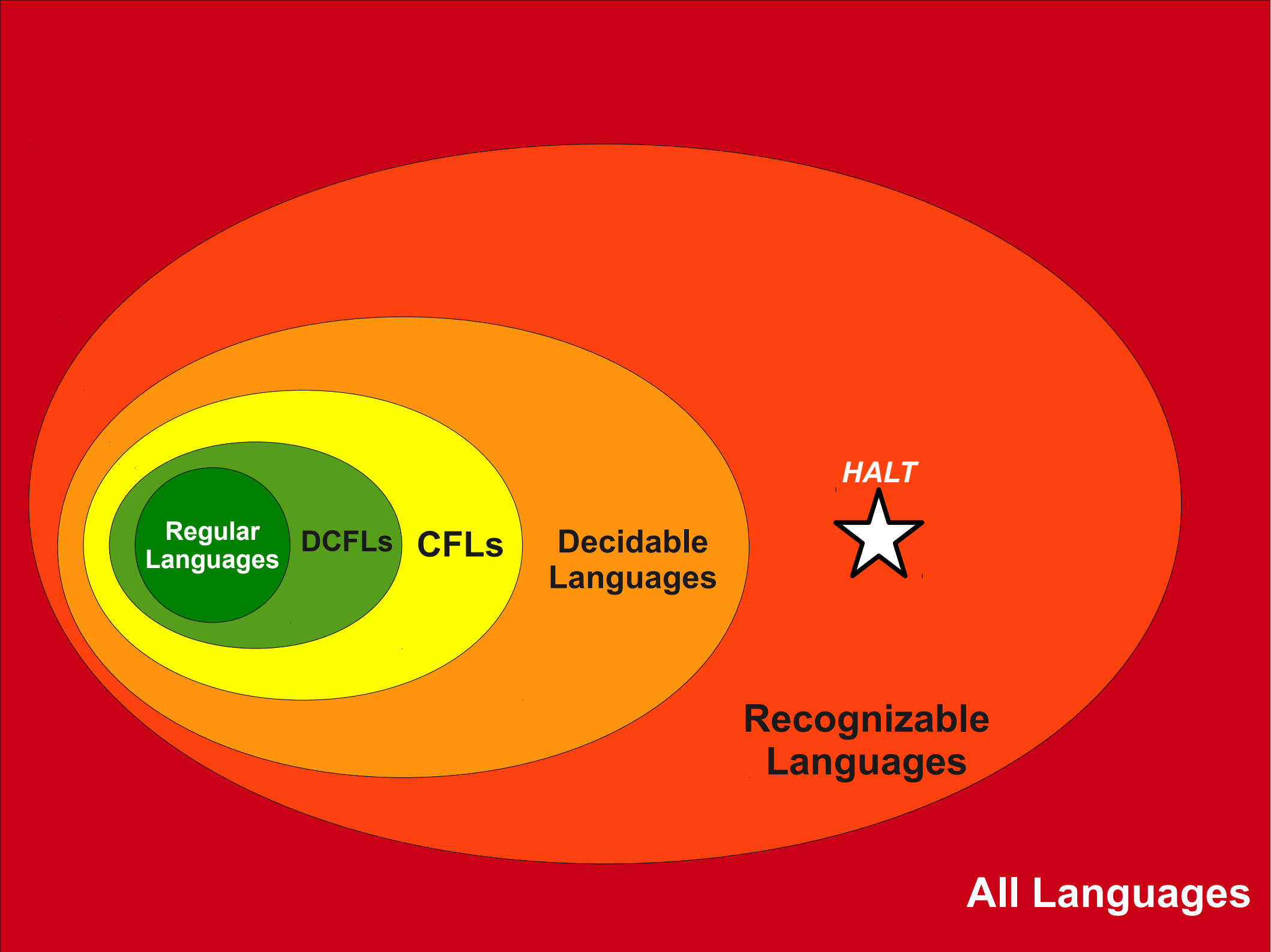
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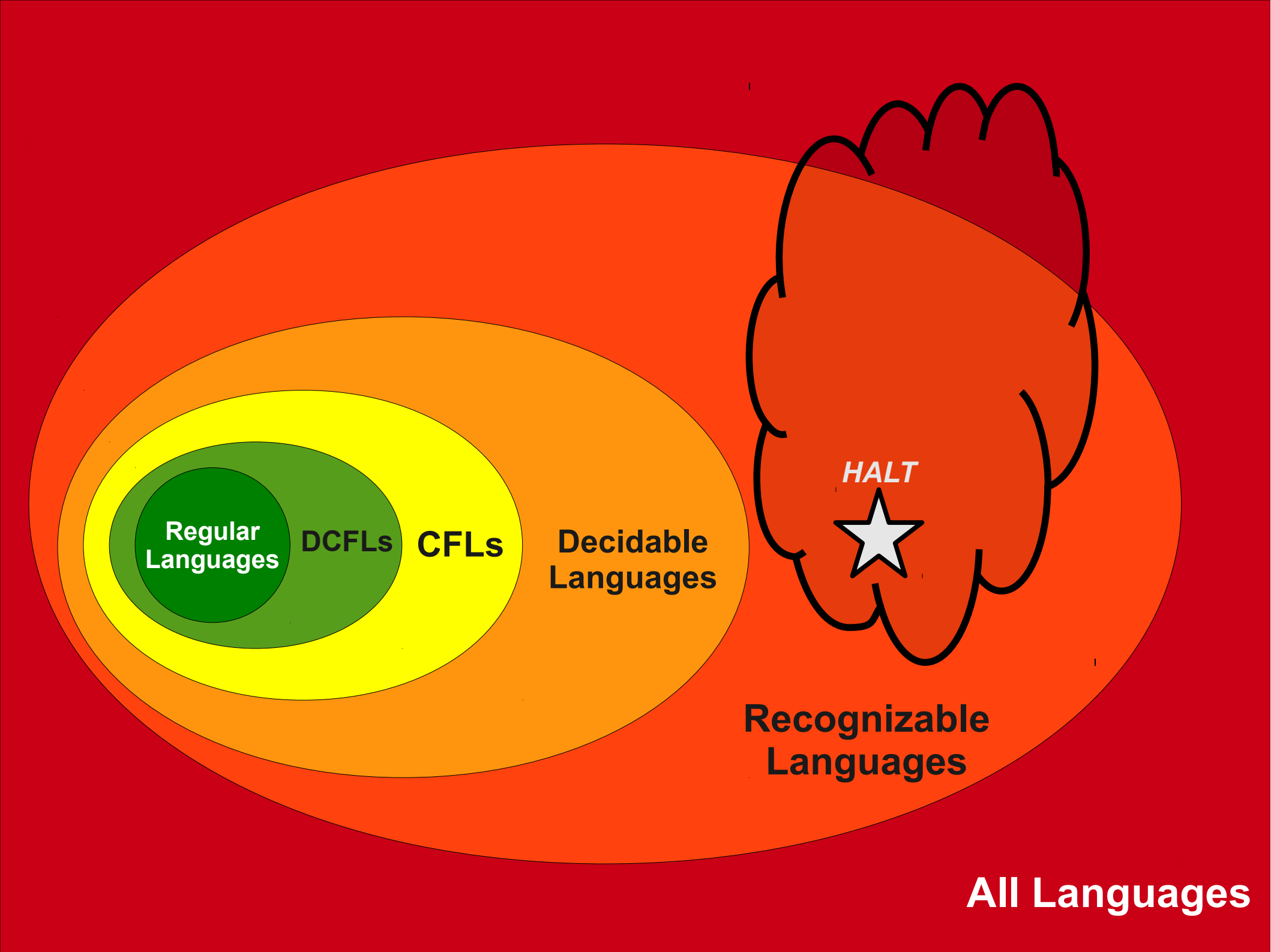
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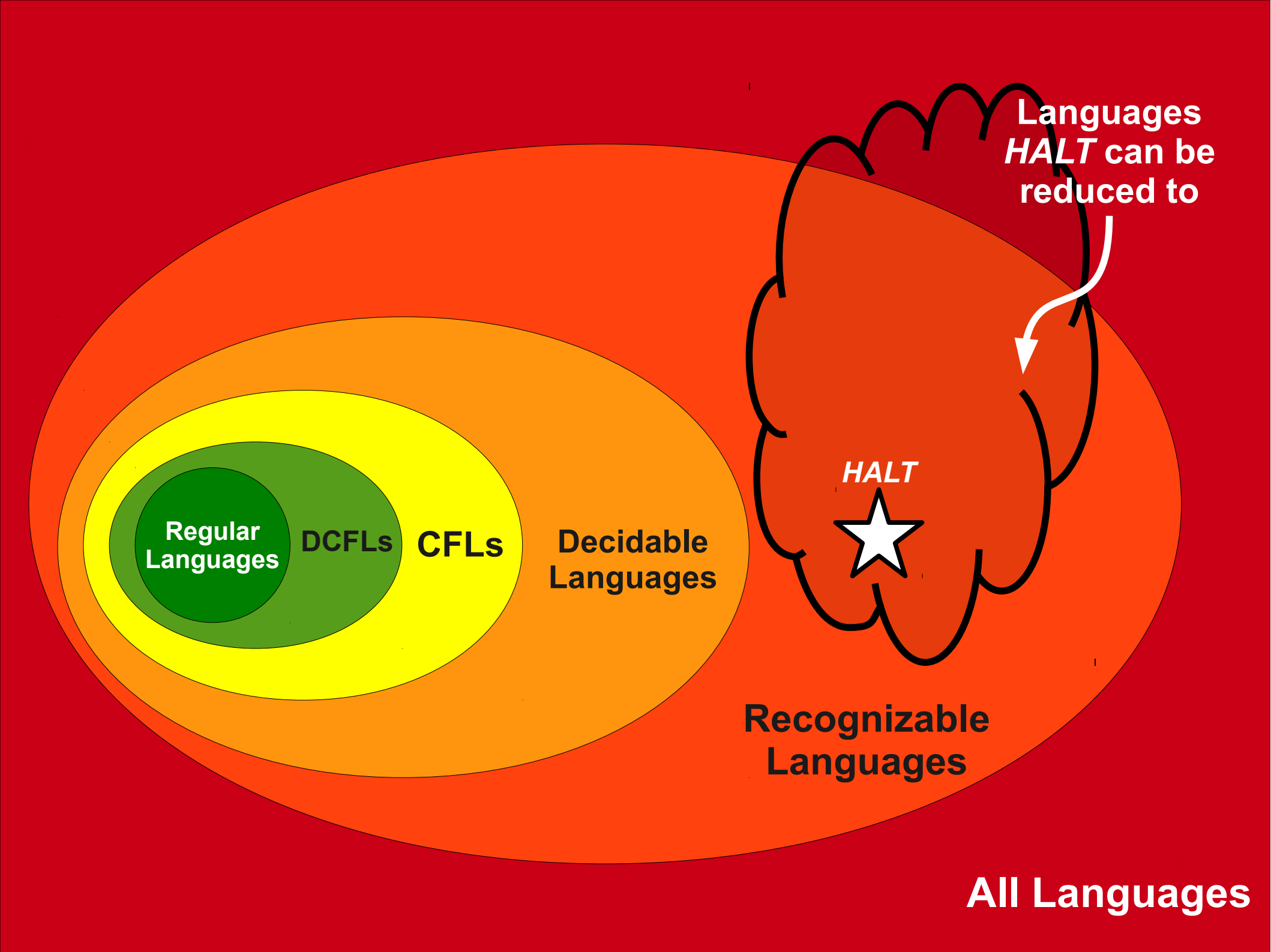
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Proof by Reduction

- Suppose that we are given a language L that we believe is undecidable.
- We can prove that this is true using the following technique:
  - Assume, for the sake of contradiction, that L is decidable.
  - Show how a decider for L could be used to construct a decider for an undecidable language.
  - Conclude that L must not be decidable.

The Totality Problem

- Last time we saw that there was no way to convert an arbitrary recognizer into a decider.
- However, is there a way to tell whether a given TM is a recognizer or a decider?
- This is called the **totality problem**.

The Totality Problem

- Let

$$TOTAL = \{ \langle M \rangle \mid M \text{ halts on all inputs.} \}$$

- This seems at least as hard as the halting problem.
- How would we prove that this is undecidable?

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- Let

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- This seems at least as hard as the halting problem.
- How would we prove that this is undecidable?
- **Idea**: Reduce *HALT* to *TOTAL*.
- Show how a decider for *TOTAL* could be used to build a decider for *HALT*.
- Conclude that no such decider can exist.

The Totality Problem

Assume, for the sake of contradiction, that TOTAL is decidable.

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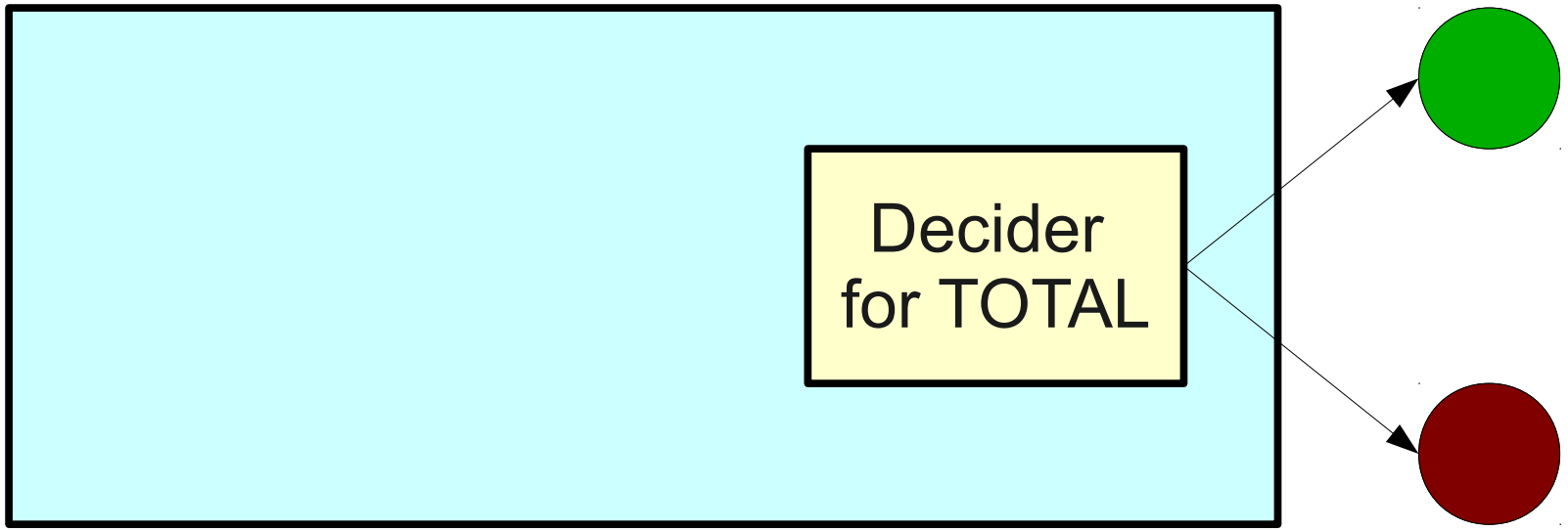

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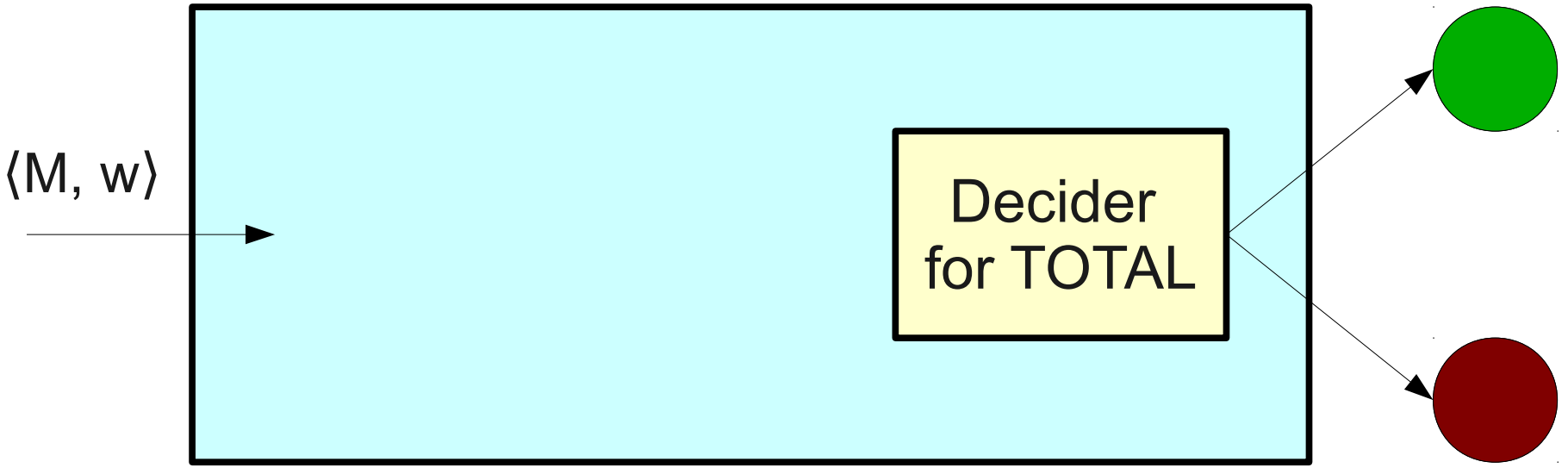

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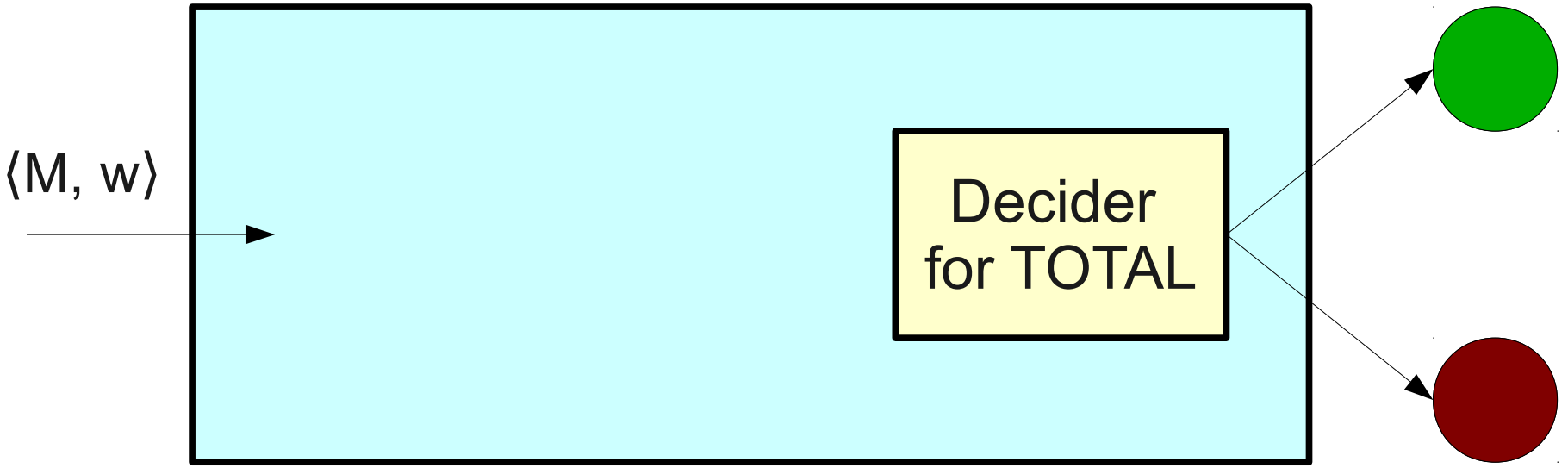

We're going to try to show how a decider for TOTAL decides HALT, so our input must be a TM and a string.

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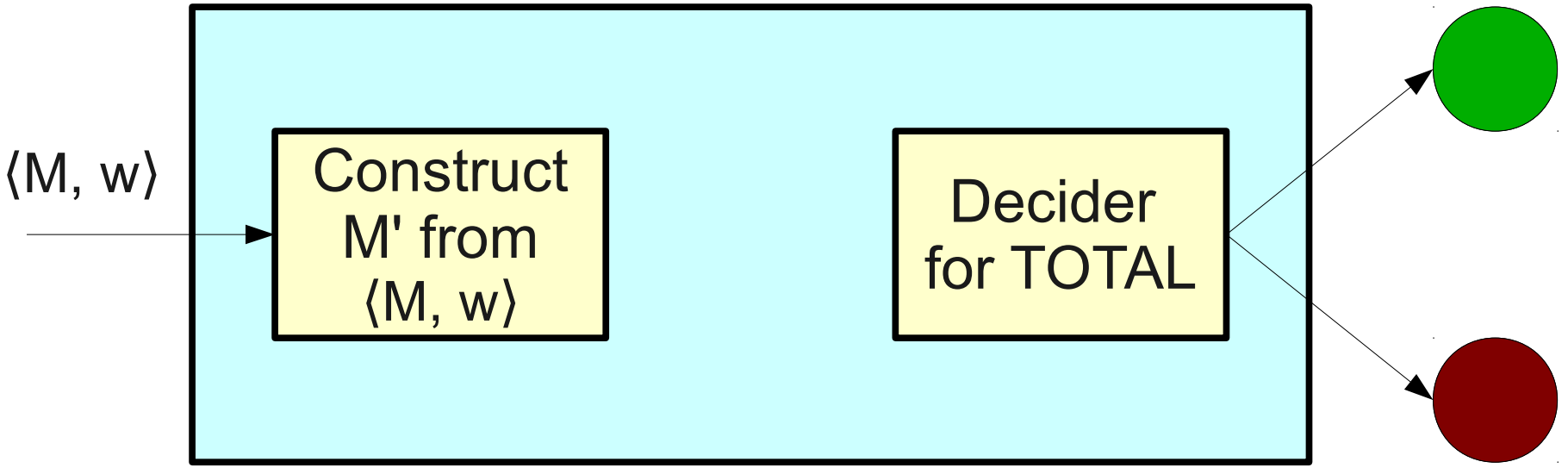

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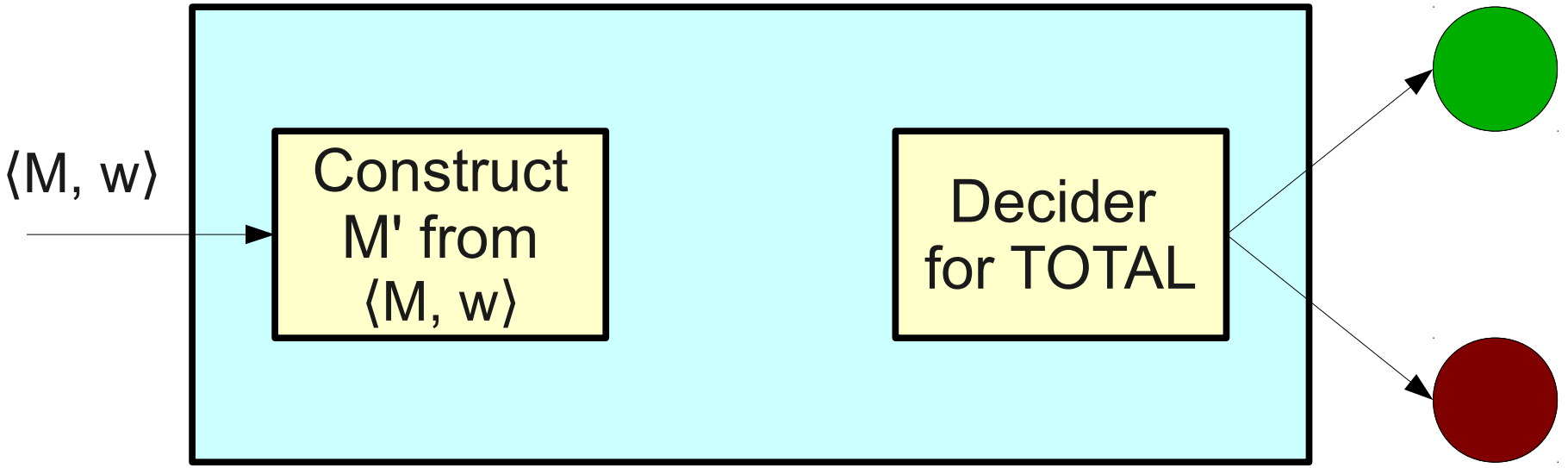

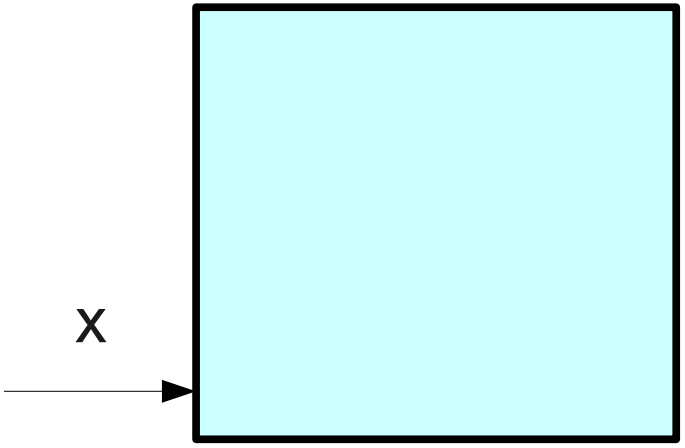

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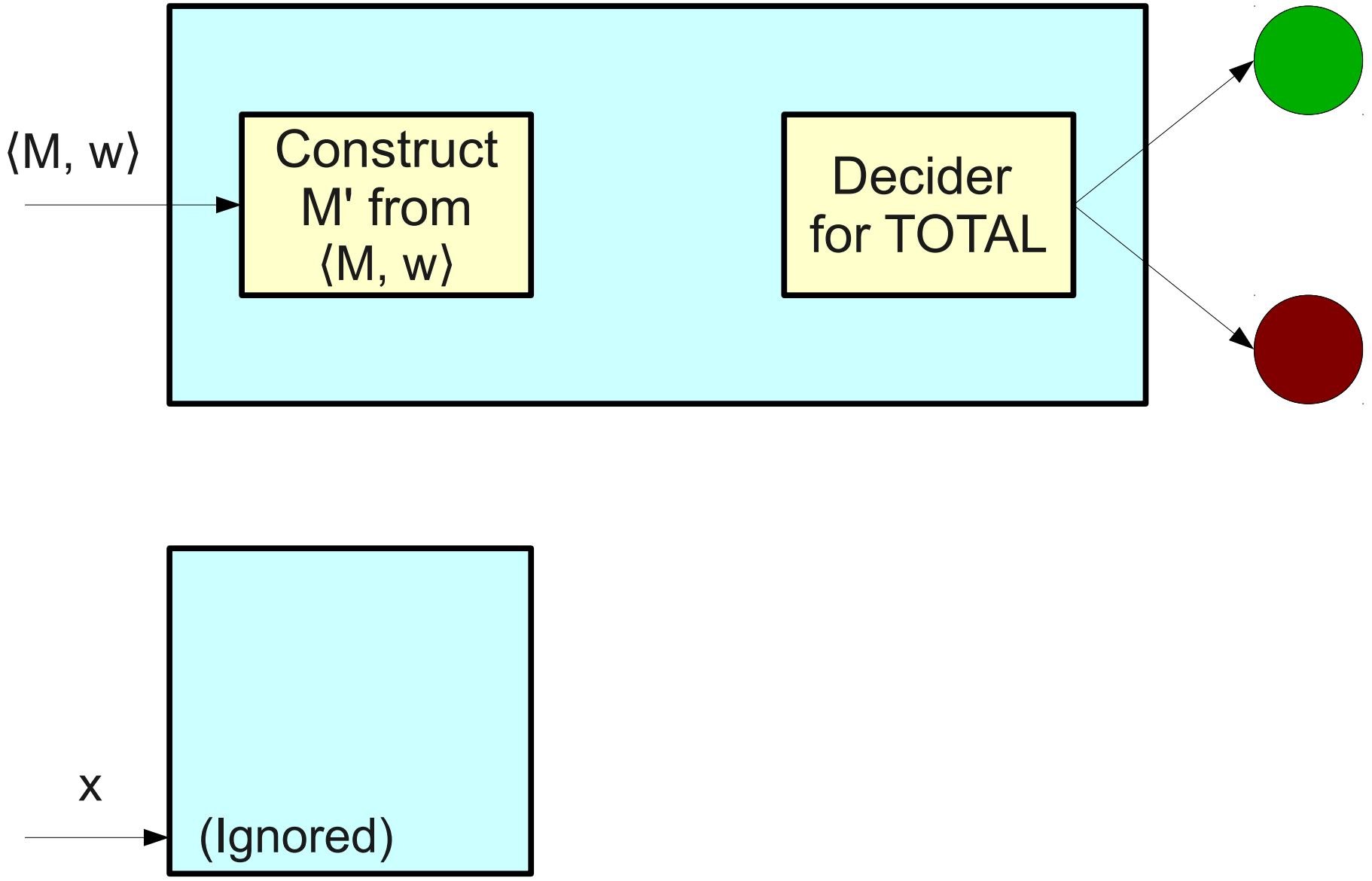

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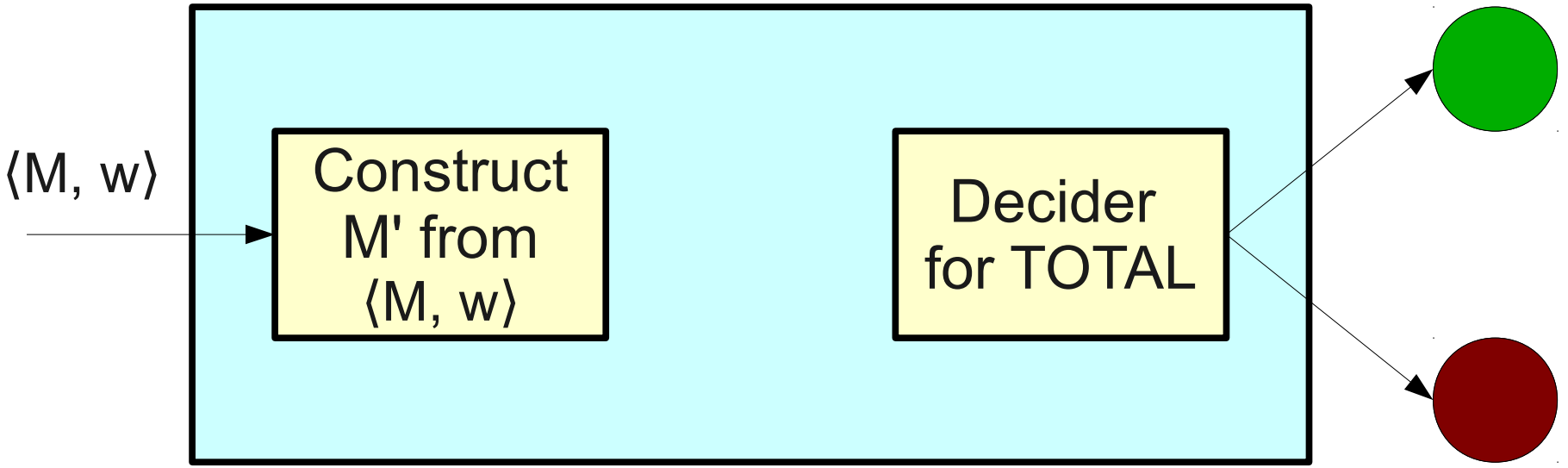

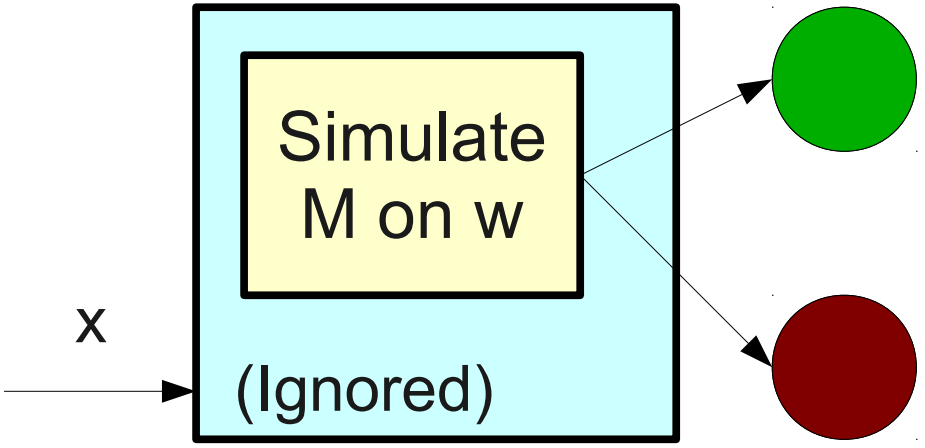

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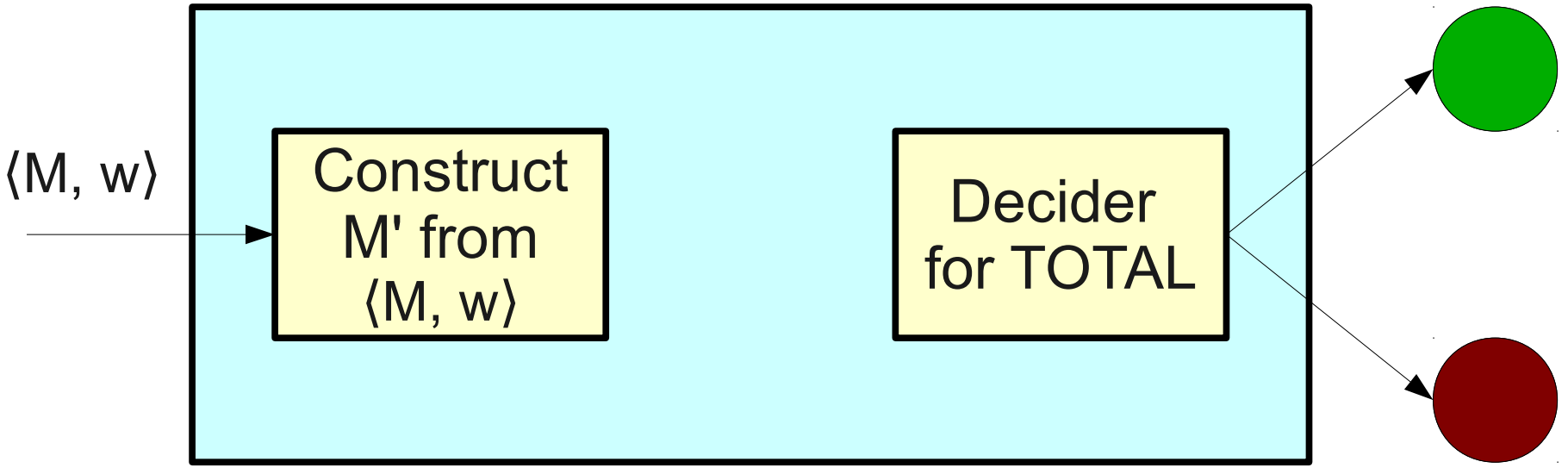

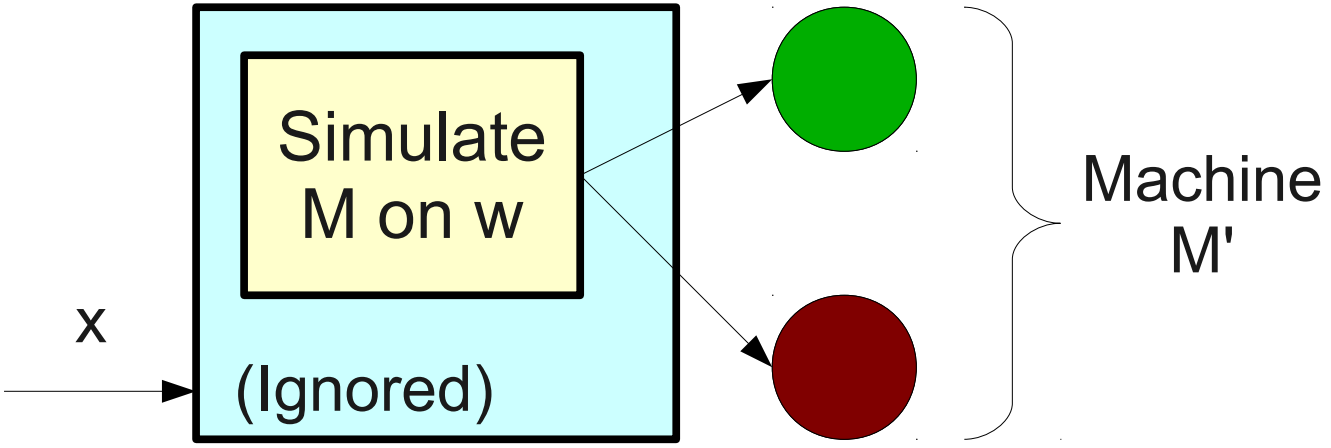

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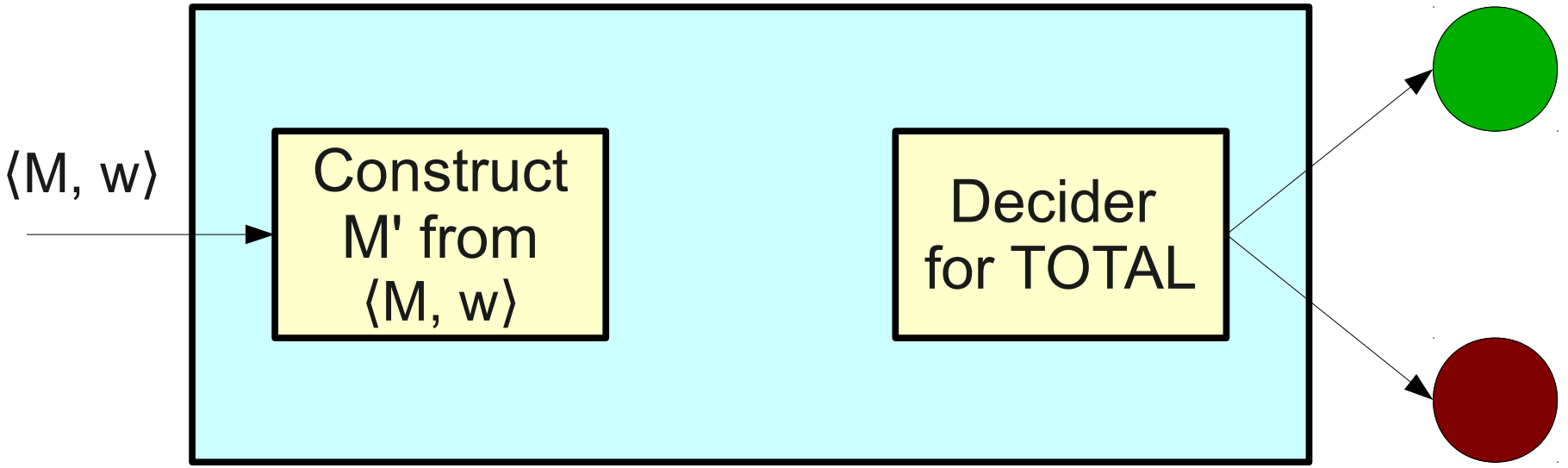
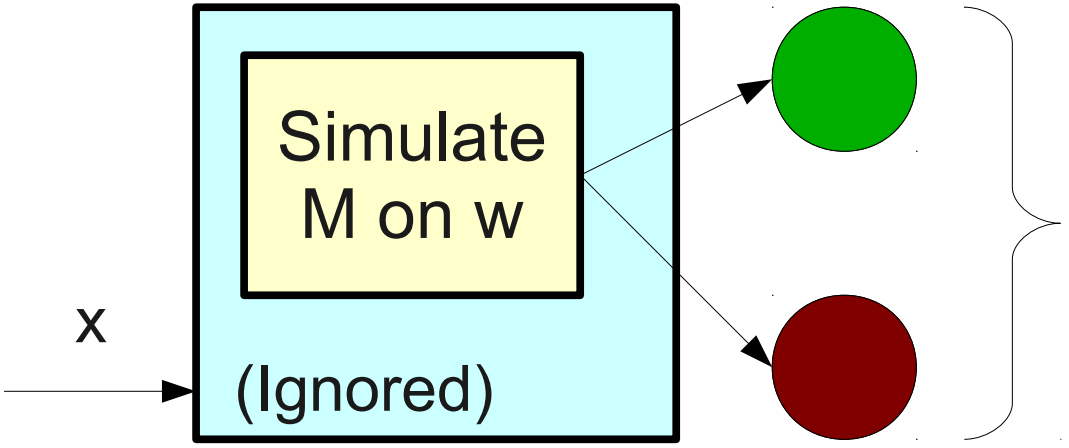
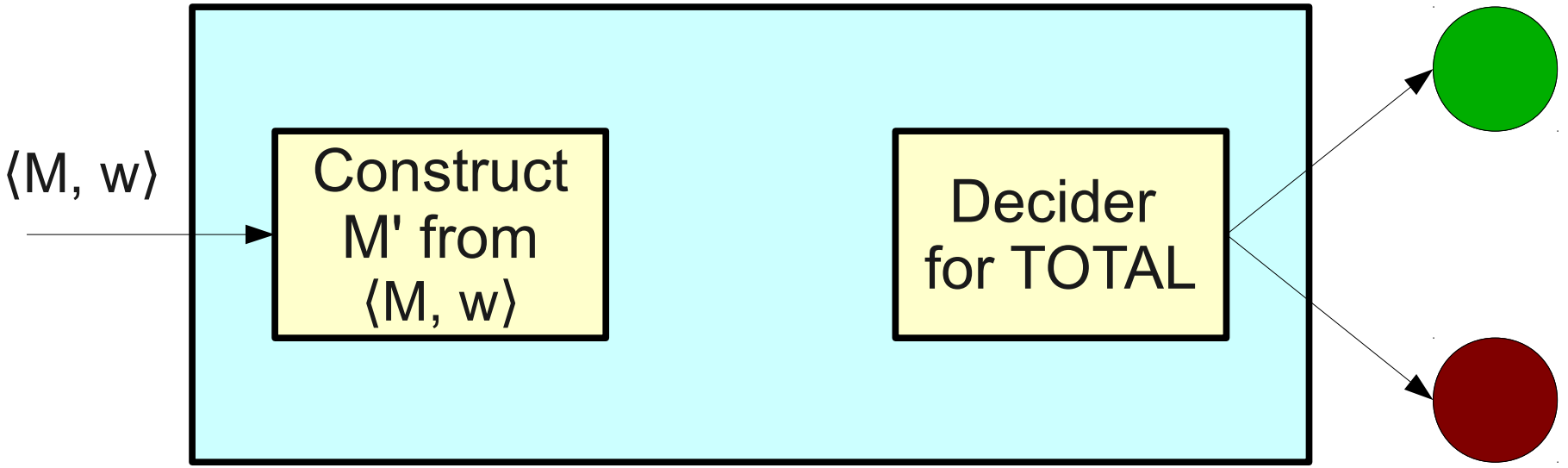


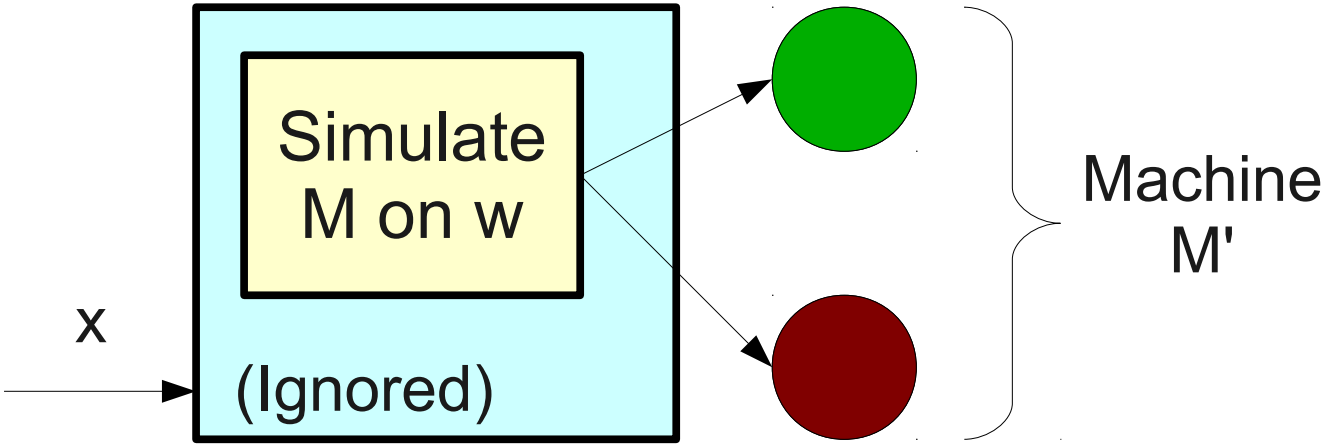


Machine M'

M' = “On input x:
Ignore x.
Run M on w.
If M accepts w, accept.
If M rejects w, reject.”

The Totality Problem





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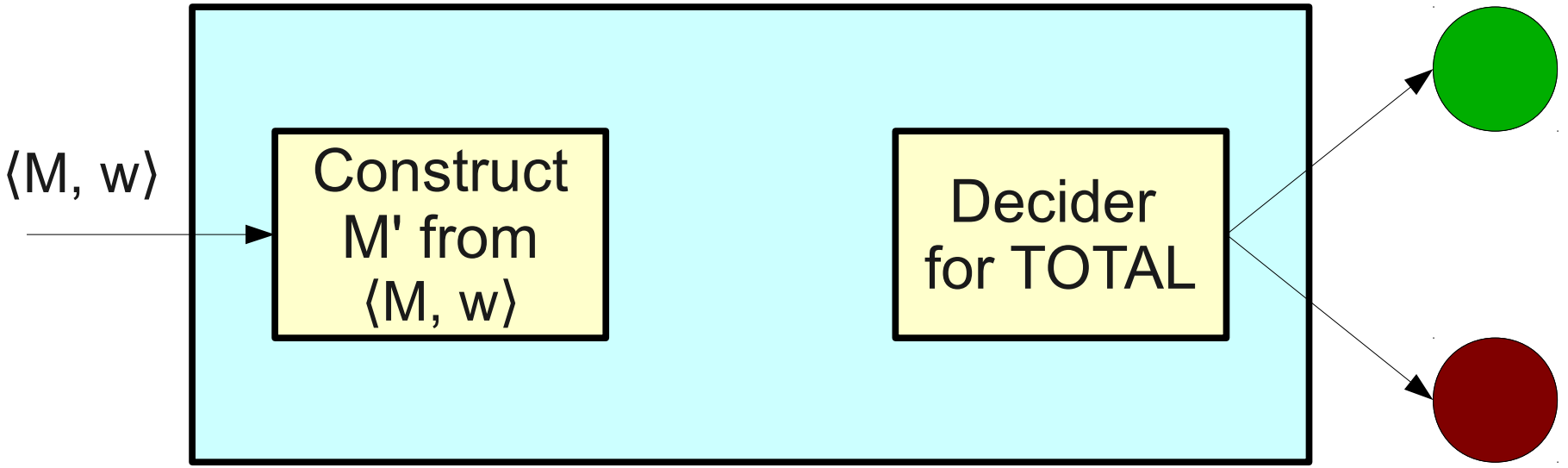
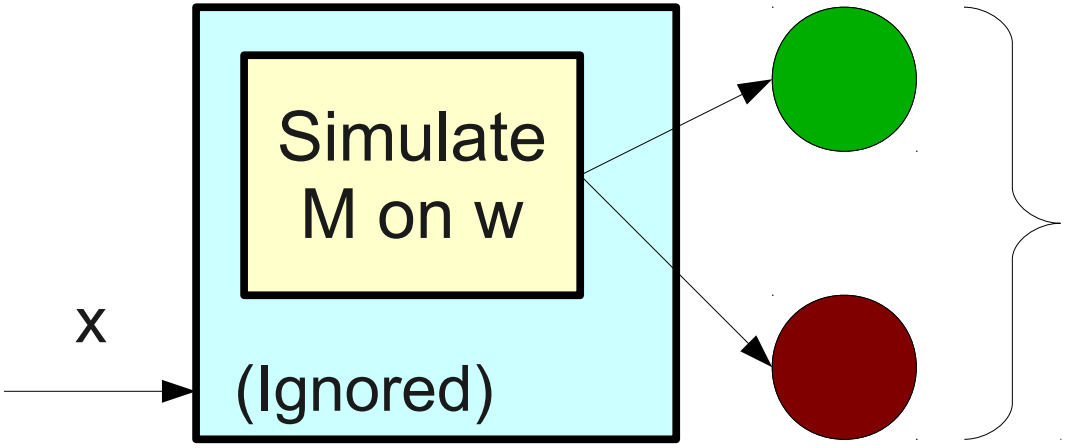
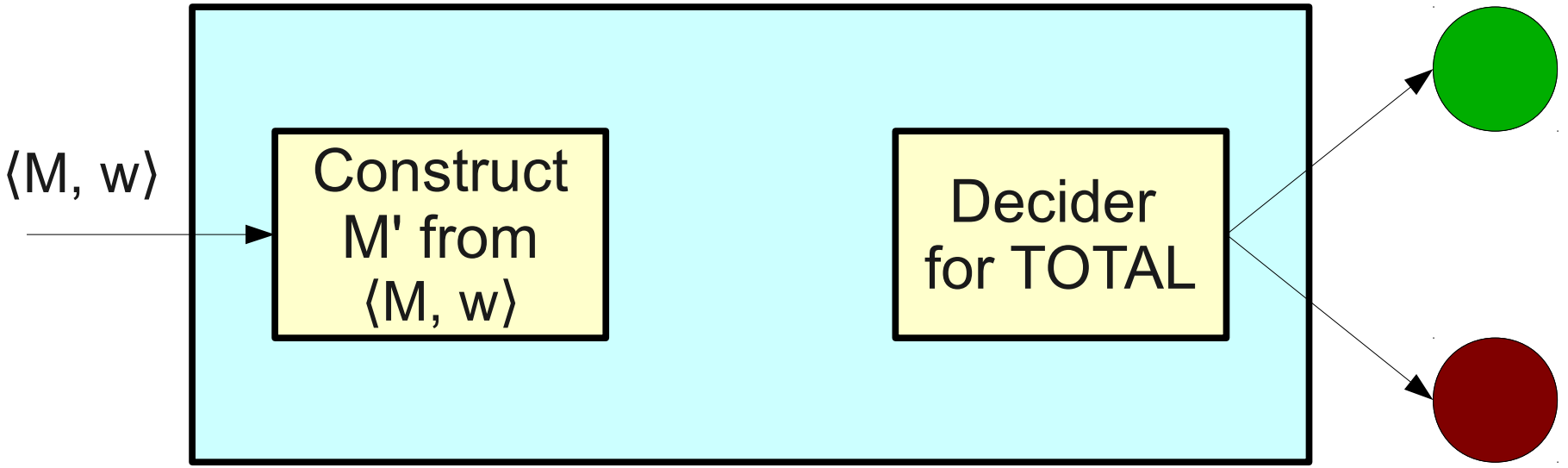
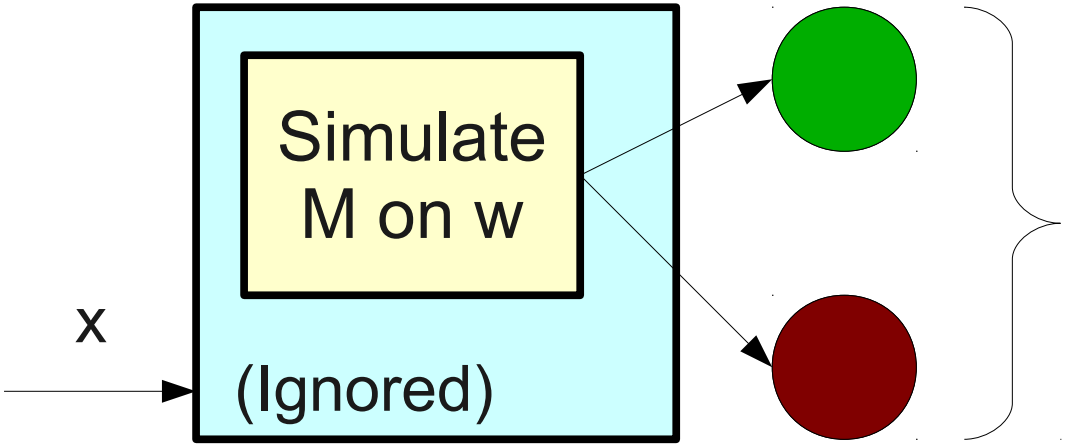




What does M' do if M halts on w?

The Totality Problem





What does M' do if M halts on w?

M' always halts

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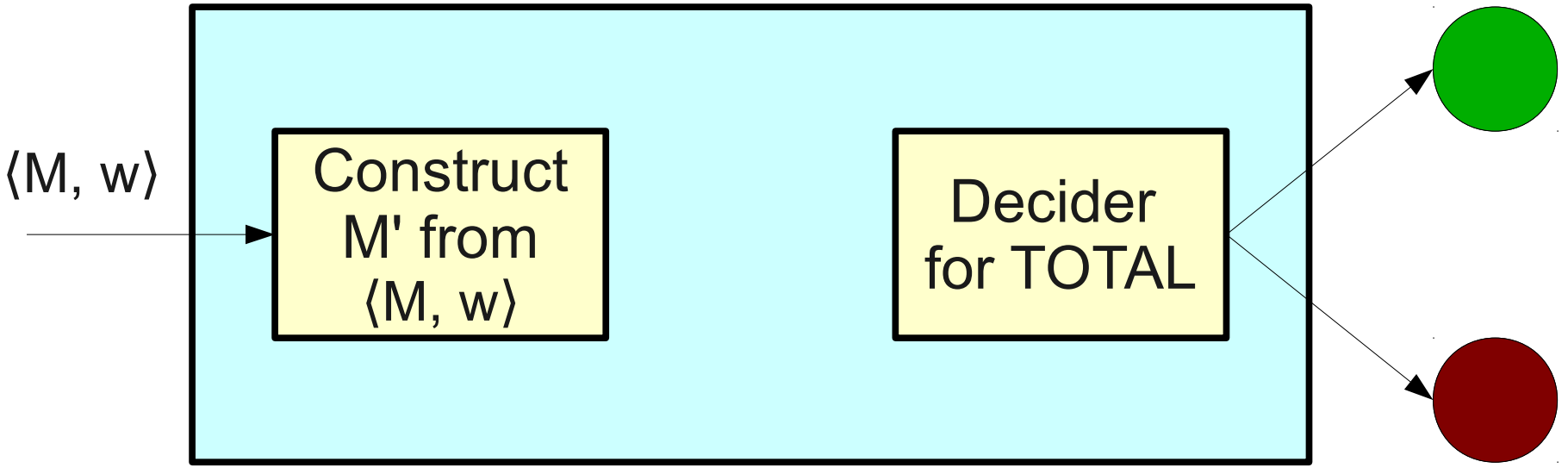
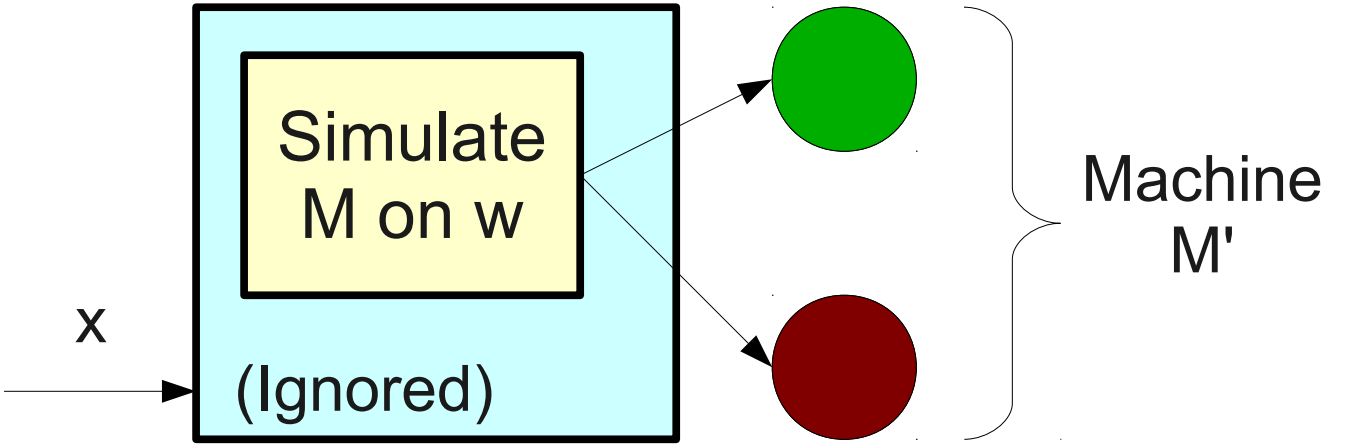
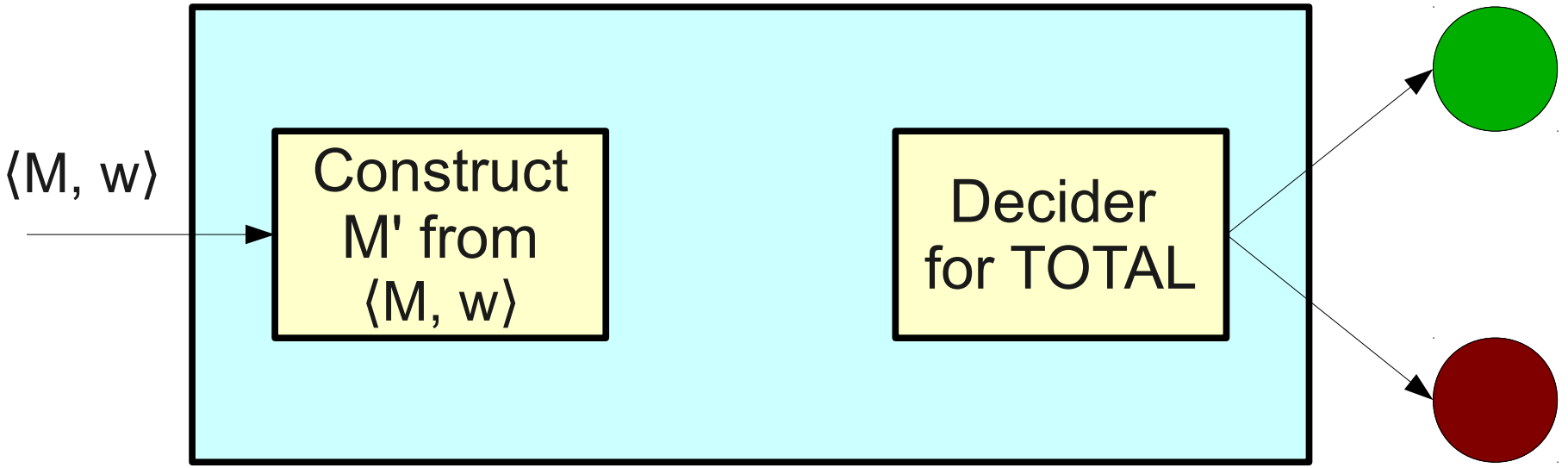
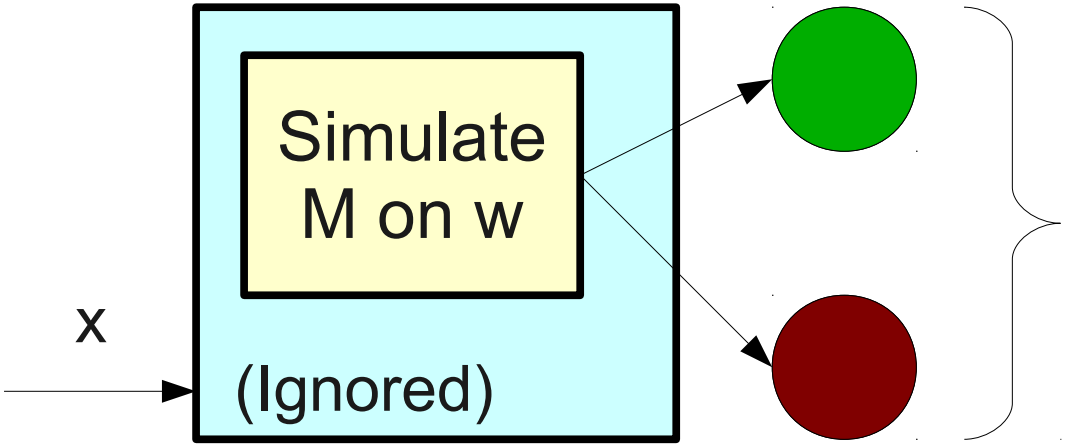




What does M' do if M loops on w?

The Totality Problem





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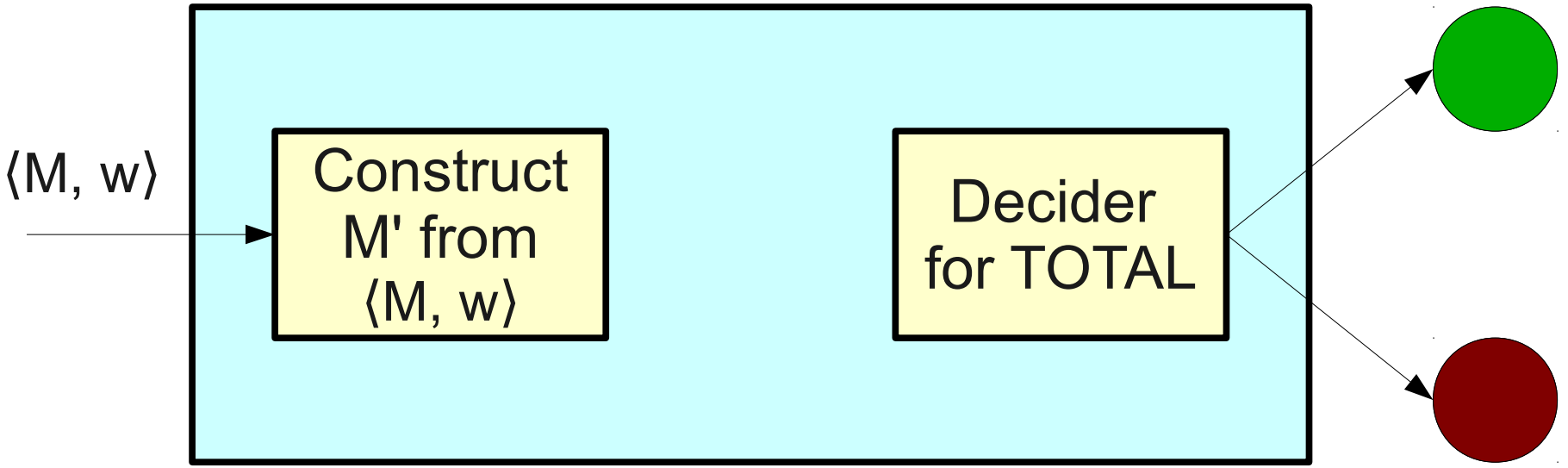
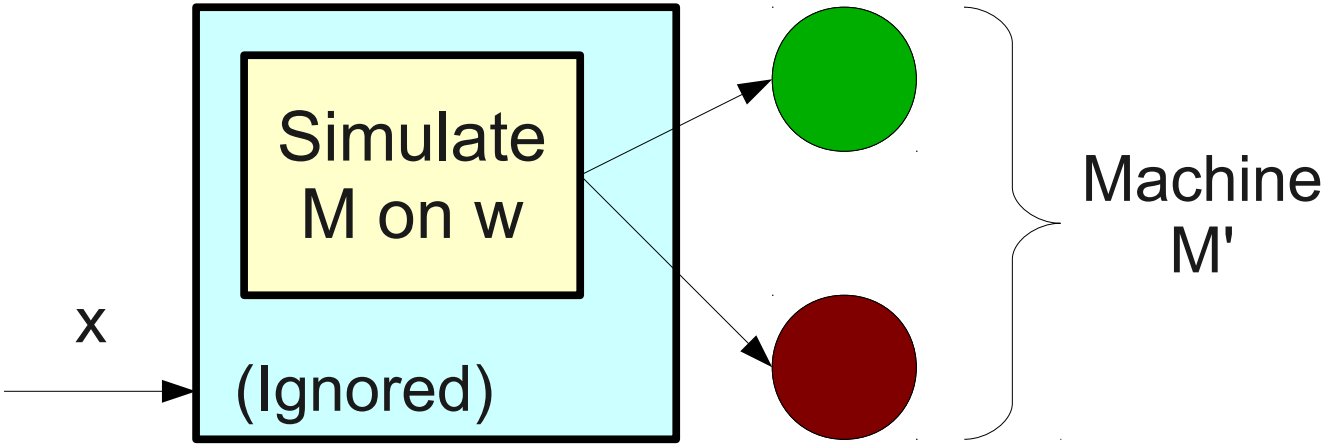
M' never halts

The Totality Problem





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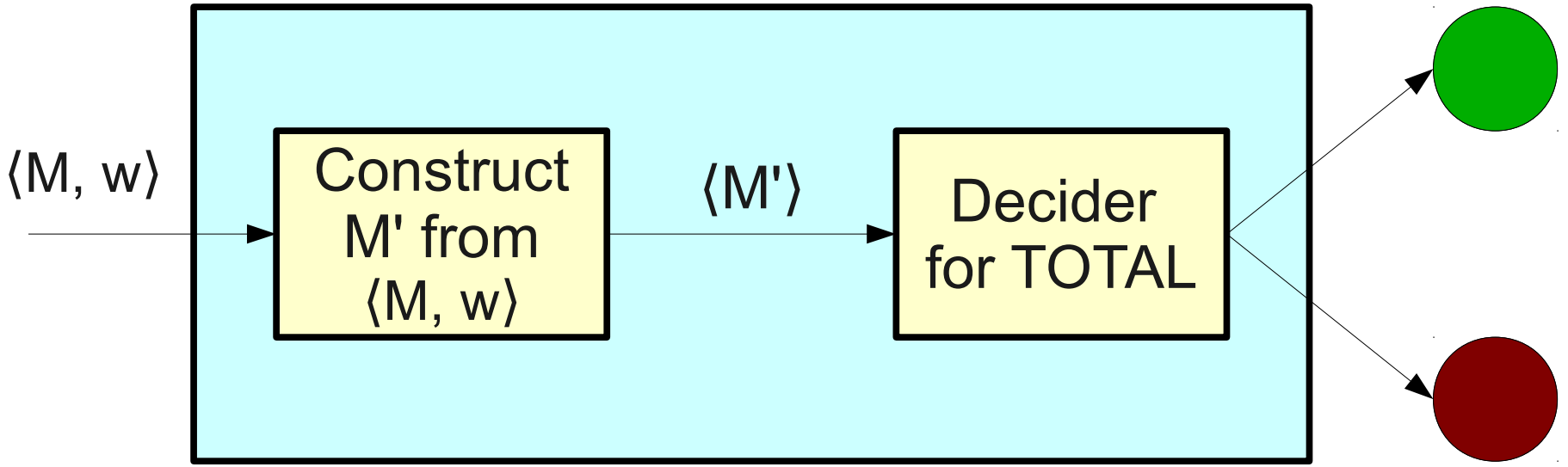

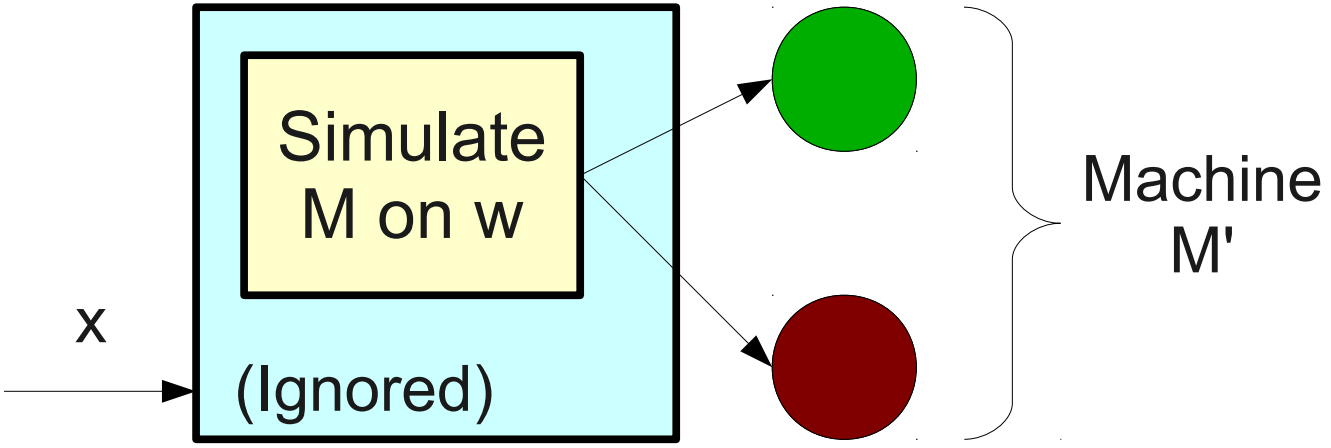

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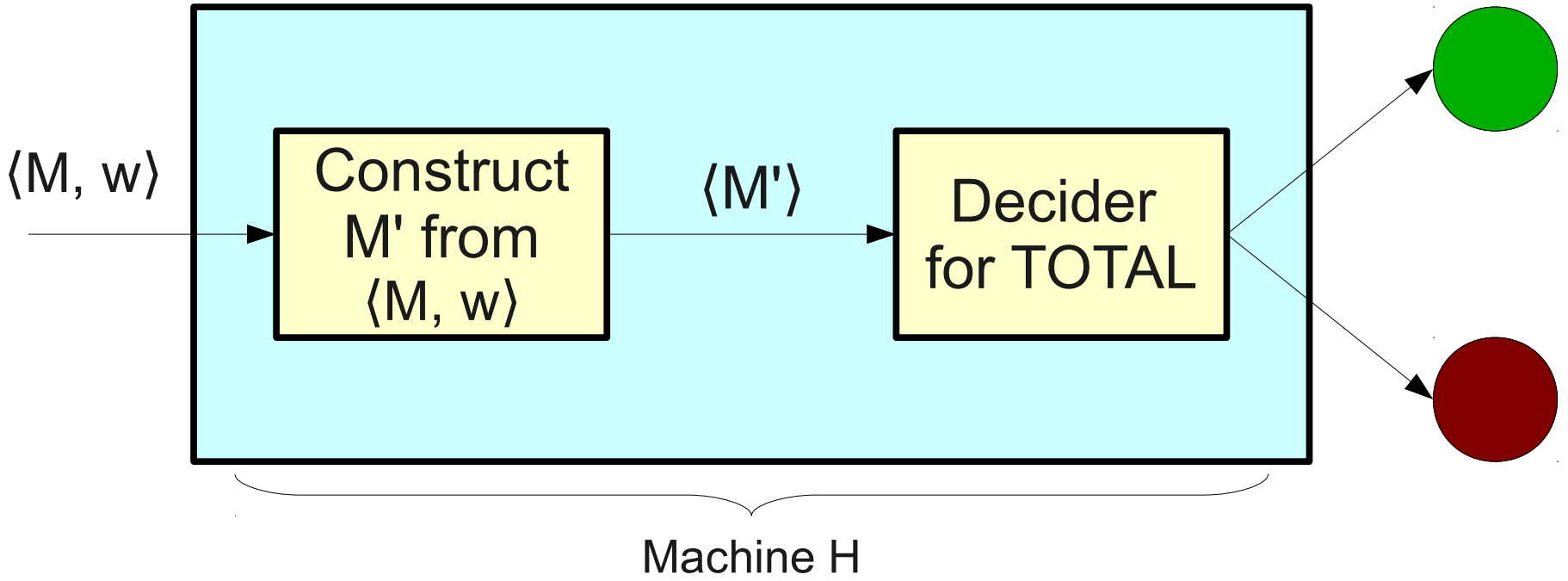

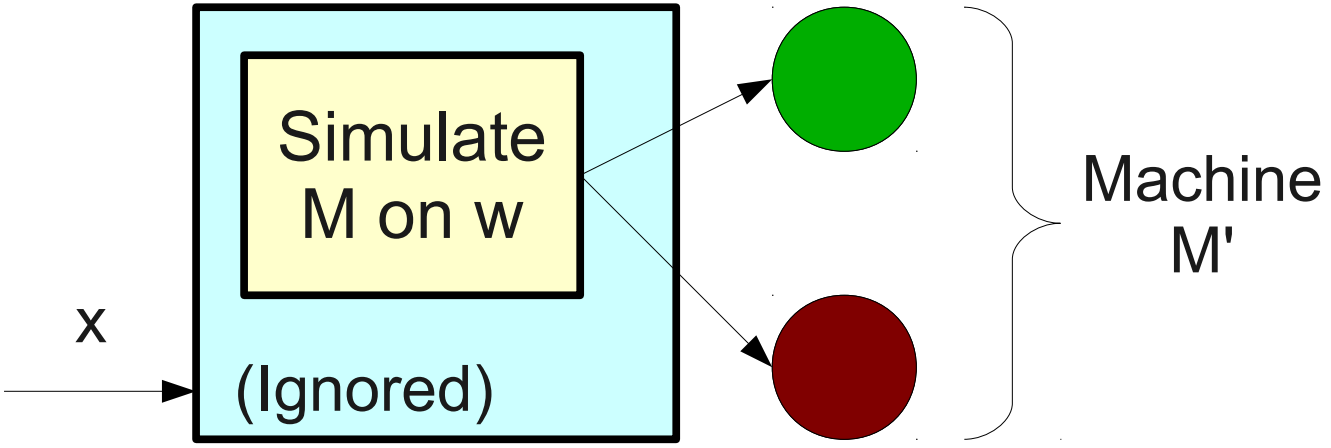

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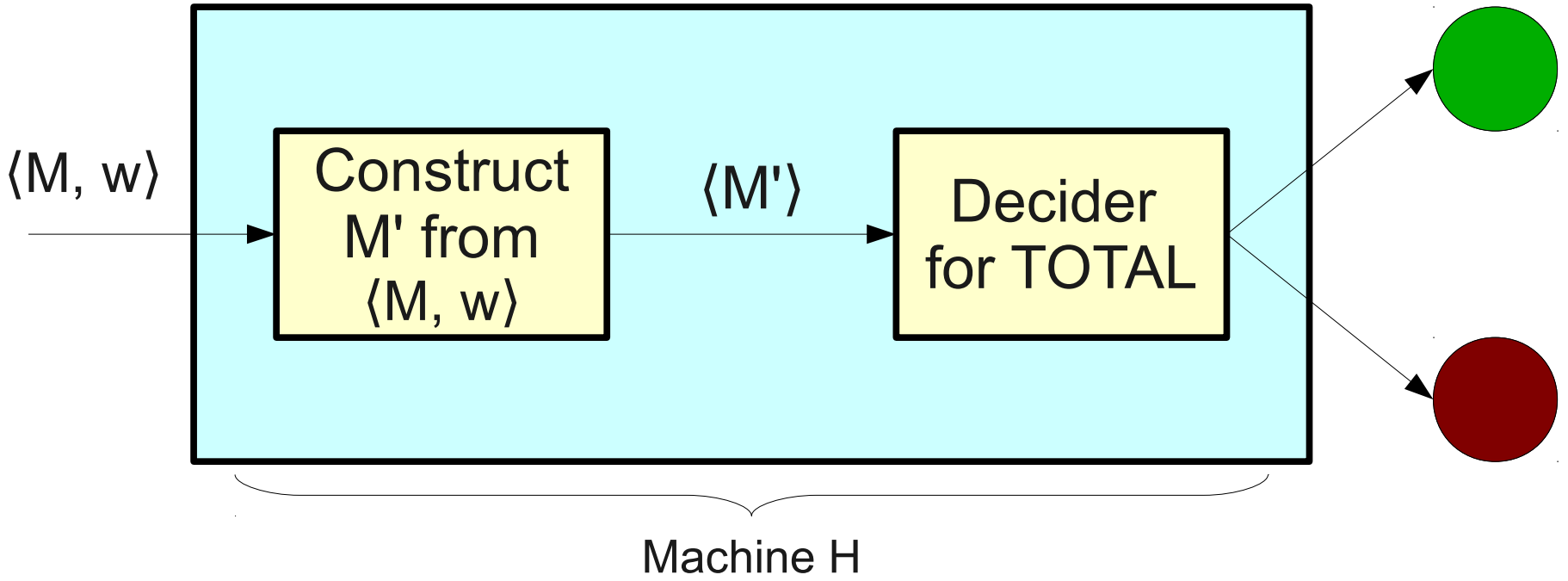

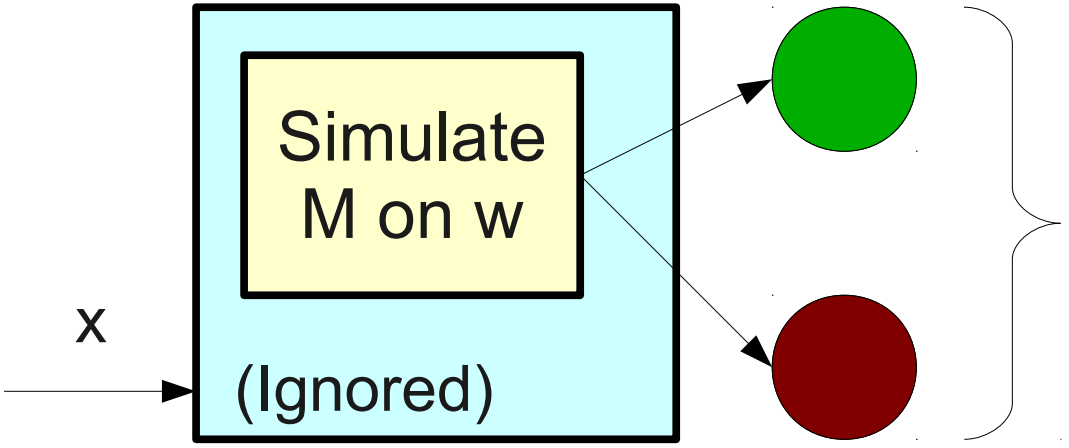

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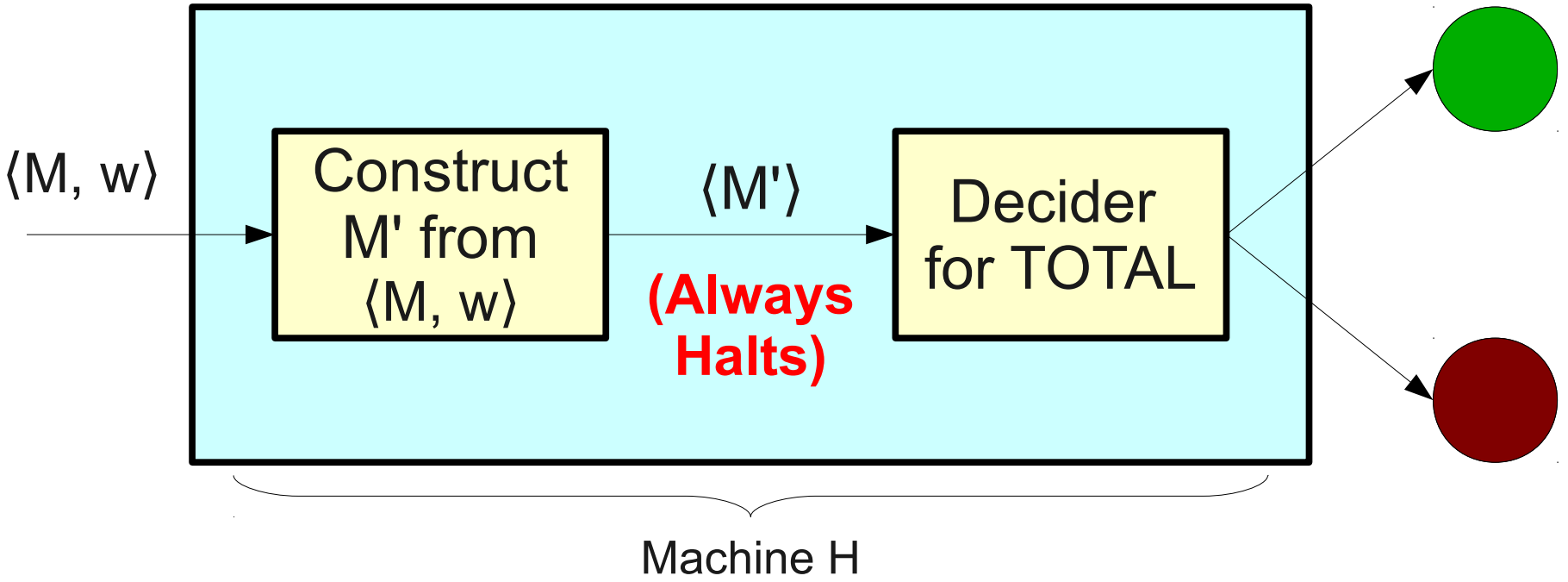

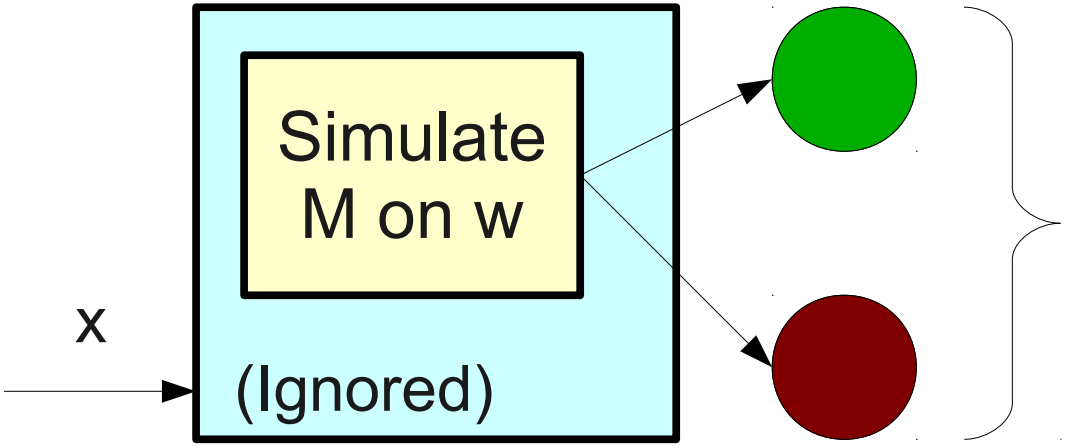

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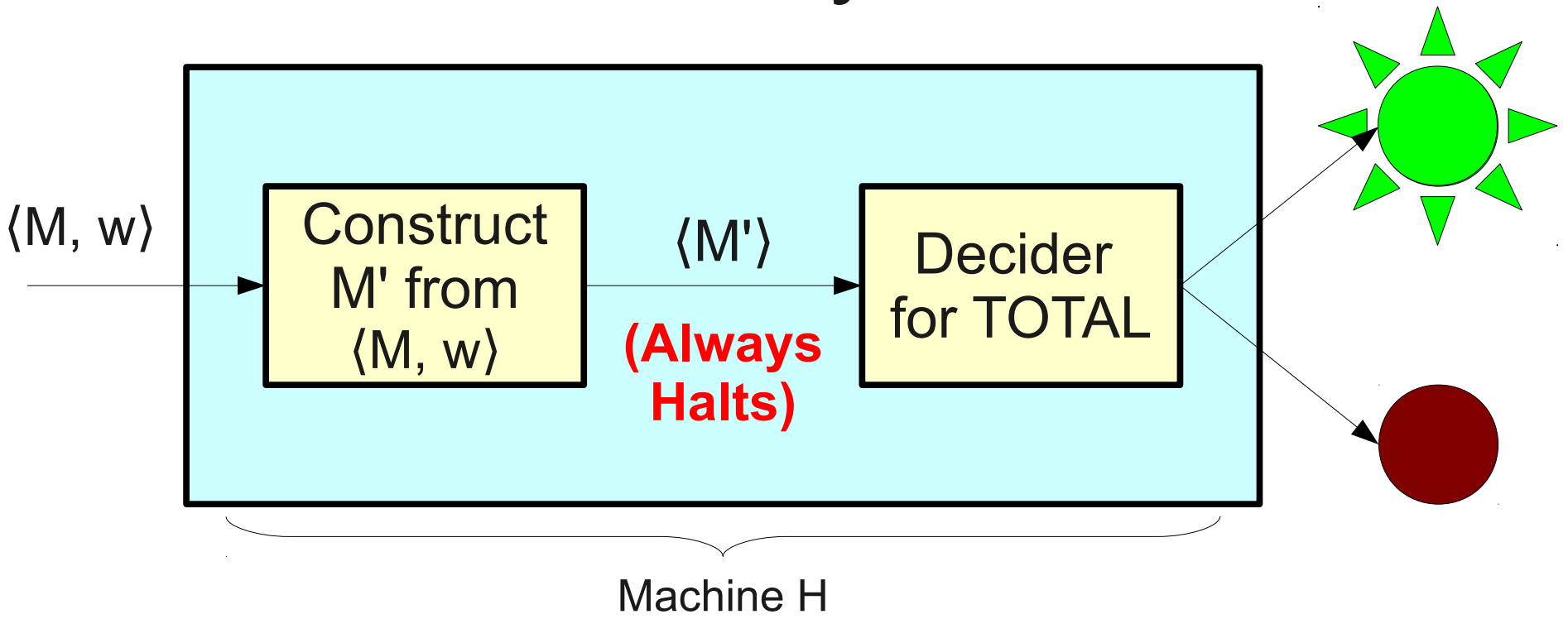

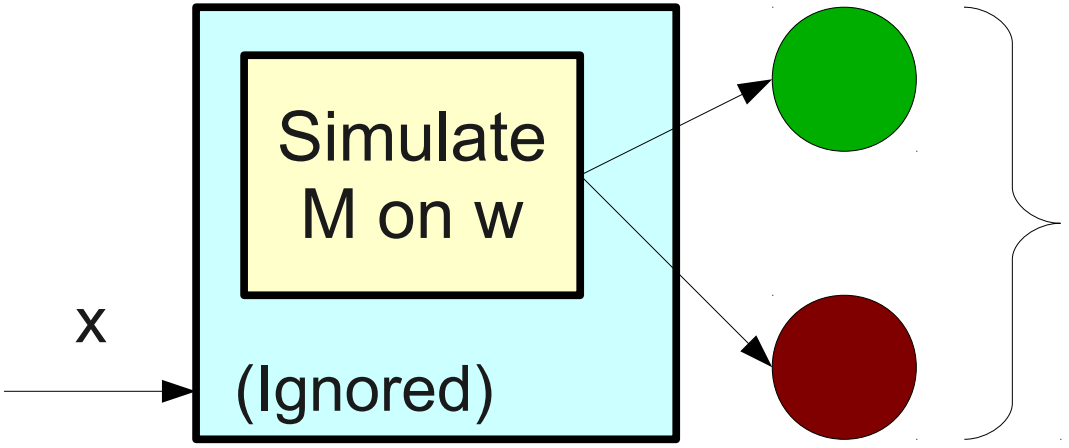

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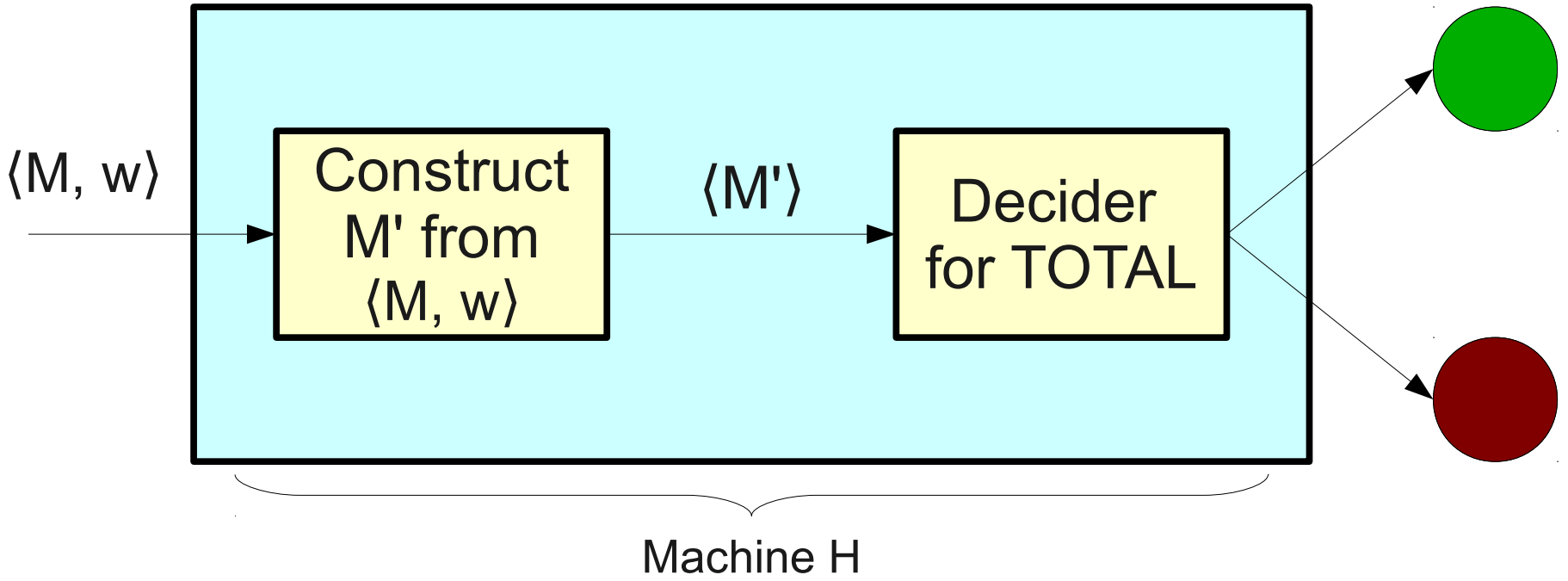

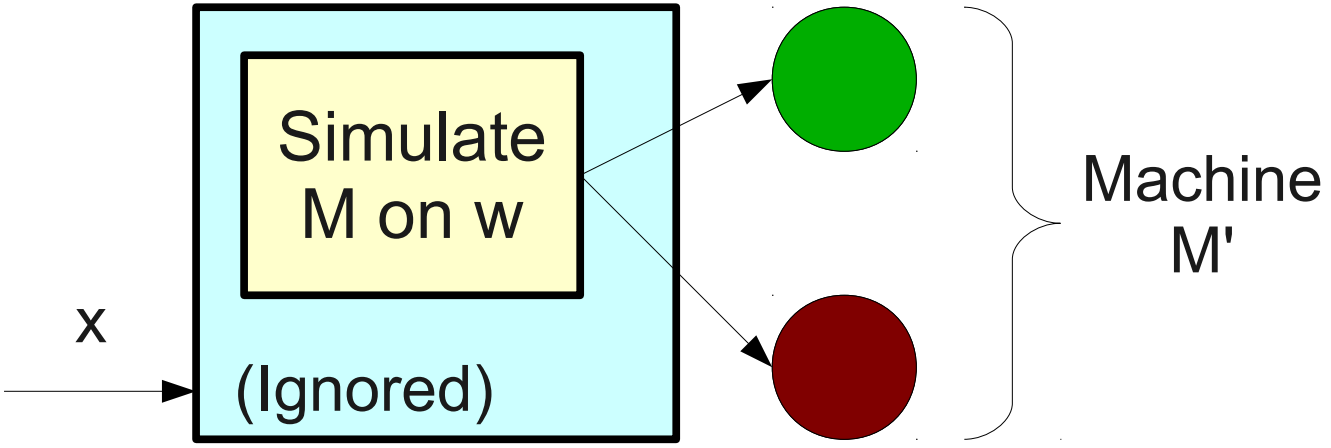

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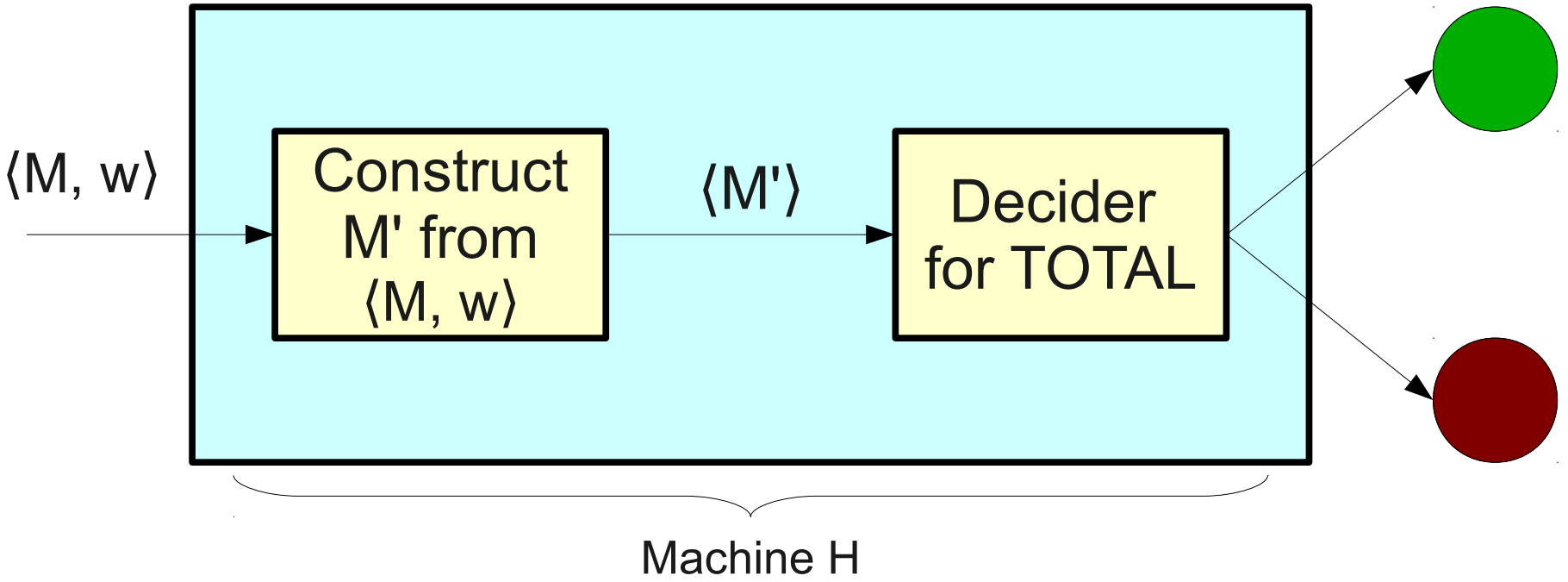

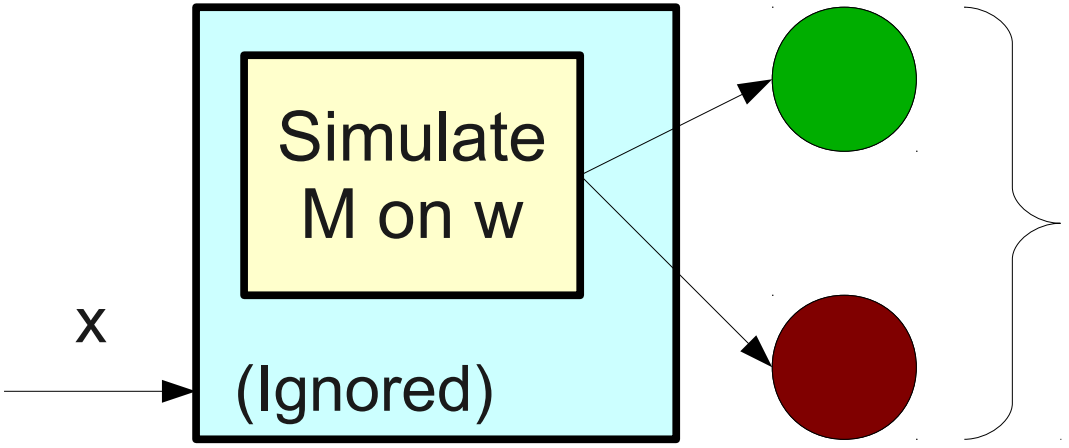

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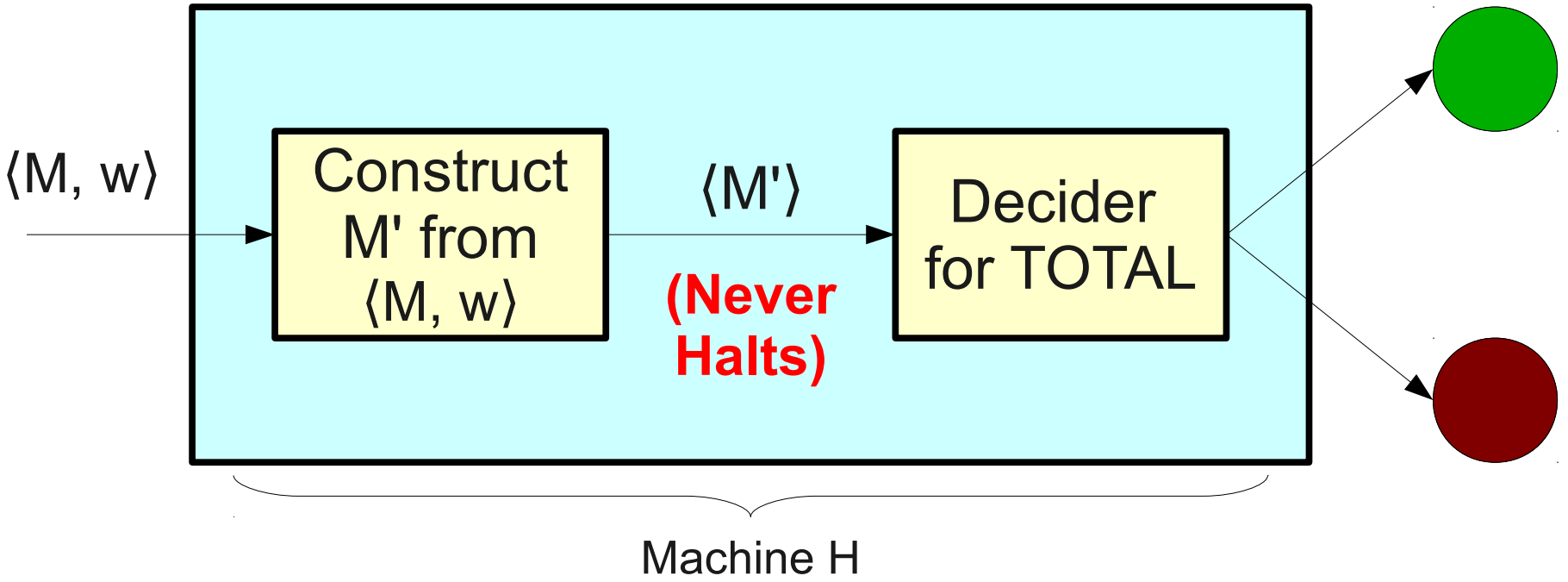

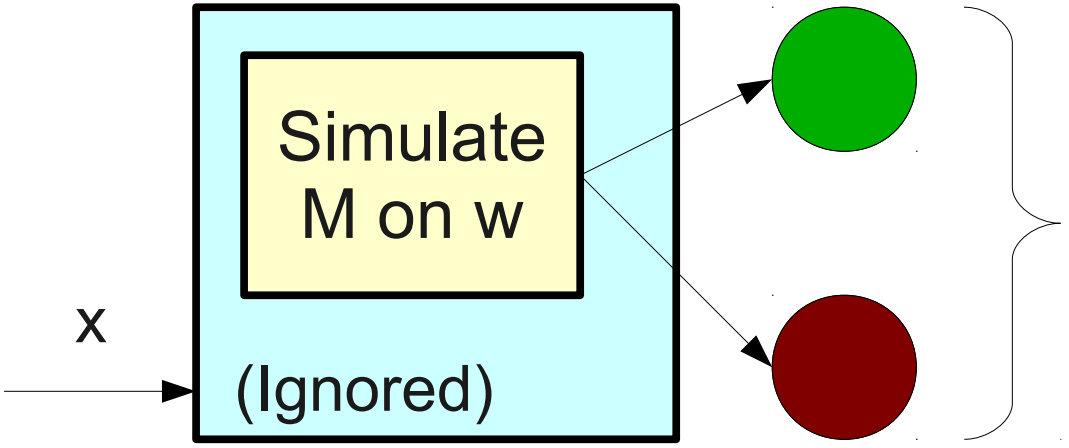

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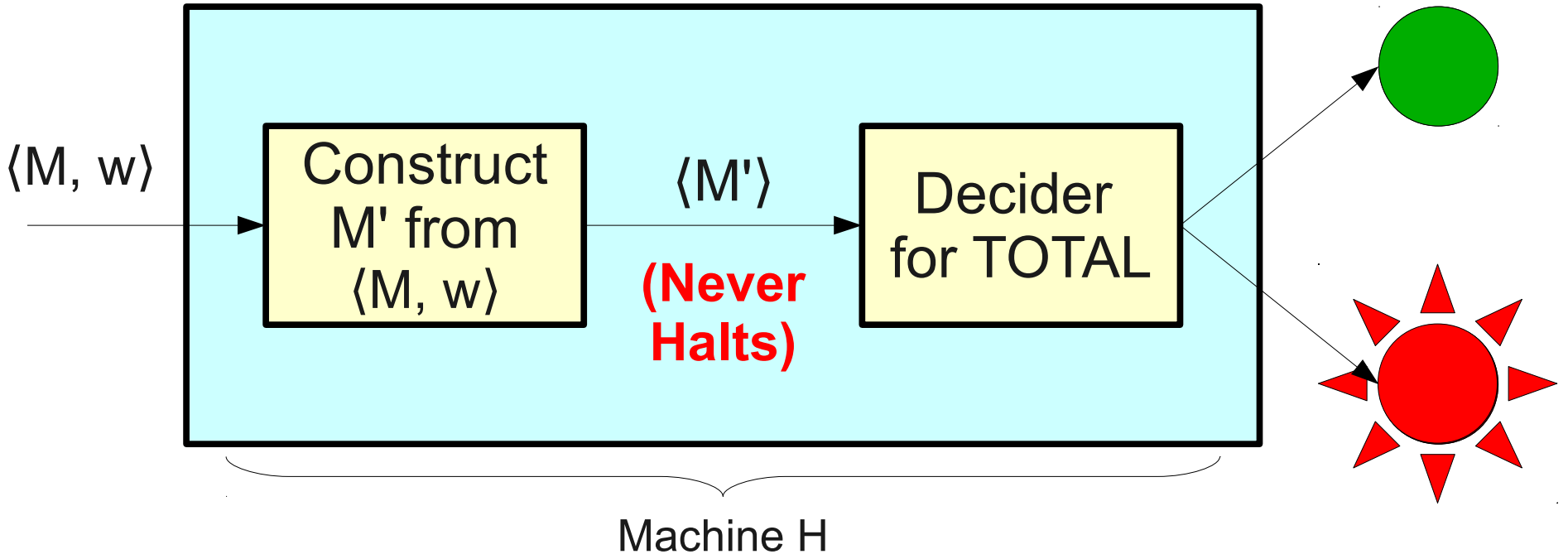

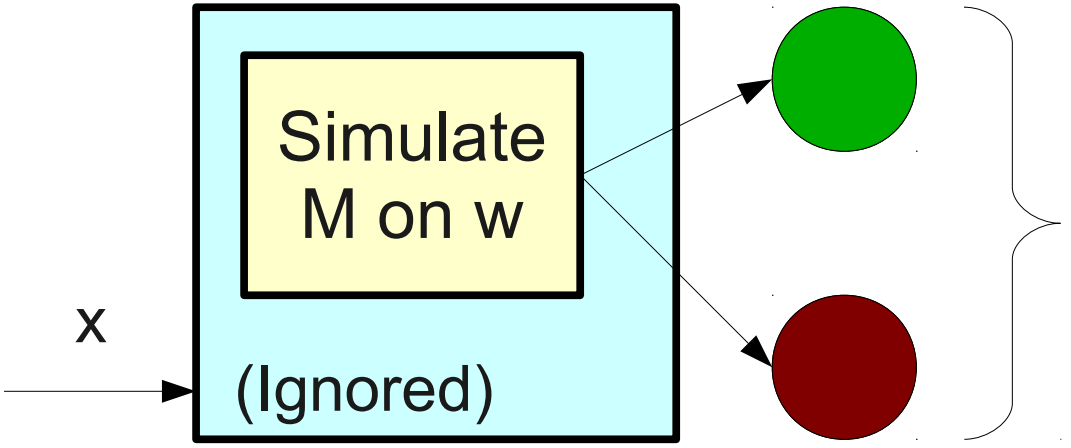

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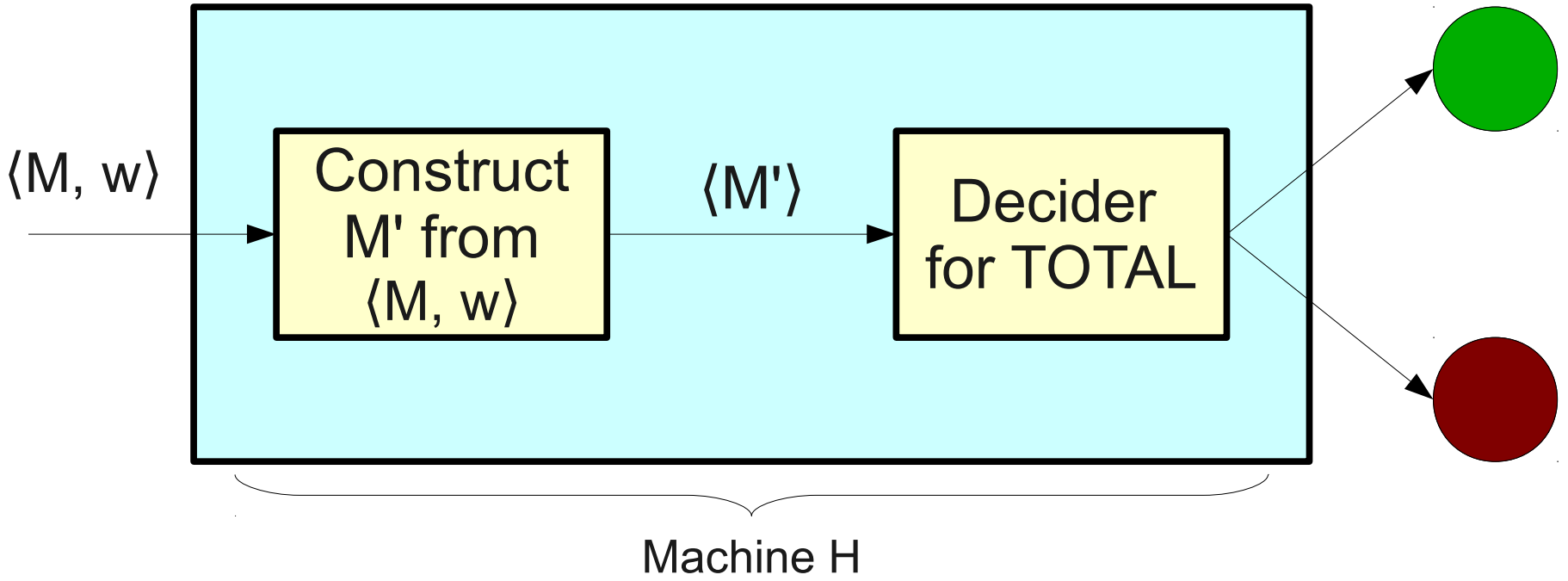

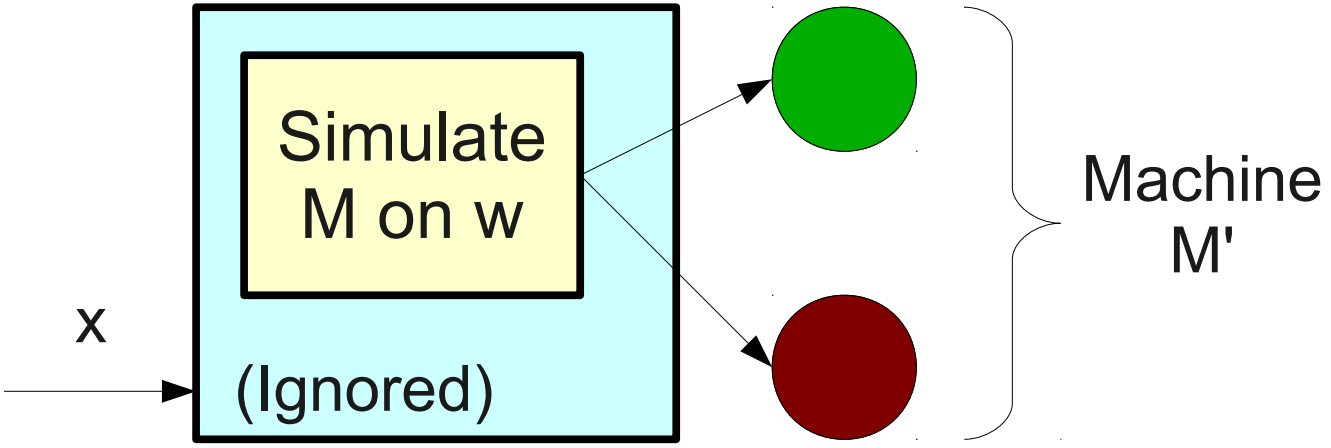

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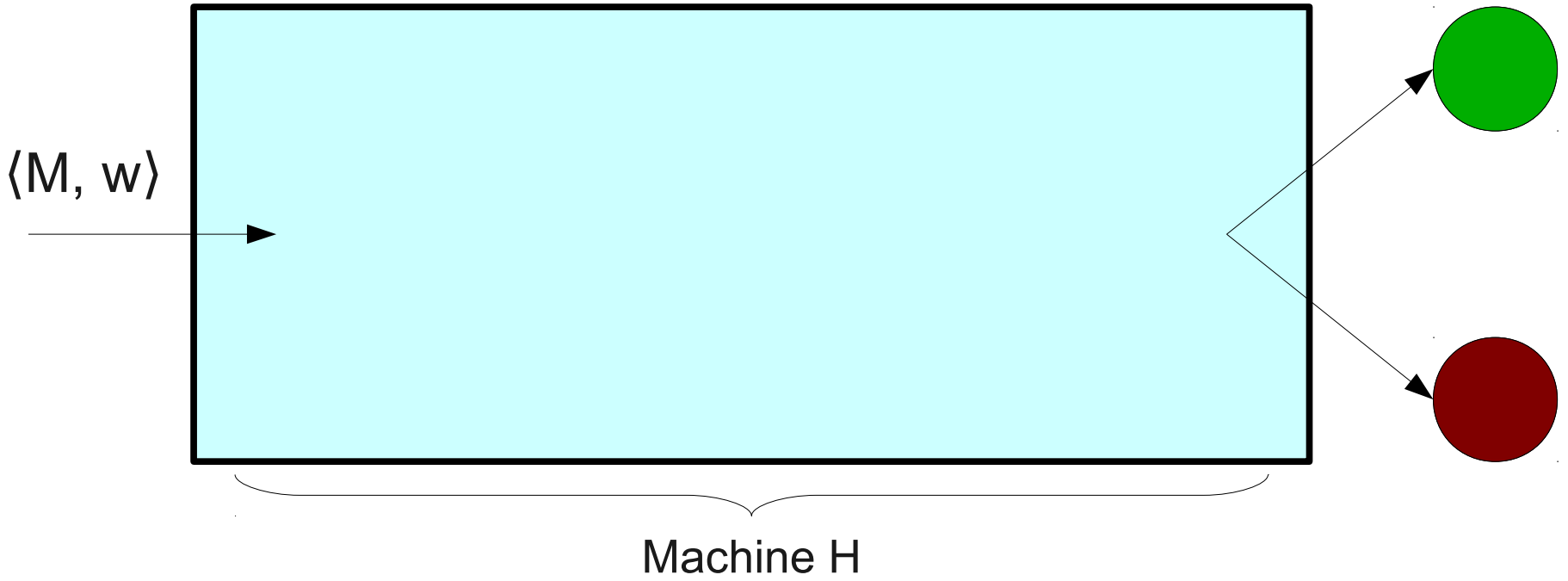

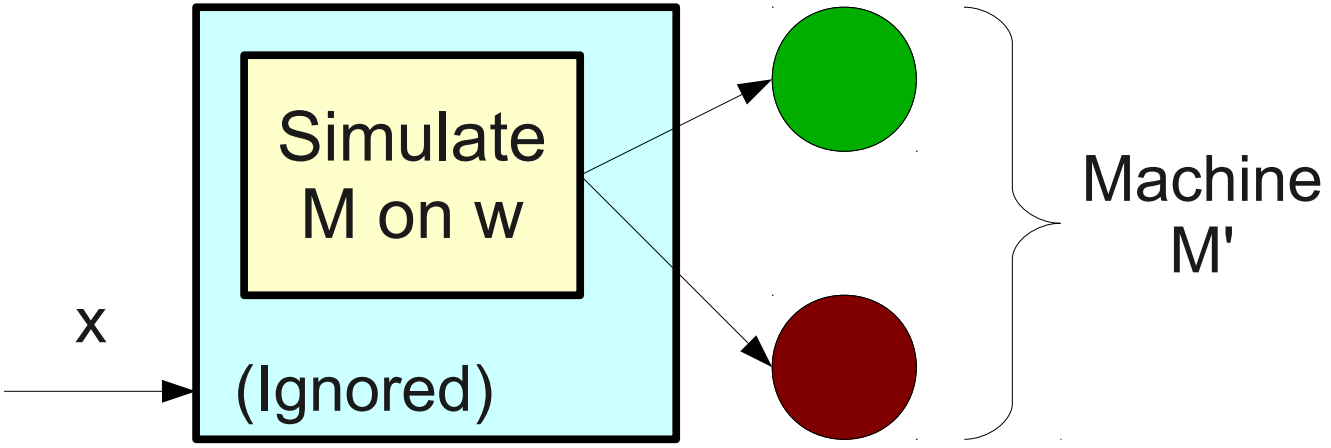

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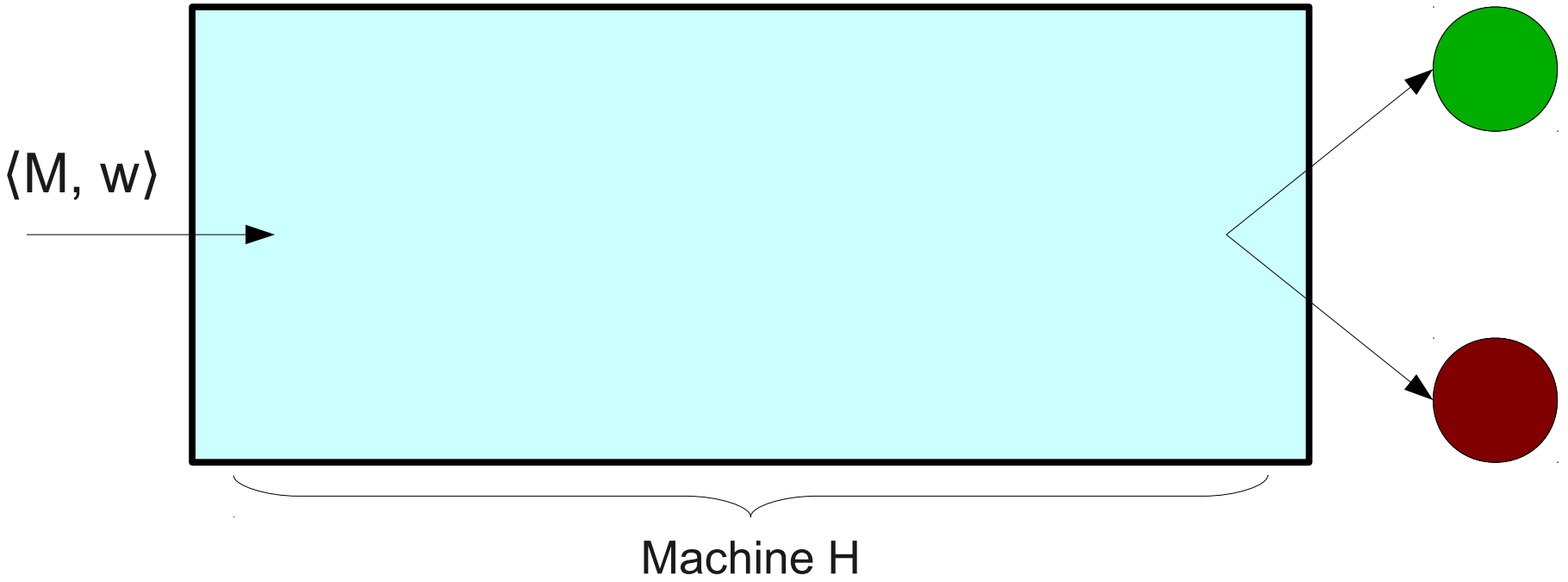

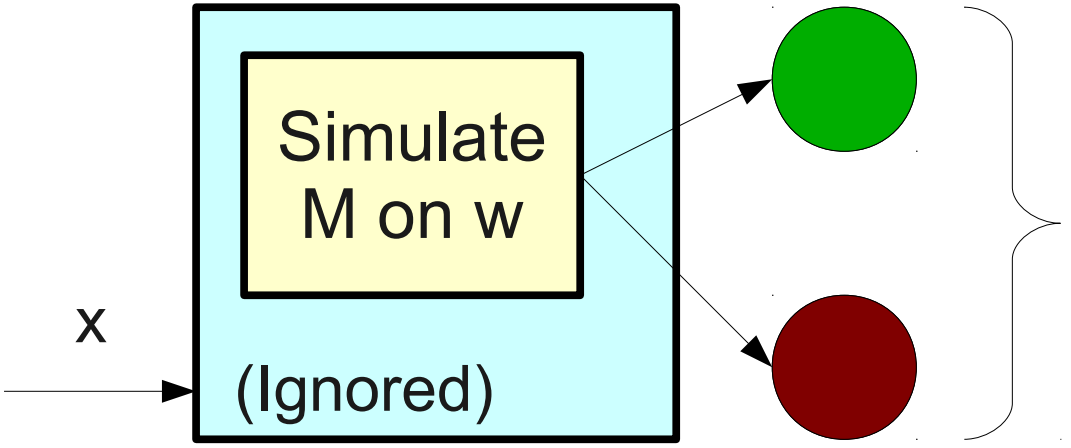

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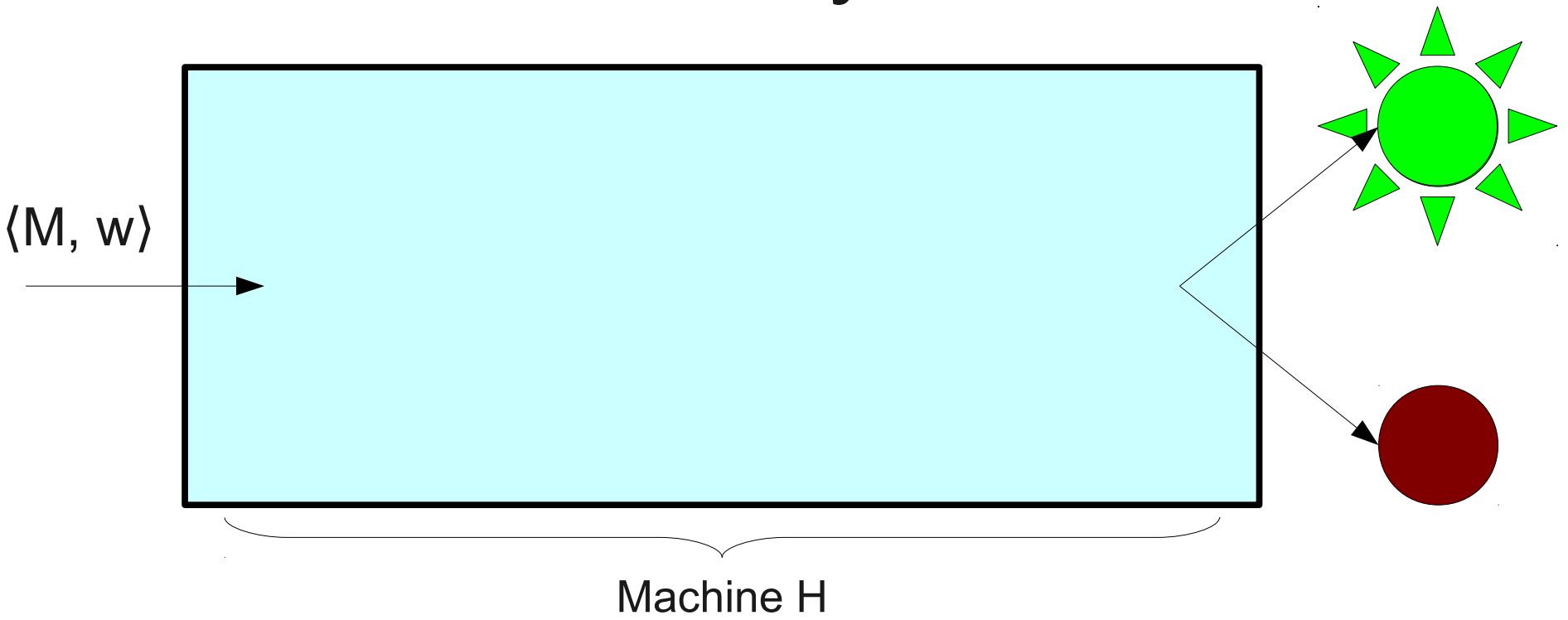

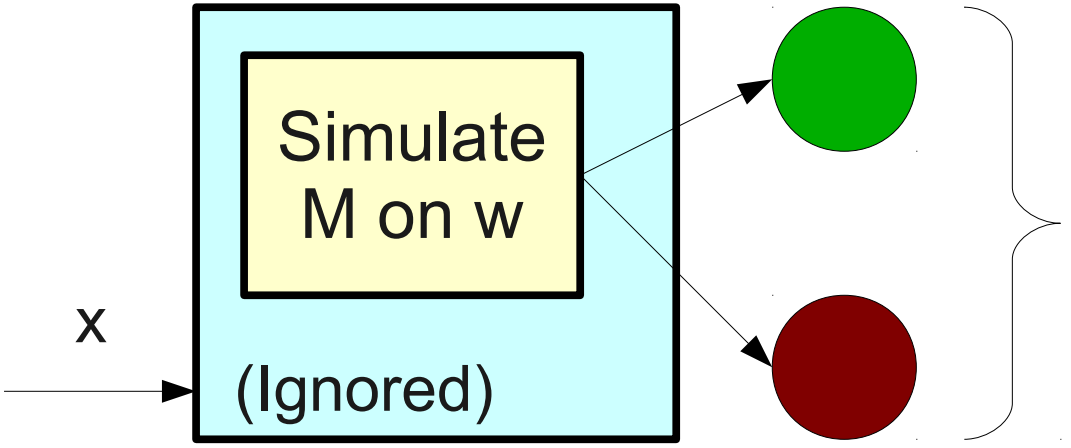

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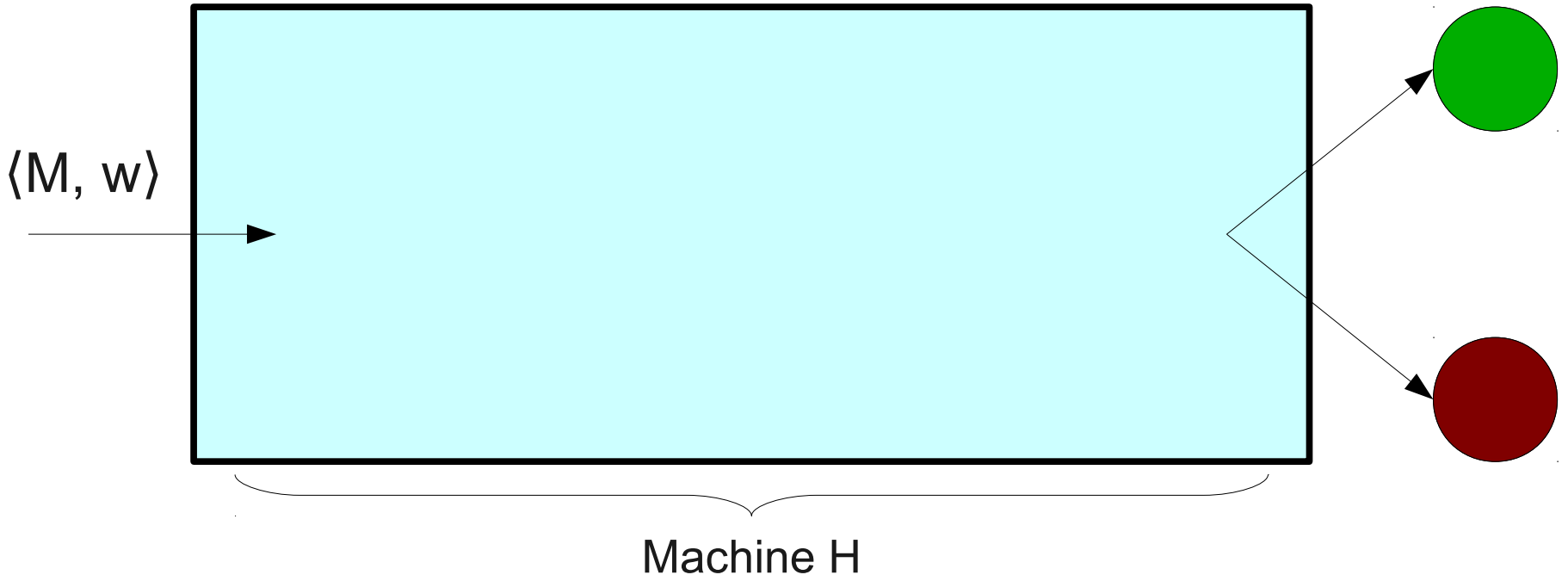

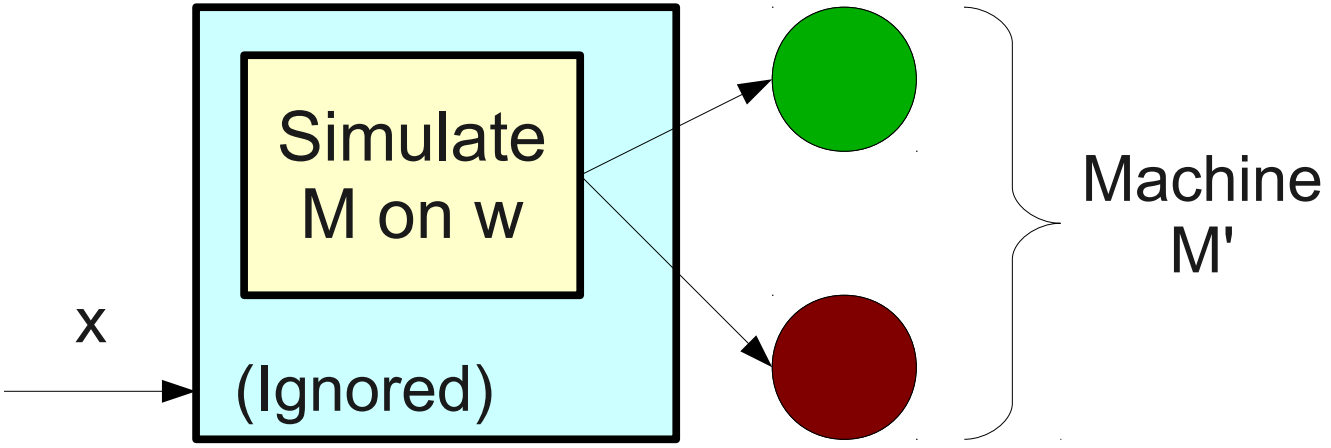

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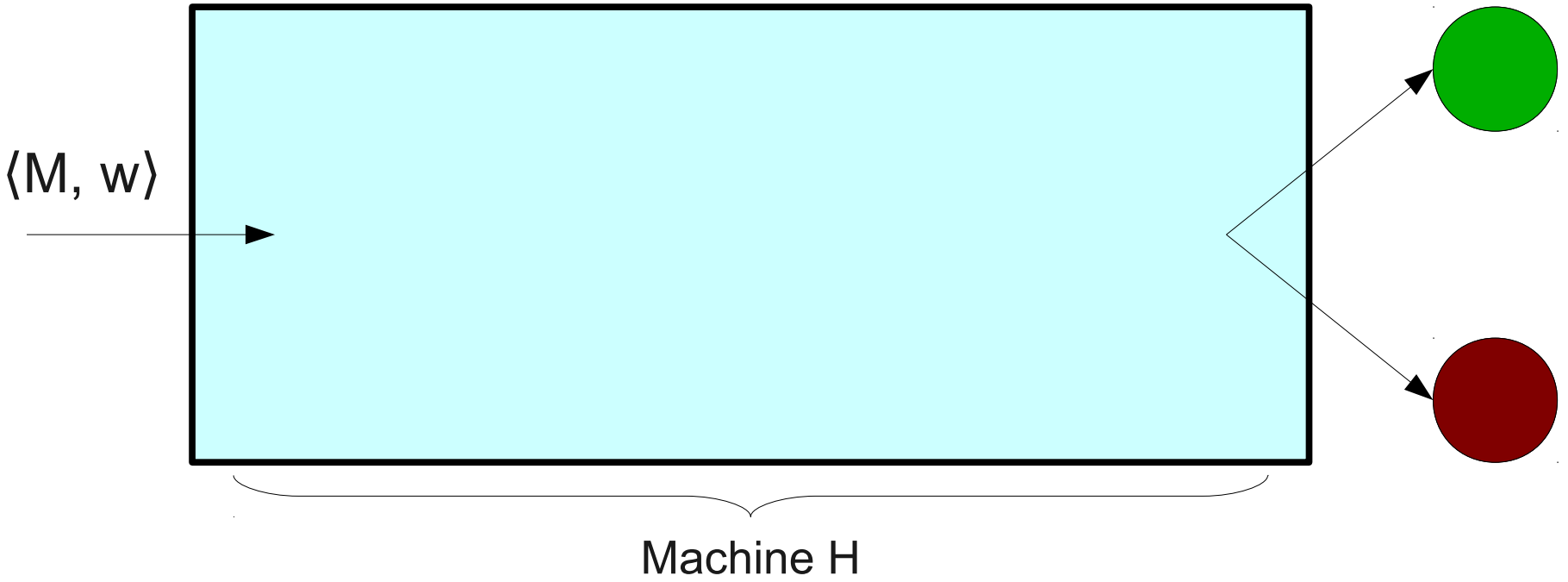

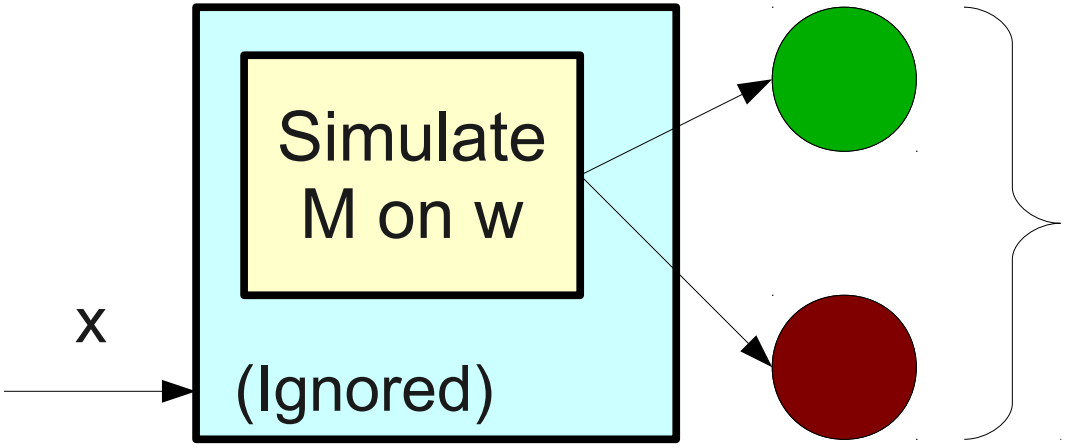

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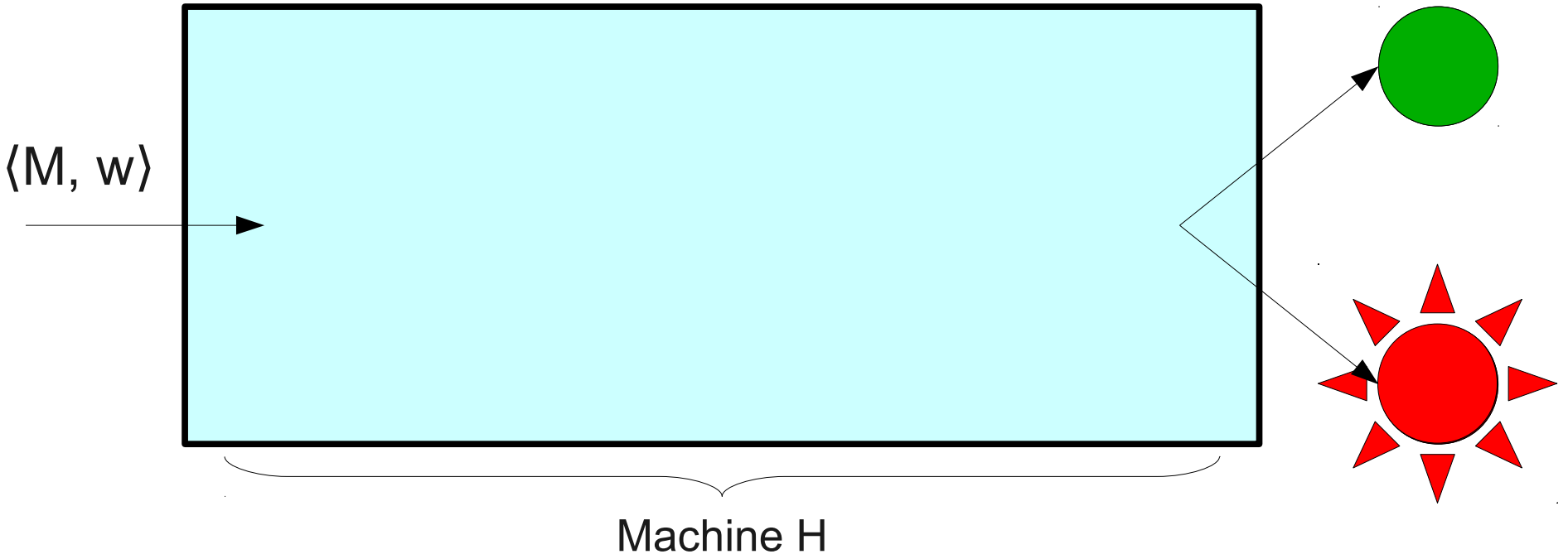

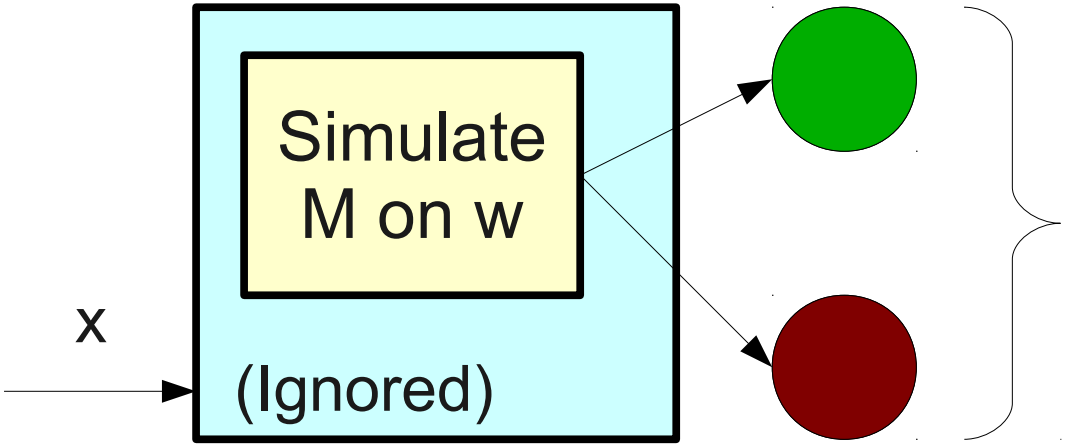

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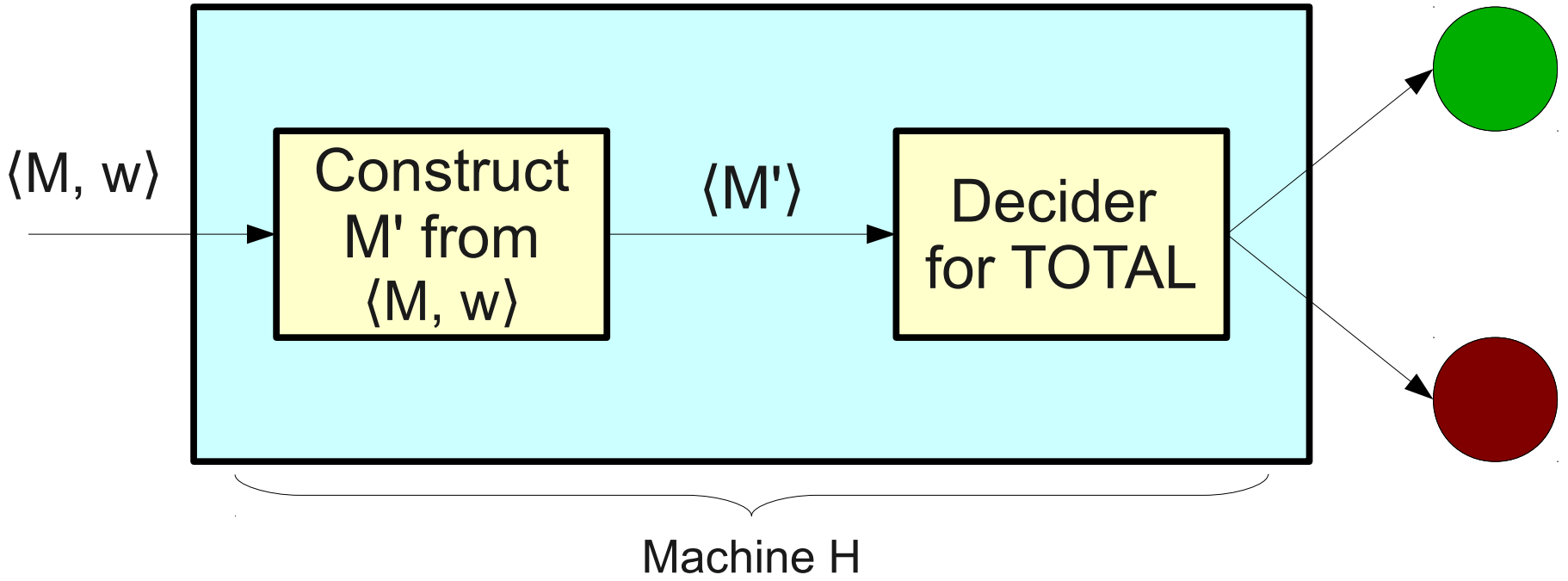

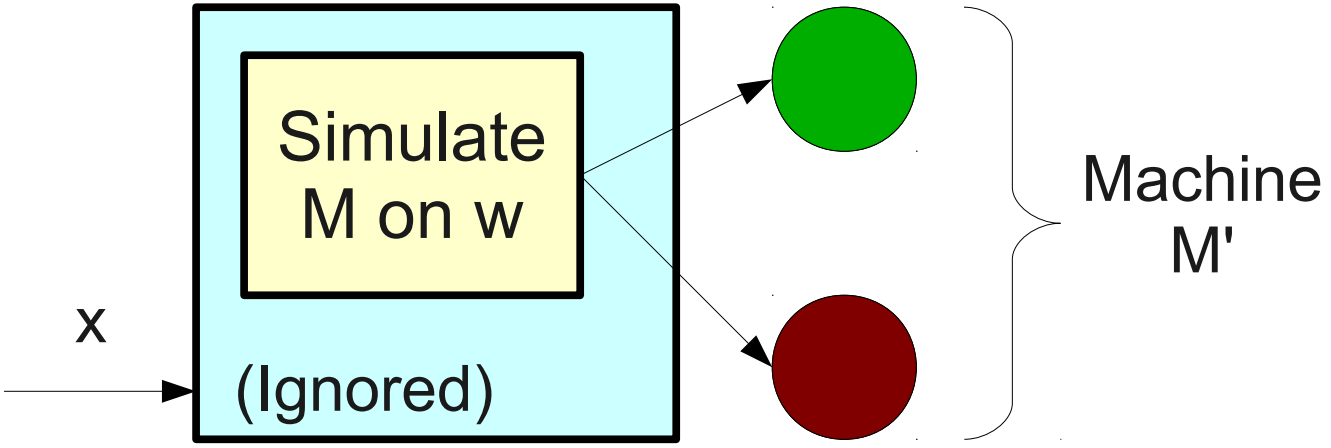

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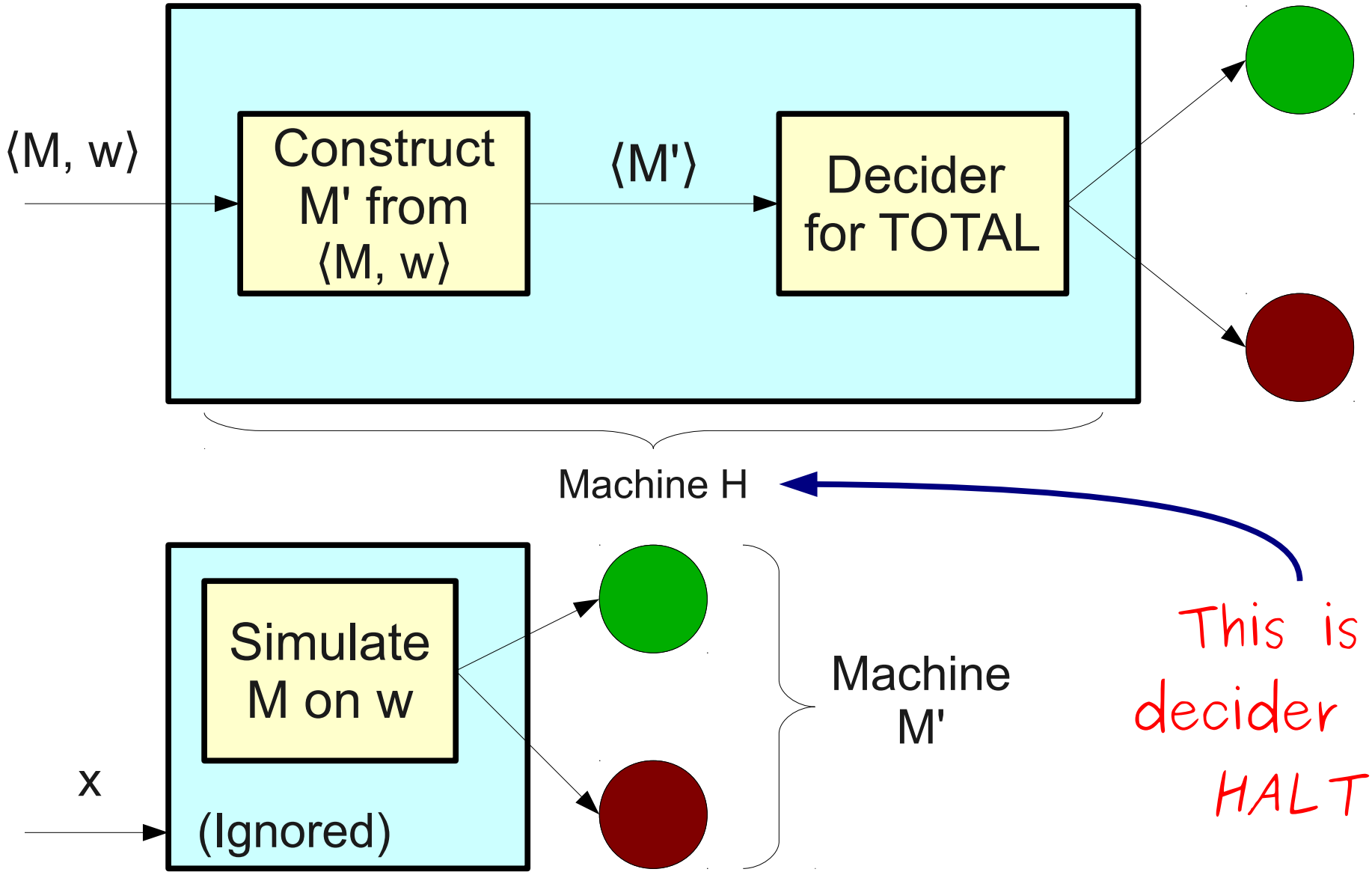

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This is the key step in most reductions. We build a TM that has a property of the new problem (here, *TOTAL*) based on whether some other TM has a property of the old problem (here, *HALT*)

Deciding whether this TM has the new property thus decides whether some other TM has the old property.

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  - Using the decider for *TOTAL*, H checks whether M' is a decider:
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  - Conclude that if *TOTAL* is decidable, then *HALT* is decidable.

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    - If M' is not a decider, then M does not halt on w.
  - Conclude that if *TOTAL* is decidable, then *HALT* is decidable.
  - But *HALT* is undecidable, so our assumption was wrong and *TOTAL* is undecidable.

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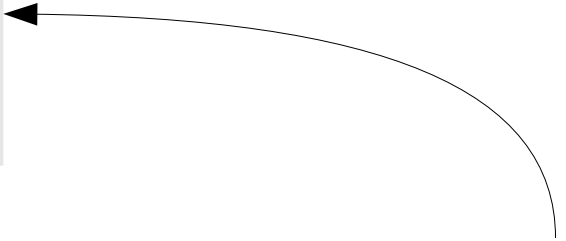

Most reduction proofs work by building a new TM out of an existing TM and a string. The new TM then has some property iff the old TM/string pair has some property.

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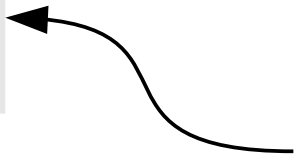

The behavior of H depends on what T does on ⟨M'⟩.

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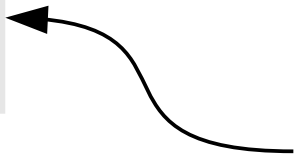

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To be completely formal in our proof, we should show that this construction can be done in finite time so that H does not loop infinitely trying to construct M'. However, conventionally this is just assumed to be true and you do not need to justify it.

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Testing Regularity

- We know that every regular language is RE.
- Thus some TMs must recognize regular languages.
- Could we detect whether a TM recognizes a regular language?

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$$REGULAR_{TM} = \{ \langle M \rangle \mid L(M) \text{ is regular.} \}$$

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- Is $REGULAR_{TM}$ decidable?
- Unfortunately, **no**, via a reduction from *HALT*.

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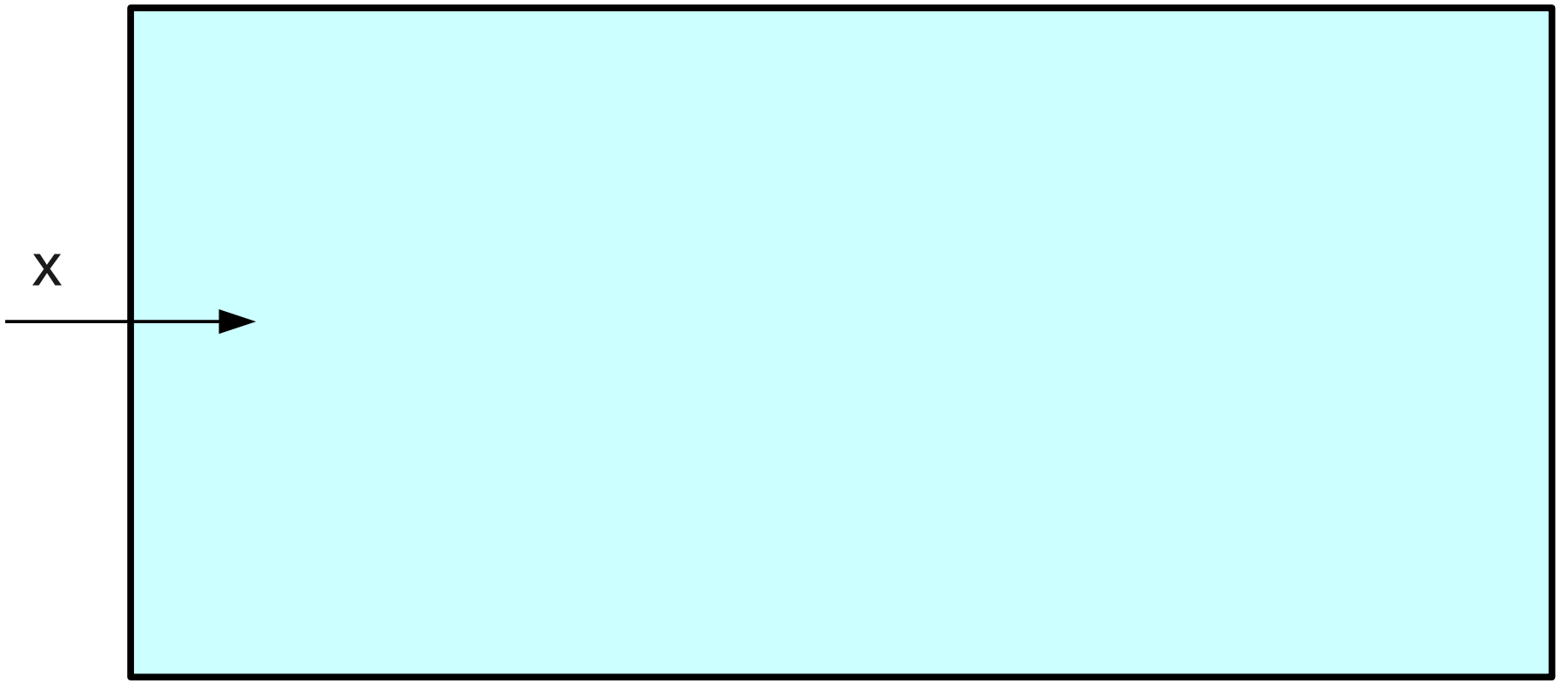

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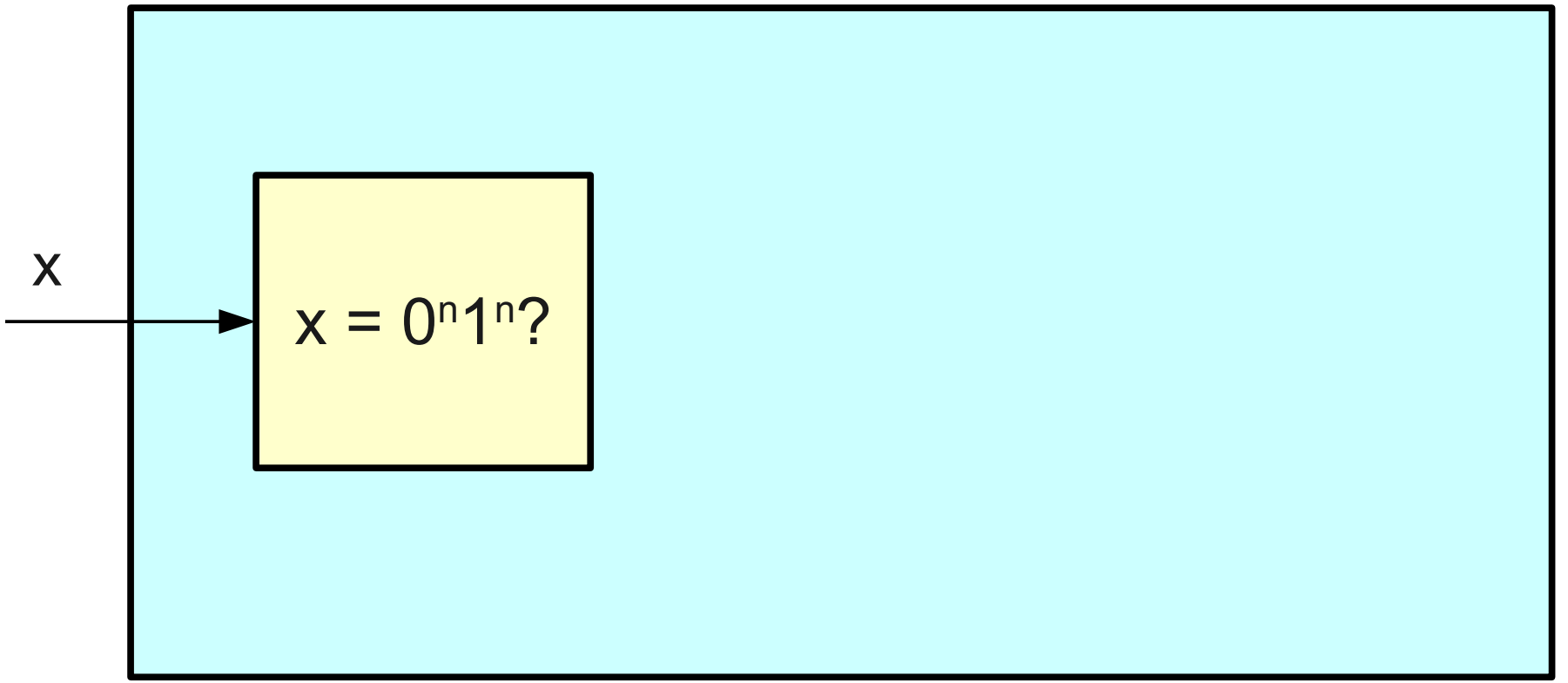

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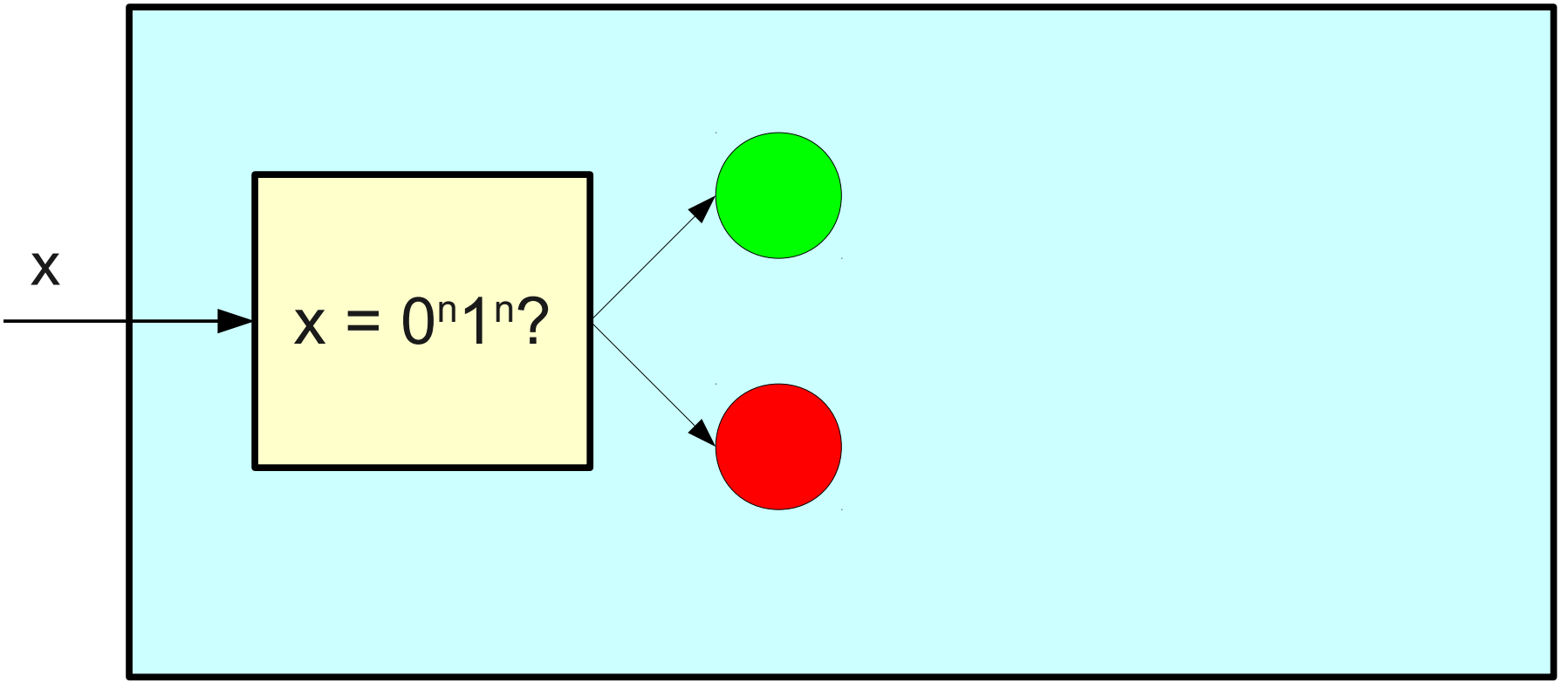

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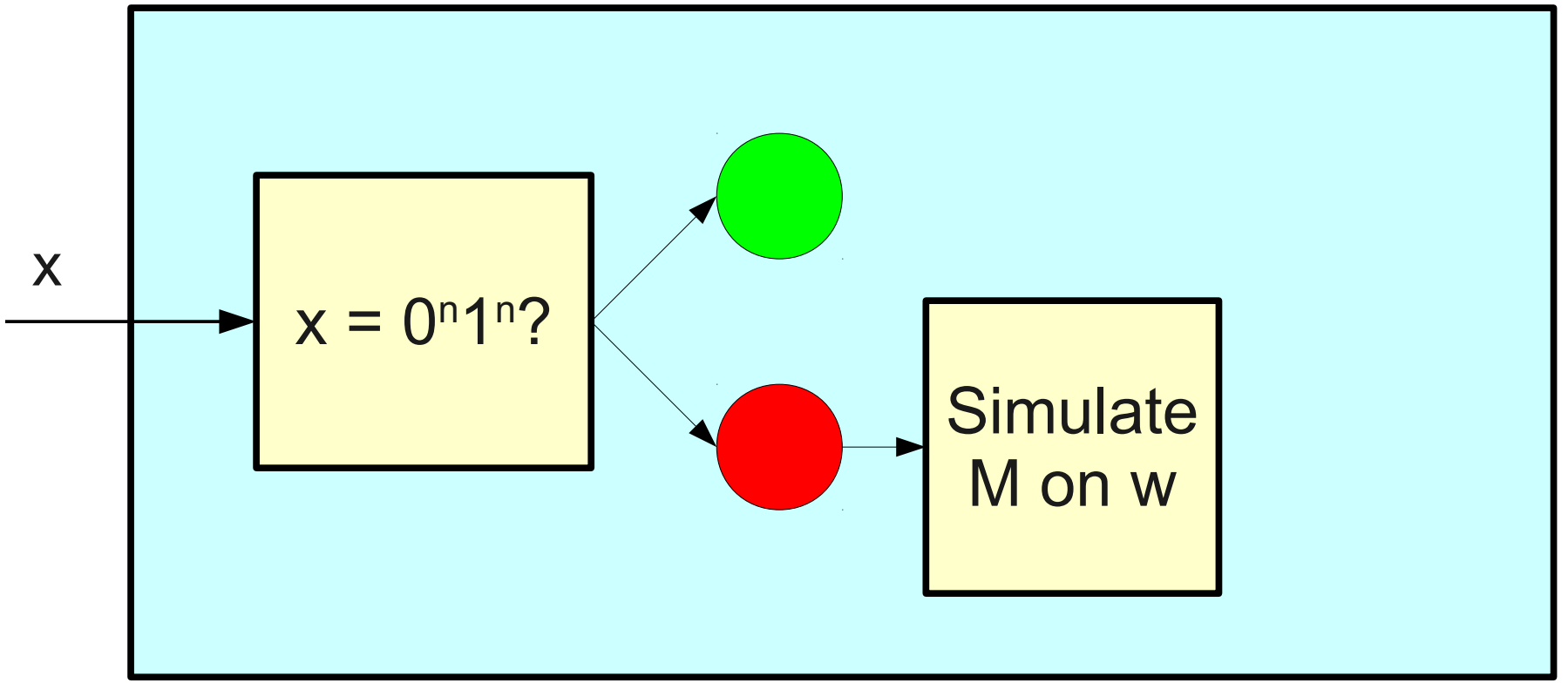

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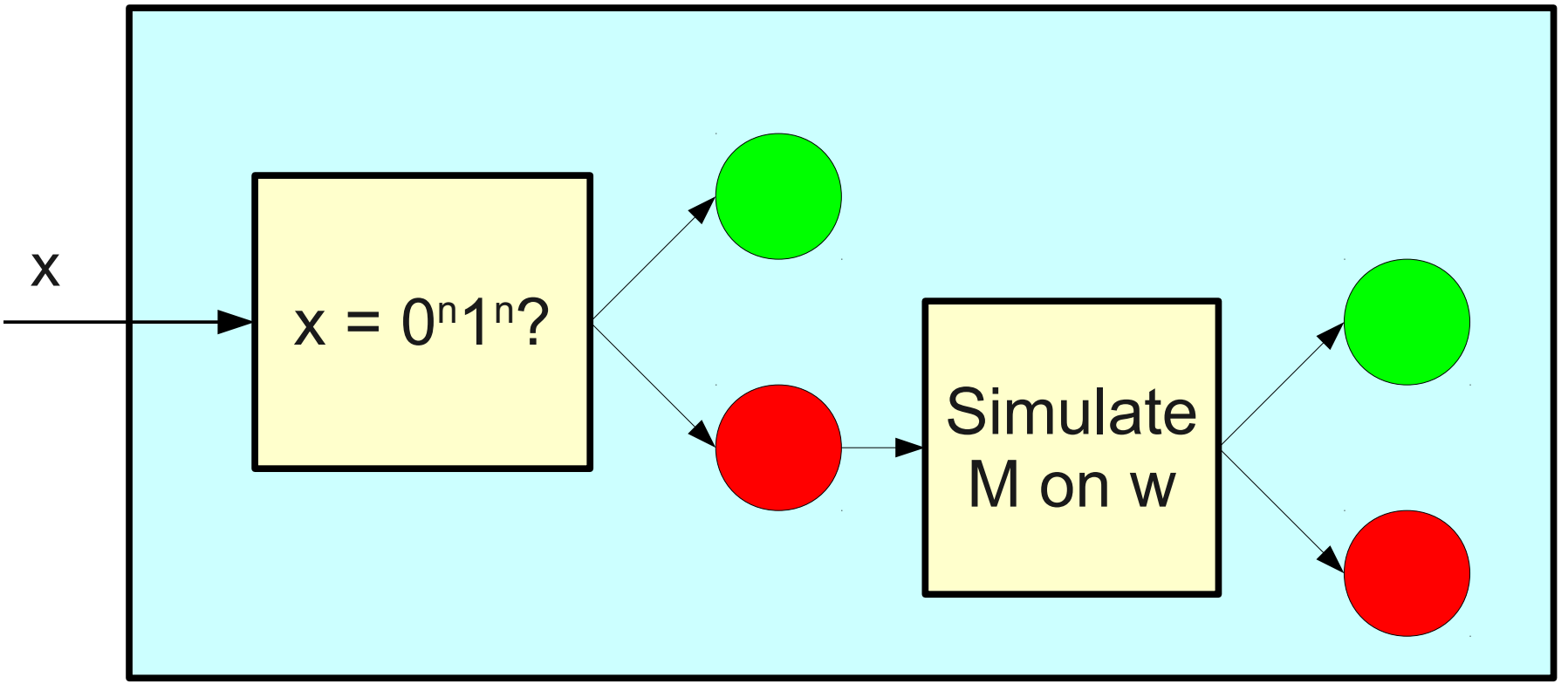

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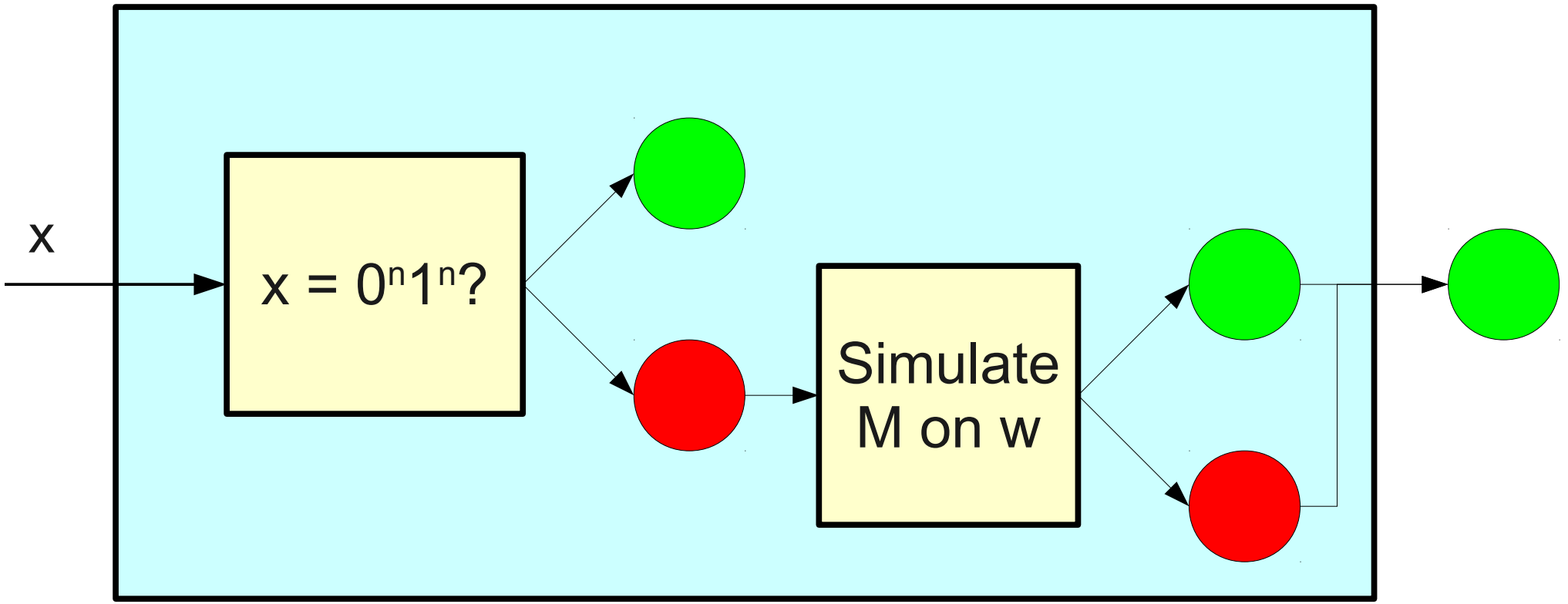

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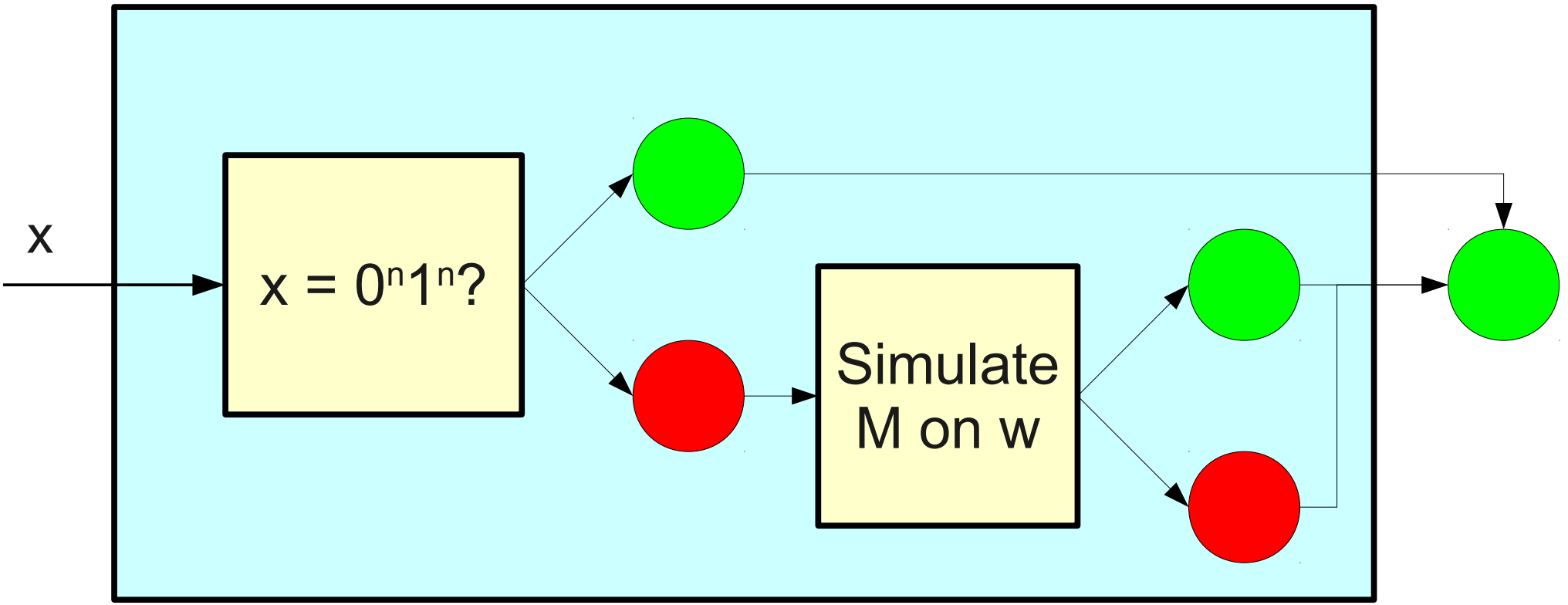

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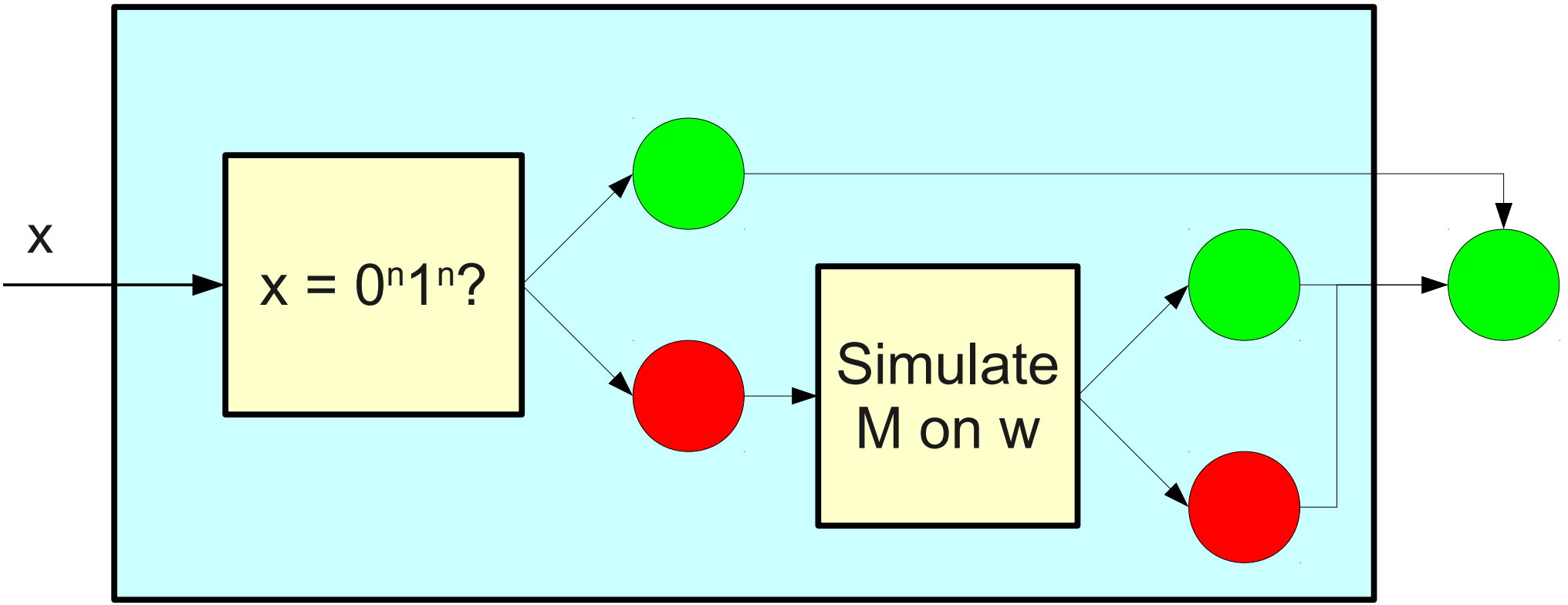

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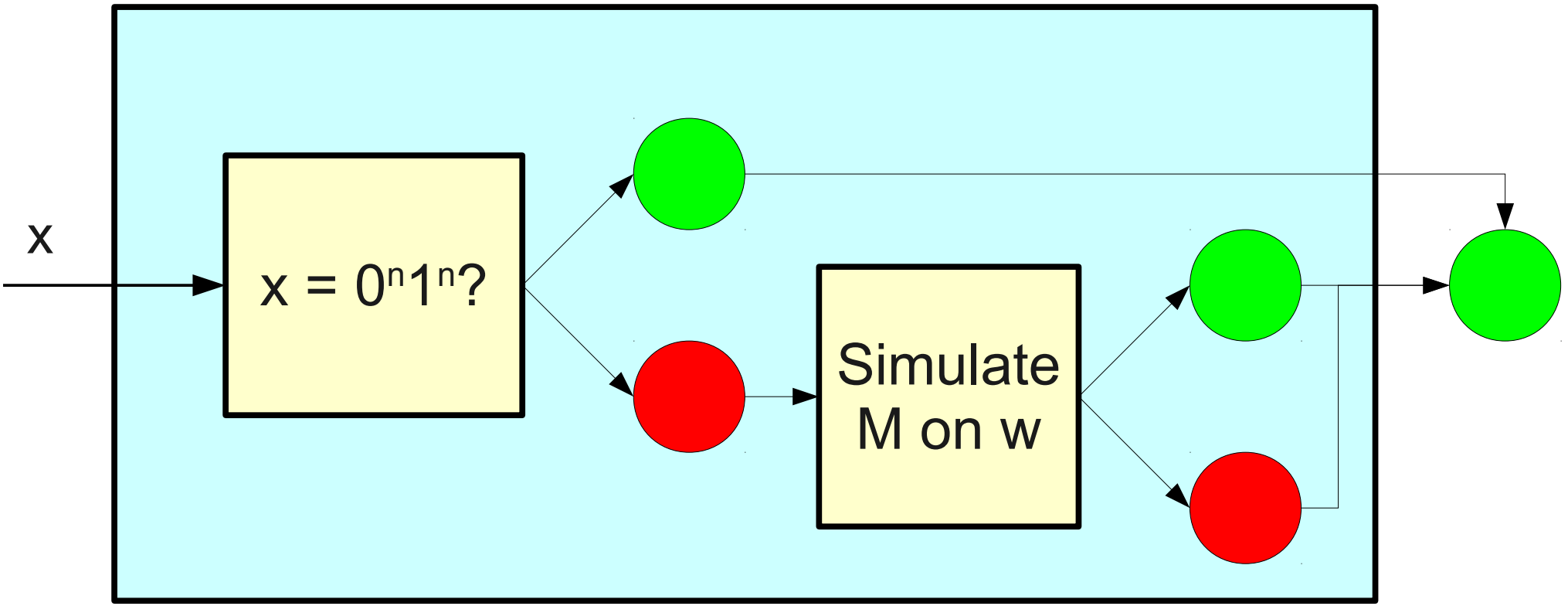

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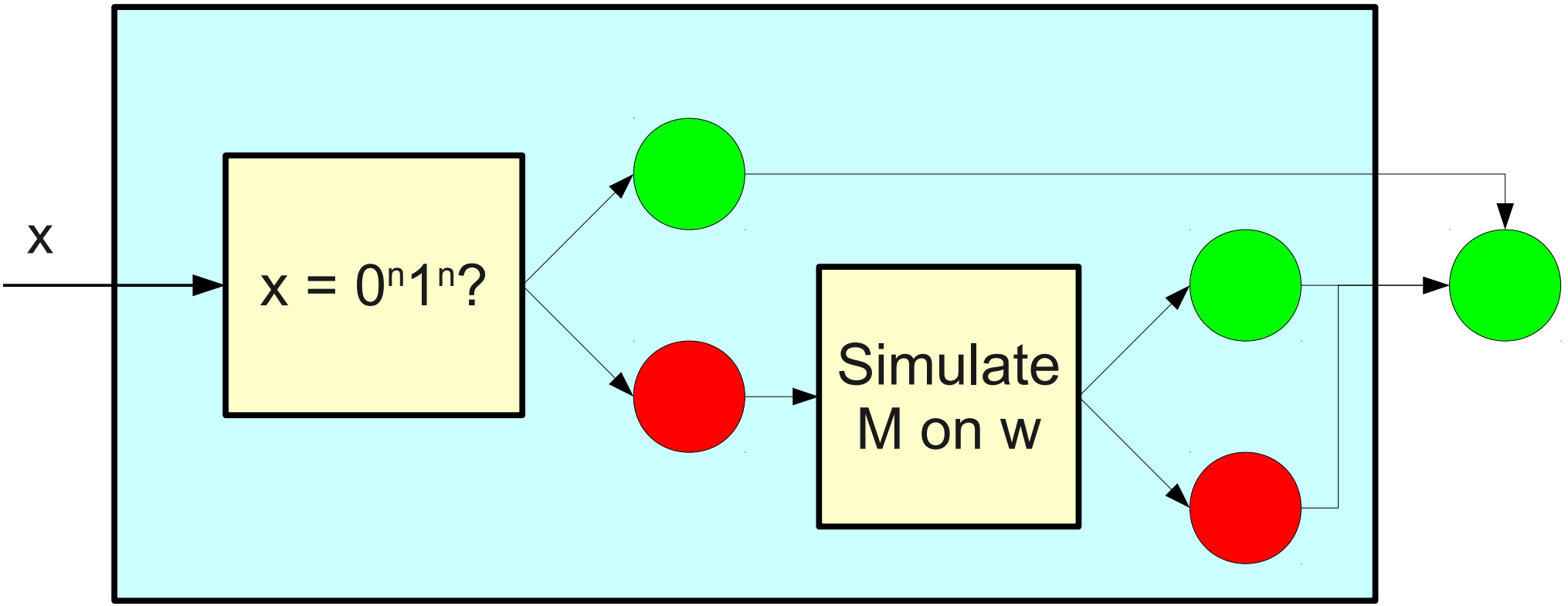

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- Consider the following problems:
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- There seems to be a trend here.
- What properties of Turing machines **are** decidable?

Rice's Theorem

Properties of RE Languages

- A **property of an RE language** is some trait that may apply to RE languages.
- For example:
  - Does L = Ø?
  - Is L regular?
  - Is L context-free?
  - Does L contain any string of length exactly 137?

Properties of RE Languages

- We can describe a property of an RE language as the set of RE languages with that property.
- If P is a property of RE languages, consider the language

$$L_P = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \in P \}$$

(The set of TMs that describe a language with property P.)

- Note that membership in L_P depends only on the **language** of a TM, not the **description** of that TM.

If $L(M_1) = L(M_2)$, then $\langle M_1 \rangle \in L_P$ iff $\langle M_2 \rangle \in L_P$

Properties of RE Languages

- The set

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  is a property of RE languages, because it depends **purely** on the language of the TM and not on the TM itself.
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  is **not** a property of RE languages, because it does not depend purely on the language of the TM.
  - Specifically: If $L(M_1) = L(M_2)$, then it may be possible for $\langle M_1 \rangle \in L_{evenQ}$ but $\langle M_2 \rangle \notin L_{evenQ}$.

Trivial and Nontrivial Properties

- A property of RE languages is called **trivial** if all RE languages have the property or no RE languages have the property.
  - { ⟨M⟩ | L(M) is RE } is trivial.
  - { ⟨M⟩ | L(M) is not RE } is trivial.
- A property of RE languages is called **nontrivial** if there exists TMs M_1 and M_2 such that $\langle M_1 \rangle \in L_P$ but $\langle M_2 \rangle \notin L_2$.
  - { ⟨M⟩ | L(M) is infinite } is nontrivial.
  - { ⟨M⟩ | L(M) is regular } is nontrivial.
  - { ⟨M⟩ | L(M) is recursive } is nontrivial.

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Any nontrivial property of the RE languages is **undecidable**.

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- Can we apply Rice's theorem to this language?

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- **Rice's theorem applies; L_{ne} is undecidable.**

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- **L_{small} is nontrivial**:

$$\exists M_1.\ \exists M_2.\ (\langle M_1 \rangle \in L_{small} \wedge \langle M_2 \rangle \notin L_{small})$$

- **L_{small} is a property of RE languages**:

$$\text{If } L(M_1) = L(M_2), \text{ then } \langle M_1 \rangle \in L_{small} \text{ iff } \langle M_2 \rangle \in L_{small}$$

Using Rice's Theorem

- Can we apply Rice's theorem to this language?

  L_{small} = { ⟨M⟩ | There is a five-state TM that accepts L(M) }

- We can apply Rice's theorem if two conditions hold:
- **L_{small} is nontrivial**:

  $$\exists M_1.\ \exists M_2.\ (\langle M_1 \rangle \in L_{small} \wedge \langle M_2 \rangle \notin L_{small})$$

- **L_{small} is a property of RE languages**:

  If $L(M_1) = L(M_2)$, then $\langle M_1 \rangle \in L_{small}$ iff $\langle M_2 \rangle \in L_{small}$

- **Rice's theorem applies; L_{small} is undecidable.**

Using Rice's Theorem

- All of the following problems are undecidable:
  - $L_{palindrome}$ = { ⟨M⟩ | every string in L(M) is a palindrome }
  - L_{allodd} = { ⟨M⟩ | every string in L(M) has odd length }
  - L_{CFL} = { ⟨M⟩ | L(M) is a context-free language }
  - L_{short} = { ⟨M⟩ | L(M) has no strings of length above 5 }
  - $L_{recursive}$ = { ⟨M⟩ | L(M) is recursive }
  - L_{empty} = { ⟨M⟩ | L(M) = Ø }

Proving Rice's Theorem

- The proof of Rice's theorem is generalization of the reductions we've seen so far.
- **General Idea**: If L_P is a nontrivial property of RE languages, show that we can reduce *HALT* to L_P.
- There are a few tricky details in the proof, so we'll take it slowly.

The General Proof Sketch

- Suppose that some nontrivial property of RE languages L_P is decidable.
- Construct a TM H that works as follows:

  On input ⟨M, w⟩:

  Construct TM M' such that ⟨M'⟩ ∈ L_P iff M halts on w.

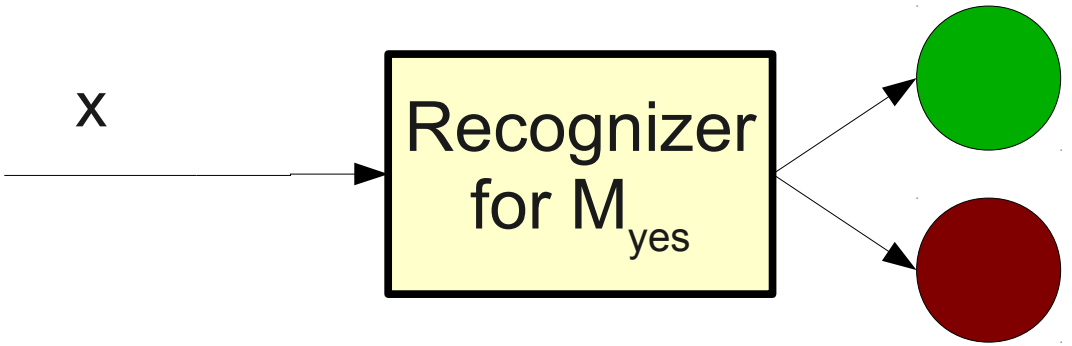
  Use the decider for L_P to decide whether ⟨M'⟩ ∈ L_P
- Conclude that H decides *HALT*, which is impossible.

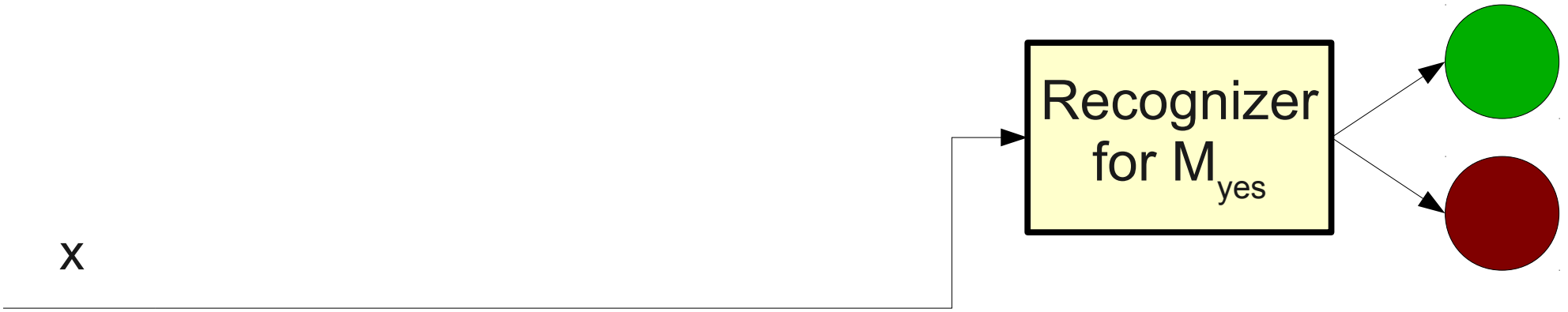
Key Idea One

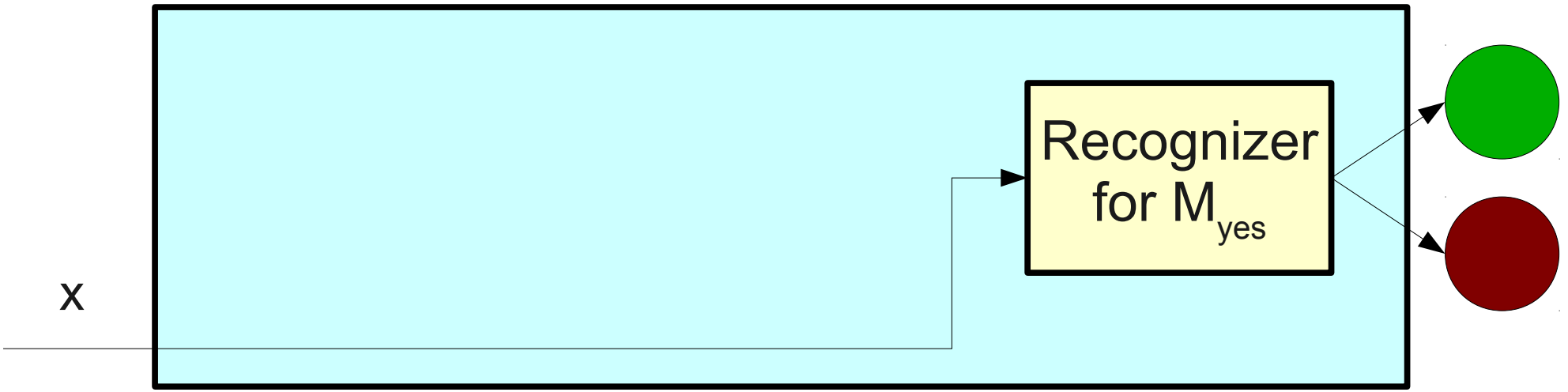
- Assume, for the sake of contradiction, that L_P is a decidable nontrivial property of RE languages.
- Since R is closed under complement, if we can show that either L_P is undecidable or that $\overline{L}_P$ is undecidable, we will have reached a contradiction.
- If $M_\varnothing$ is a TM that accepts the empty language, then either $\langle M_\varnothing \rangle \in L_P$ or $\langle M_\varnothing \rangle \notin L_P$.
  - If $\langle M_\varnothing \rangle \notin L_P$, we prove that L_P is undecidable.
  - If $\langle M_\varnothing \rangle \in L_P$, we prove that $\overline{L}_P$ is undecidable.

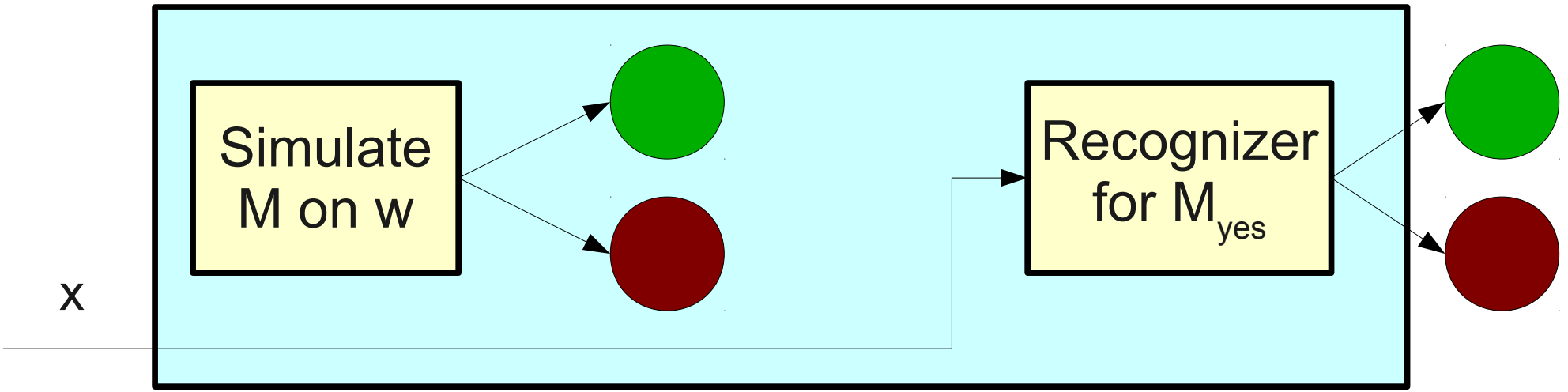
Key Idea Two

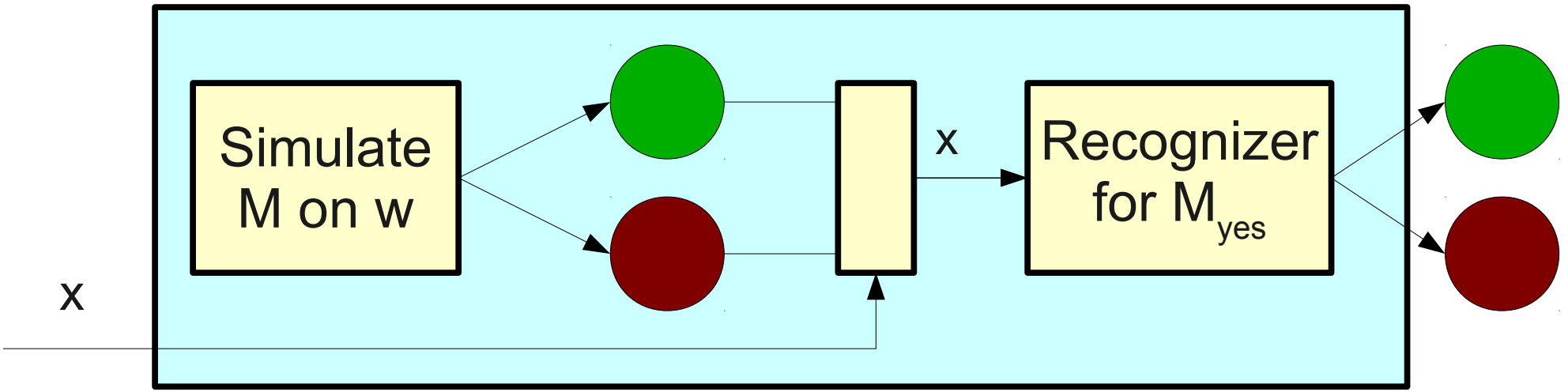
- Assume without loss of generality that $\langle M_\varnothing \rangle \notin L_P$.
- L_P is a *nontrivial property of RE languages*, so there must be some TM M_{yes} for which $\langle M_{yes} \rangle \in L_P$.
- L_P is a *nontrivial property of RE languages*, so $L(M_{yes}) \neq L(M_\varnothing)$.

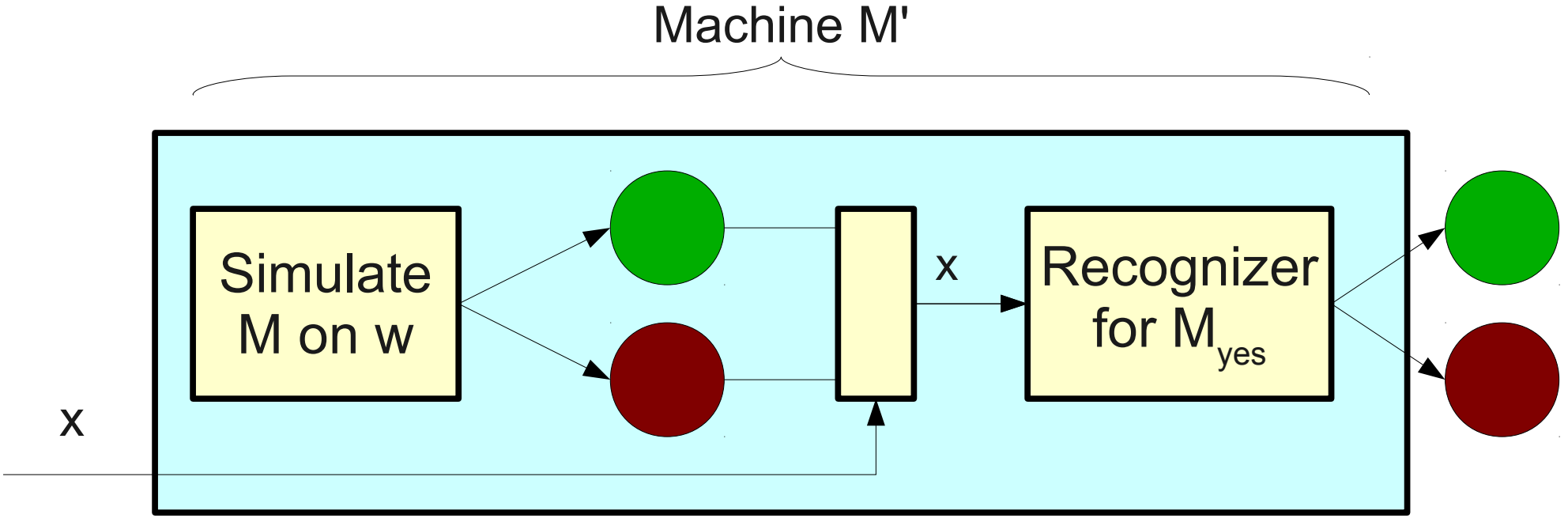

x
Recognizer
for M
yes

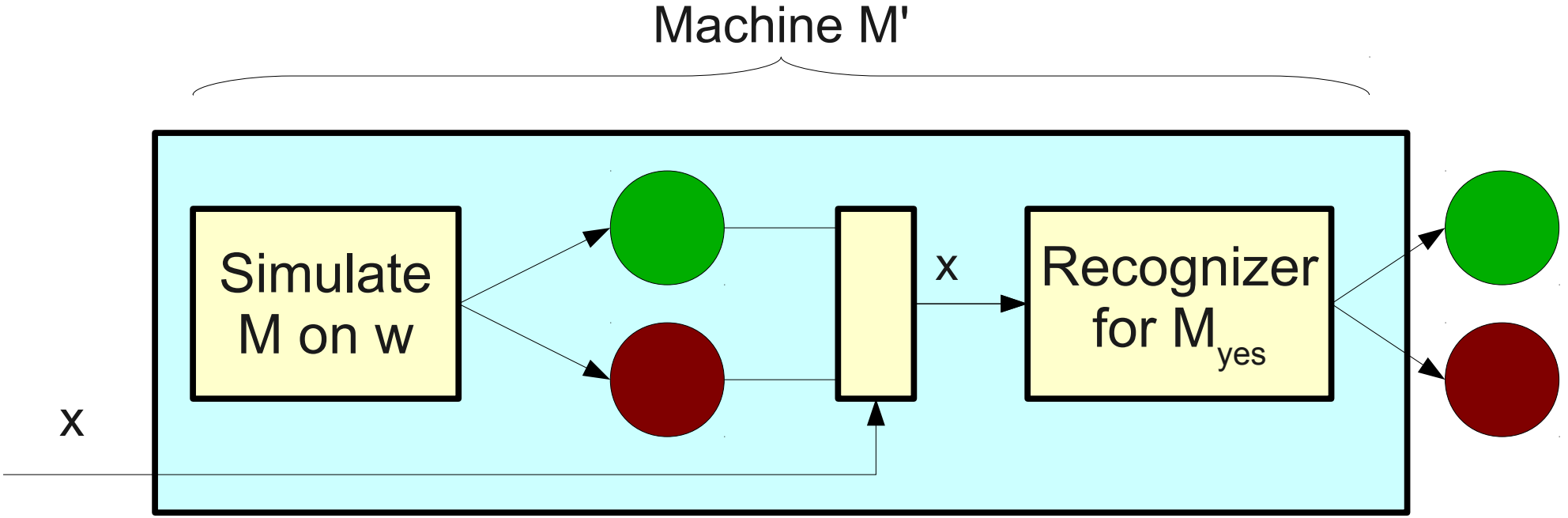

Recognizer
for M_{yes}
x


Recognizer
for M yes
x


Simulate
M on w
Recognizer
for M yes
x


Simulate
M on w
x
Recognizer
for M yes
x


Machine M'
Simulate
M on w
x
Recognizer
for M yes
x



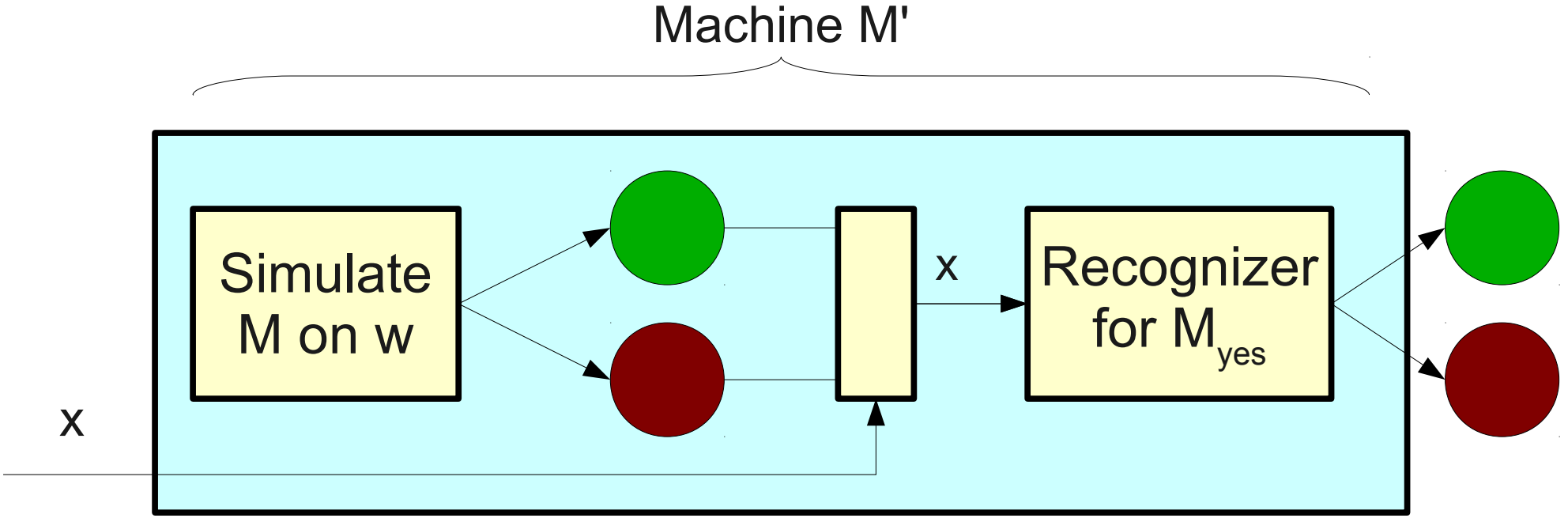
M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”



M' = “On input x:

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If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

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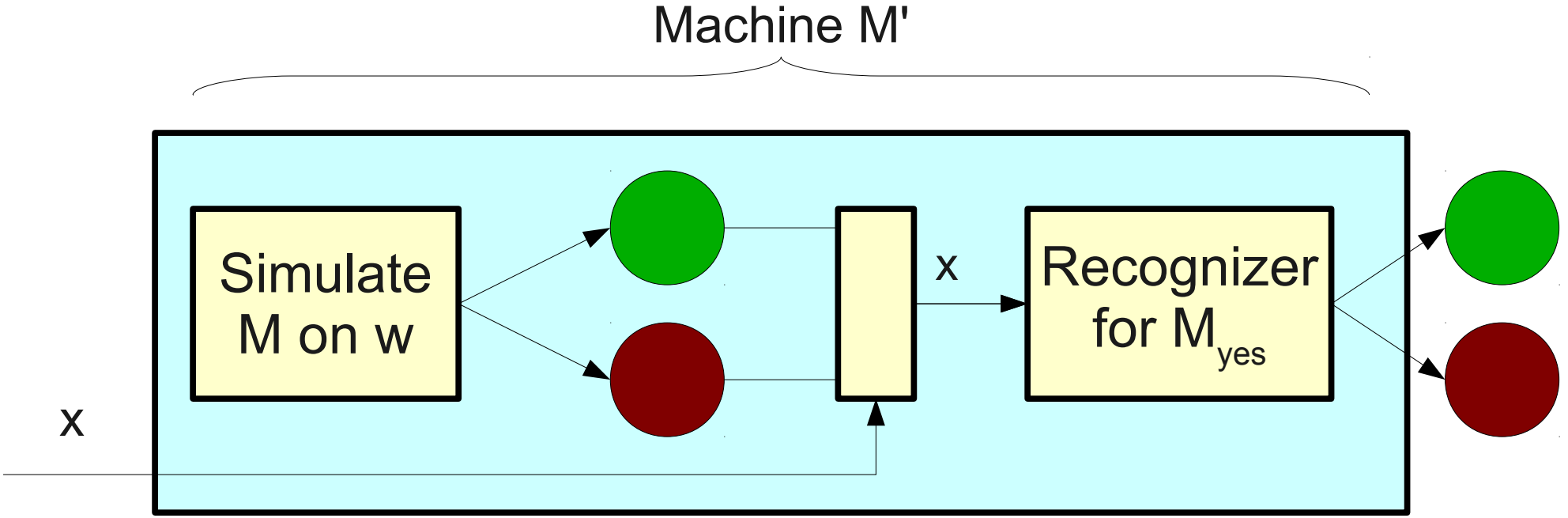
If M loops on w, what does M' do on x?



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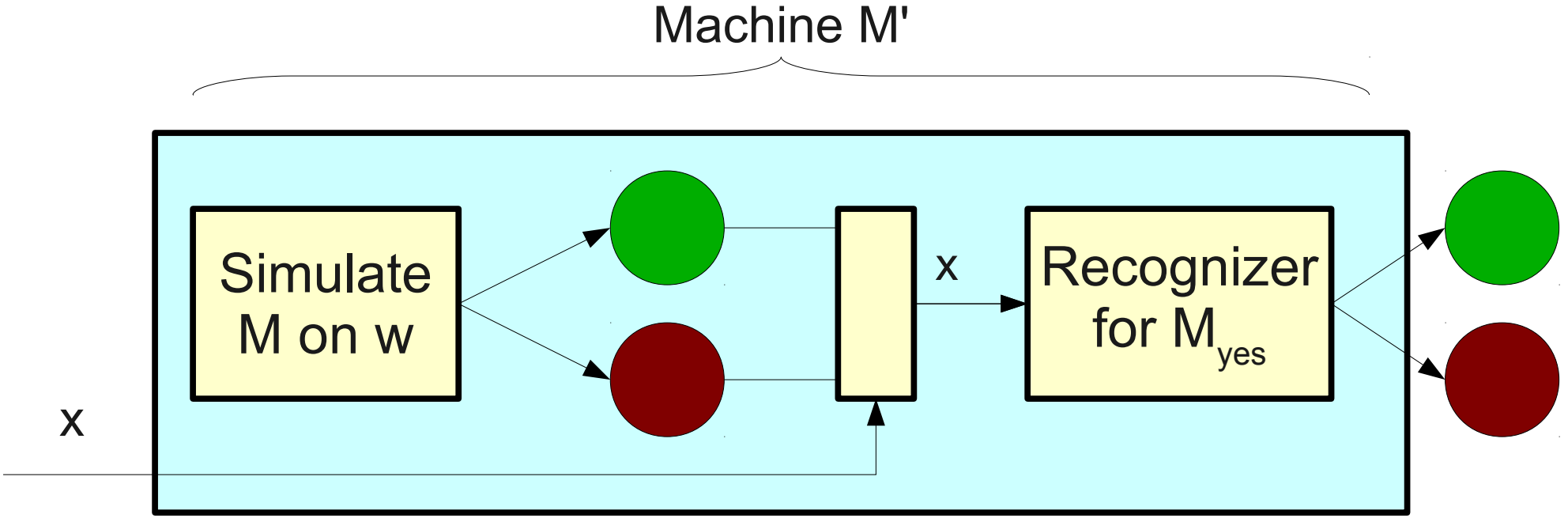


M' = "On input x:

Run M on w.

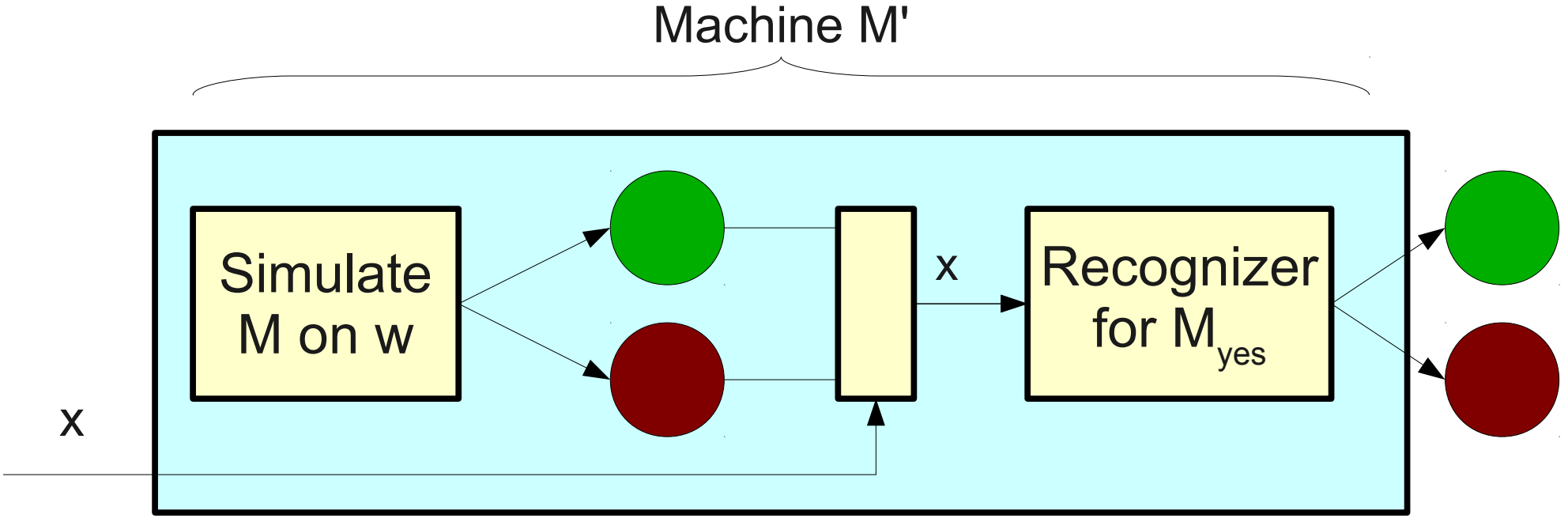
If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x."

If M loops on w, what does M' do on x?

M' always loops.



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Run M on w.

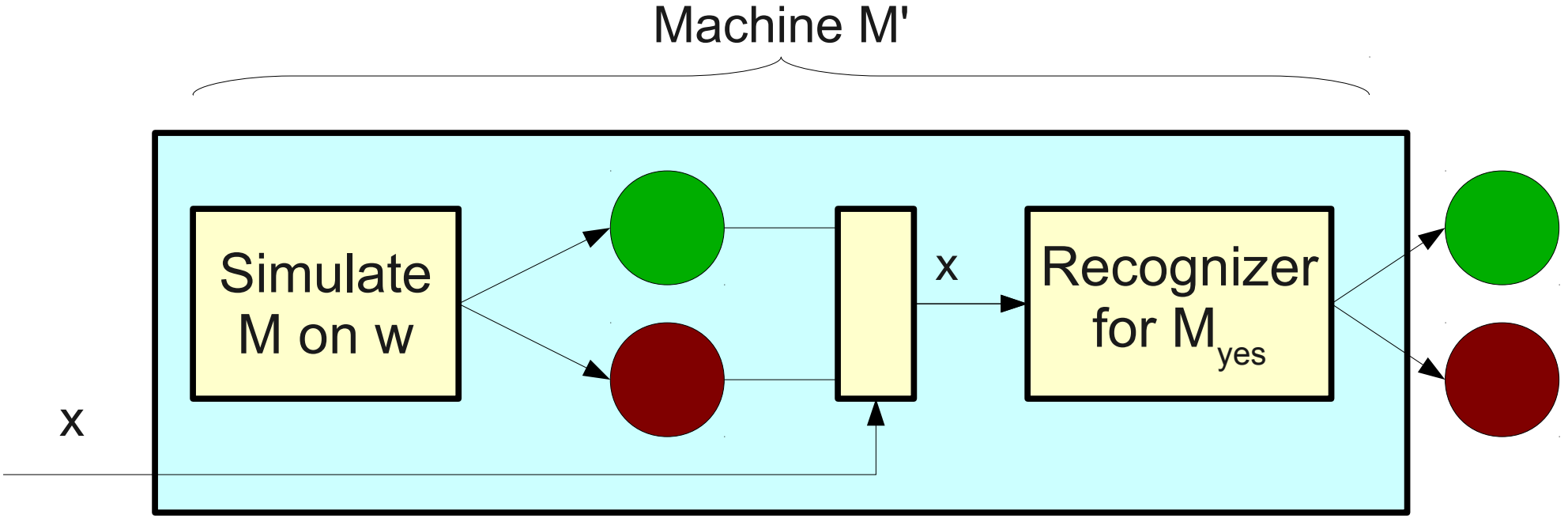
If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x."

If M loops on w, what does M' do on x?

$L(M') = \emptyset$



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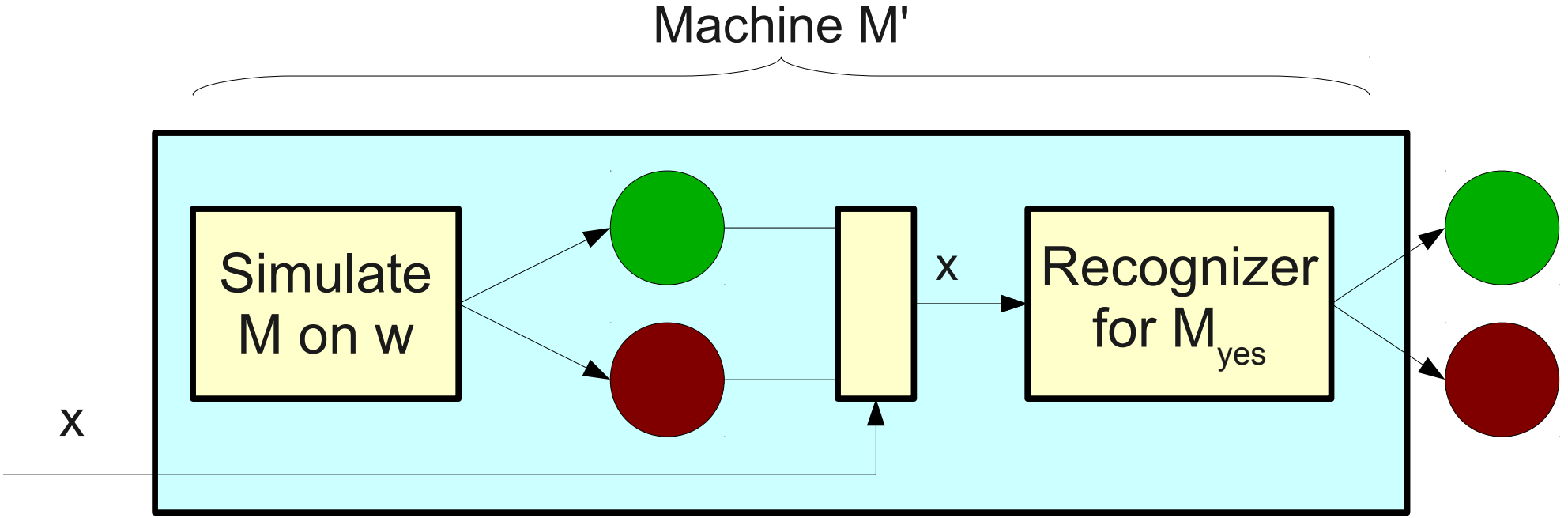

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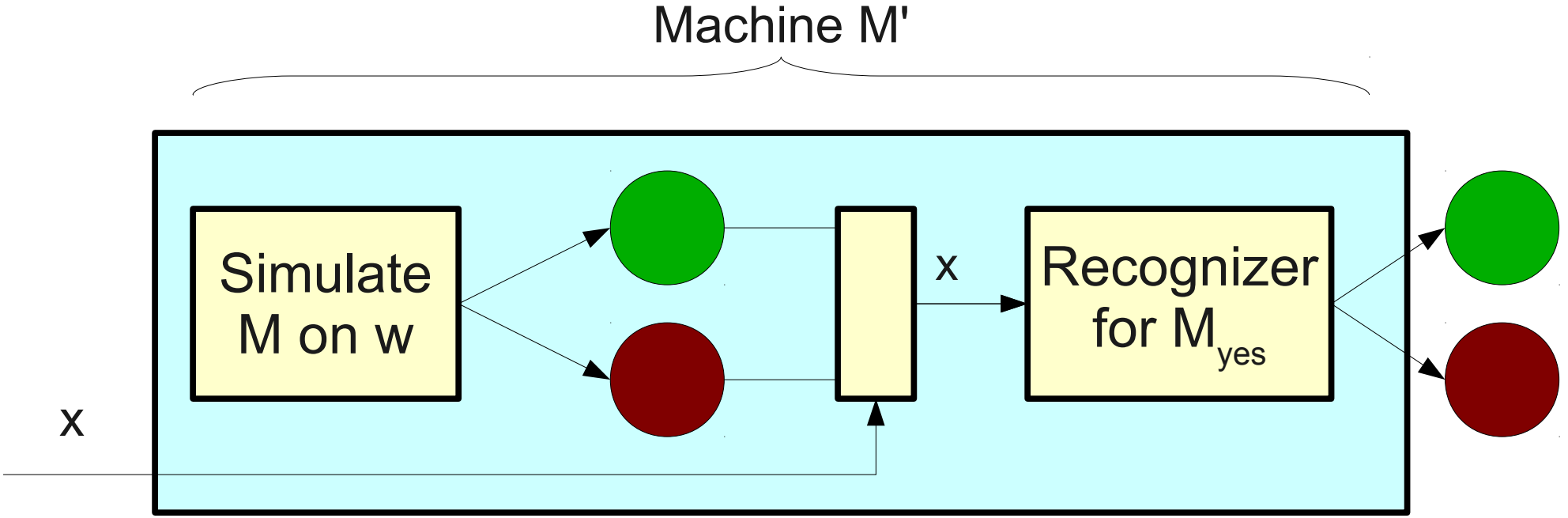
If M halts on w, what does M' do on x?



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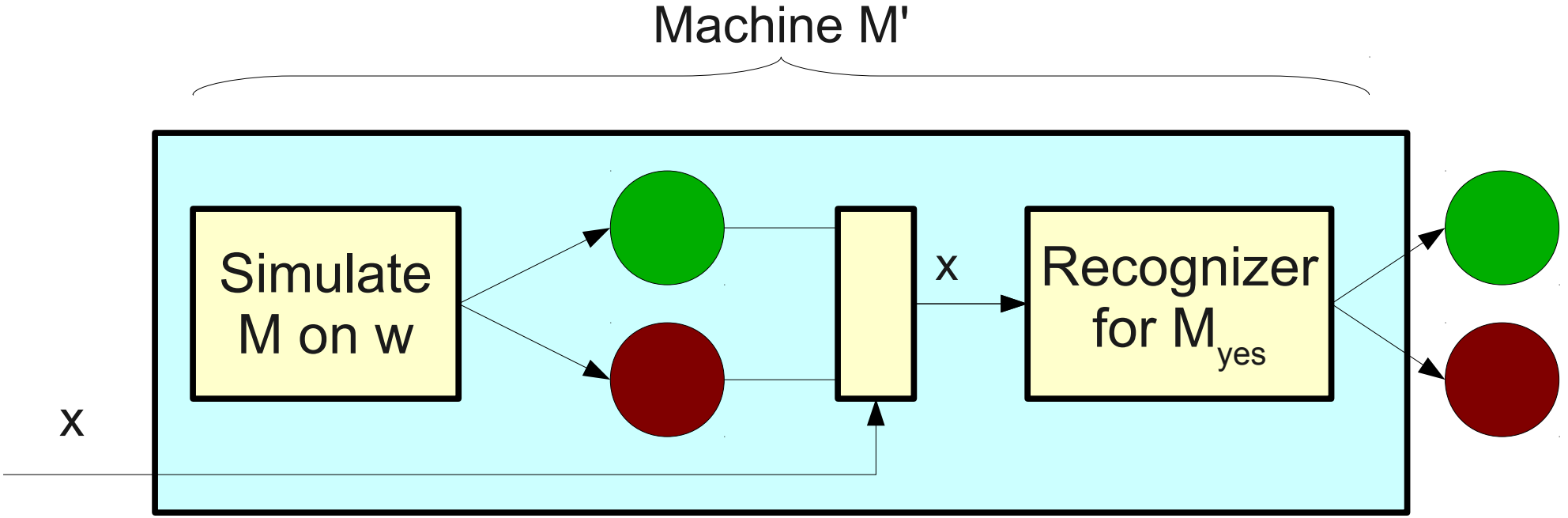


M' = "On input x:

Run M on w.

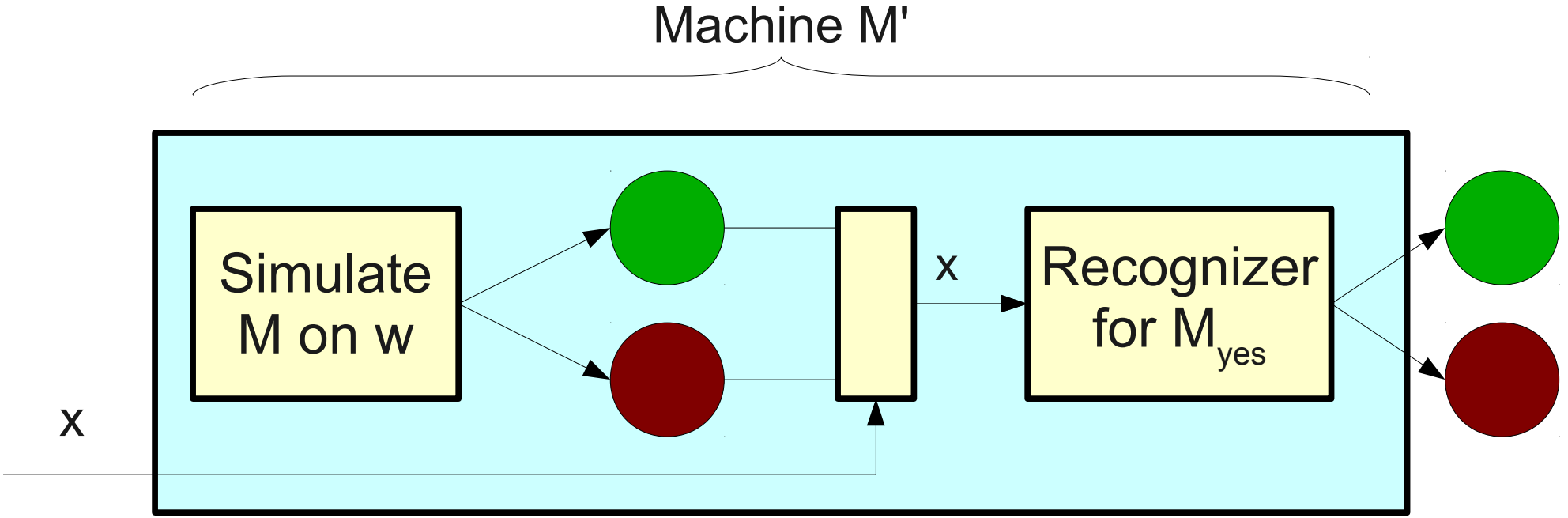
If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x."

If M halts on w, what does M' do on x?

M' accepts x iff M_{yes} accepts x



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If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x."

If M halts on w, what does M' do on x?

$L(M') = L(M_{yes})$

Key Idea Three

M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”

If M halts on w, $L(M') = L(M_{yes})$

If M loops on w, $L(M') = \varnothing$

Key Idea Three

M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”

If M halts on w, $L(M') = L(M_{yes})$

If M loops on w, **$L(M') = \emptyset$**

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M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”

If M halts on w, $L(M') = L(M_{yes})$

If M loops on w, **$L(M') = L(M_{\emptyset})$**

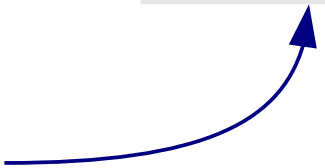
Key Idea Three

M' = “On input x:
Run M on w.
If M halts, run M_{yes} on x.
Accept if M_{yes} accepts x.
Reject if M_{yes} rejects x.”

If M halts on w, $L(M') = L(M_{yes})$

If M loops on w, **$L(M') = L(M_{\emptyset})$**

Remember that $M_{\emptyset}$ is a machine that never accepts, so $L(M_{\emptyset}) = \emptyset$



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If M halts, run M_{yes} on x.

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If M loops on w, $L(M') = L(M_{\varnothing})$

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If M halts, run M_{yes} on x.

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Reject if M_{yes} rejects x.”

If M halts on w, **$L(M') = L(M_{yes})$**

If M loops on w, $L(M') = L(M_{\varnothing})$

Key Idea Three

M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”

If M halts on w, $\langle M' \rangle \in L_P$

If M loops on w, $L(M') = L(M_\varnothing)$

Key Idea Three

M' = "On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x."

Since we know that $L(M') = L(M_{yes})$ and we know that $\langle M_{yes} \rangle \in L_P$ we know that $\langle M' \rangle \in L_P$.

If M halts on w, $\langle M' \rangle \in L_P$

If M loops on w, $L(M') = L(M_\varnothing)$

Key Idea Three

M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

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If M loops on w, **$L(M') = L(M_{\emptyset})$**

Key Idea Three

M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”

If M halts on w, $\langle M' \rangle \in L_P$

If M loops on w, $\langle M' \rangle \notin L_P$

Key Idea Three

M' = "On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x."

Since we know that

$L(M') = L(M_\emptyset)$

and we know that

$\langle M_\emptyset \rangle \notin L_P$

we know that

$\langle M' \rangle \notin L_P$.

If M halts on w, $\langle M' \rangle \in L_P$

If M loops on w, **$\langle M' \rangle \notin L_P$**

Key Idea Three

M' = “On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

Reject if M_{yes} rejects x.”

If M halts on w, $\langle M' \rangle \in L_P$

If M loops on w, $\langle M' \rangle \notin L_P$

Key Idea Three

M' = "On input x:

Run M on w.

If M halts, run M_{yes} on x.

Accept if M_{yes} accepts x.

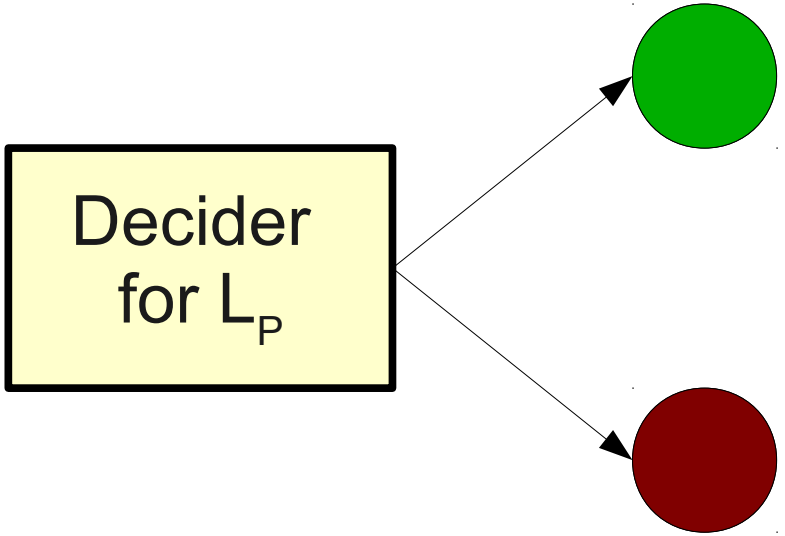
Reject if M_{yes} rejects x."

This is the key step of the construction. If we can decide whether $\langle M' \rangle \in L_P$, we would be deciding whether M halts on w.

If M halts on w, $\langle M' \rangle \in L_P$

If M loops on w, $\langle M' \rangle \notin L_P$

The Complete Construction



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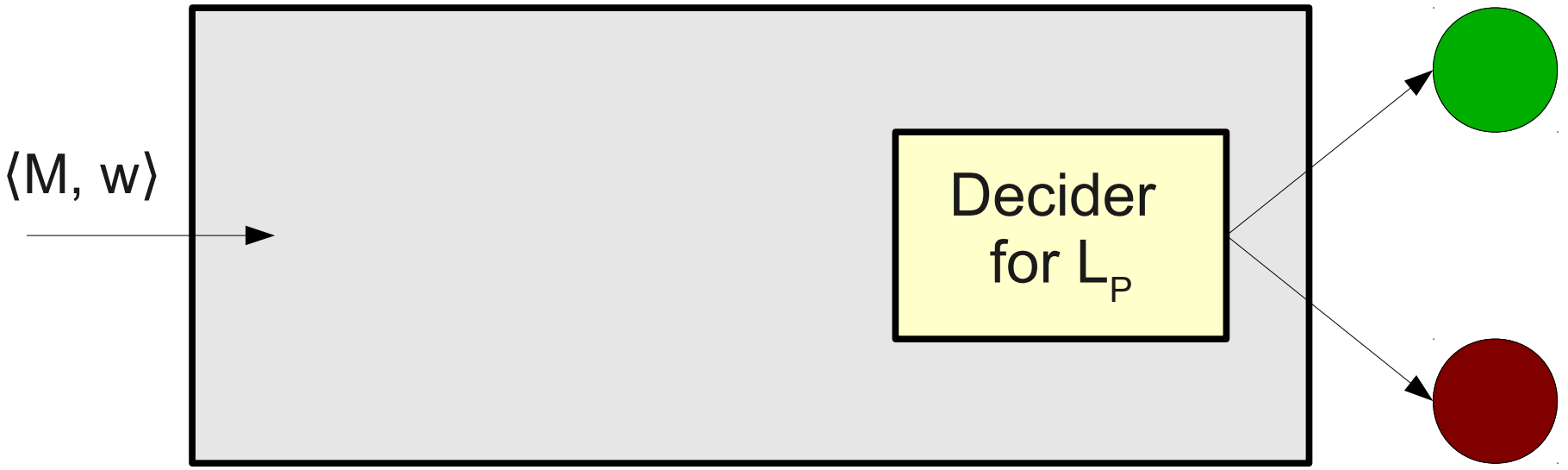

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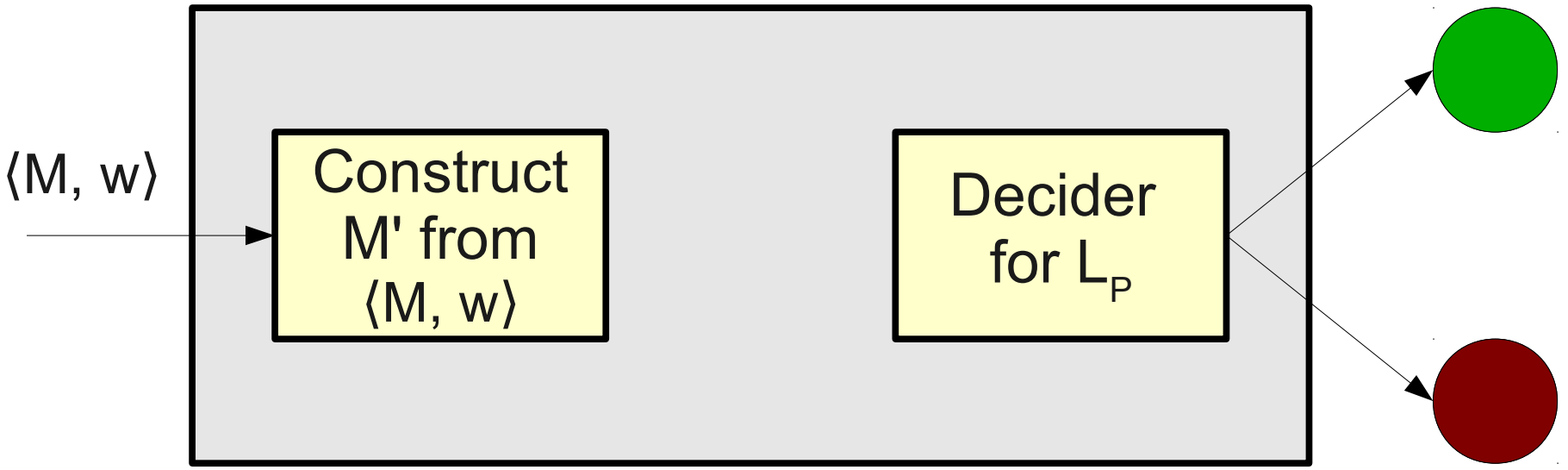

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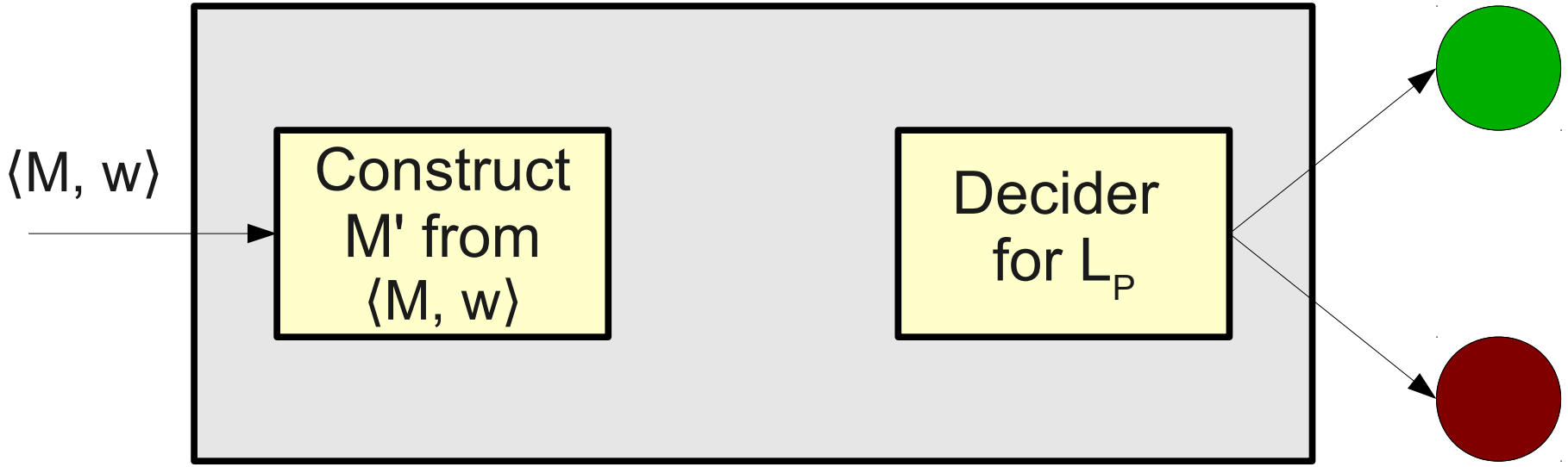

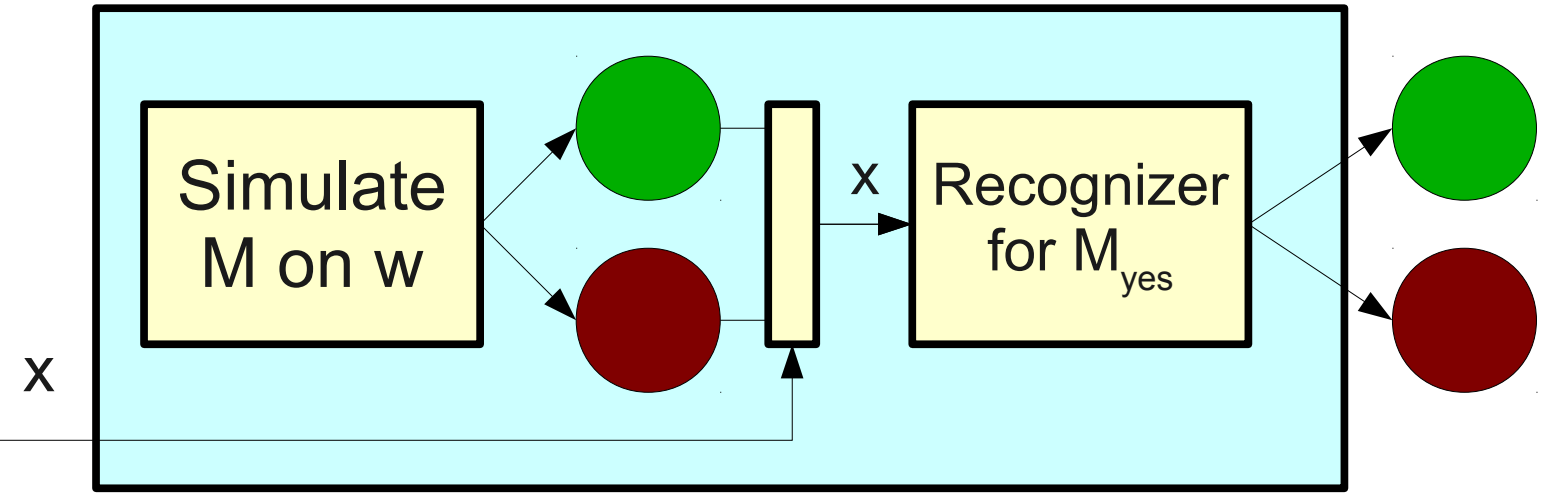

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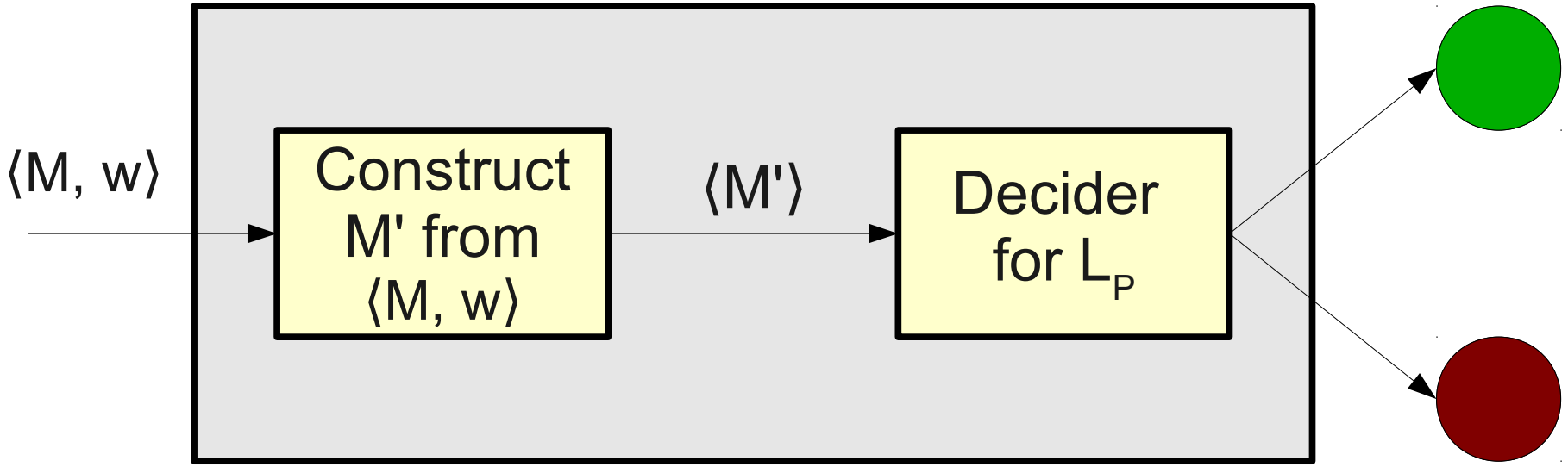

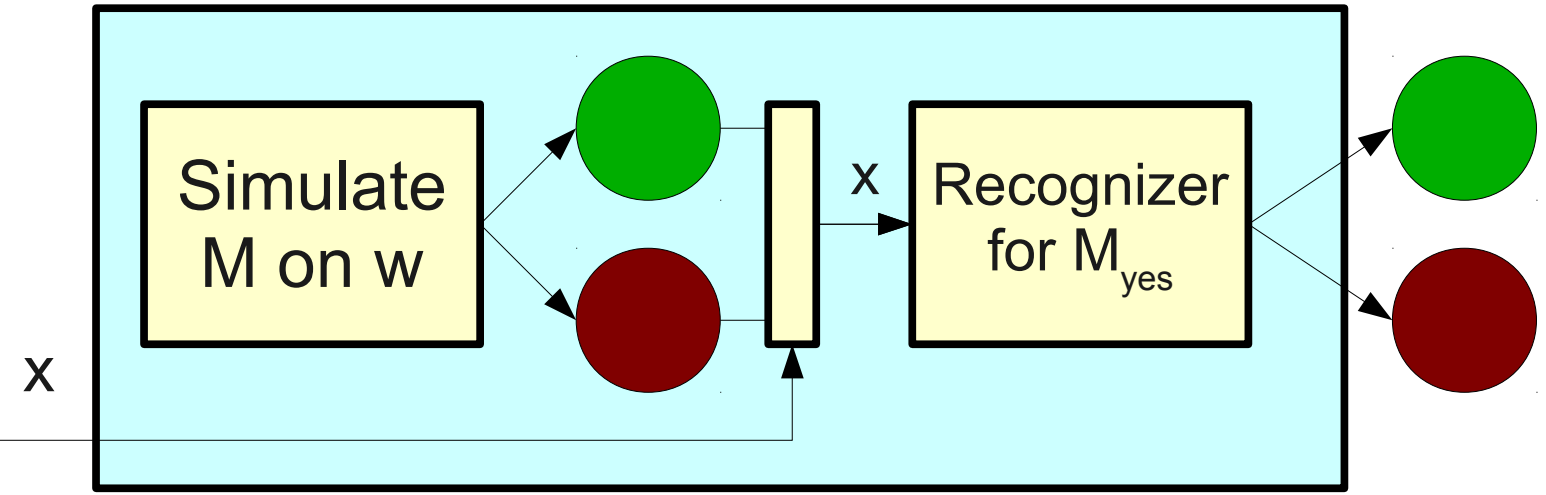

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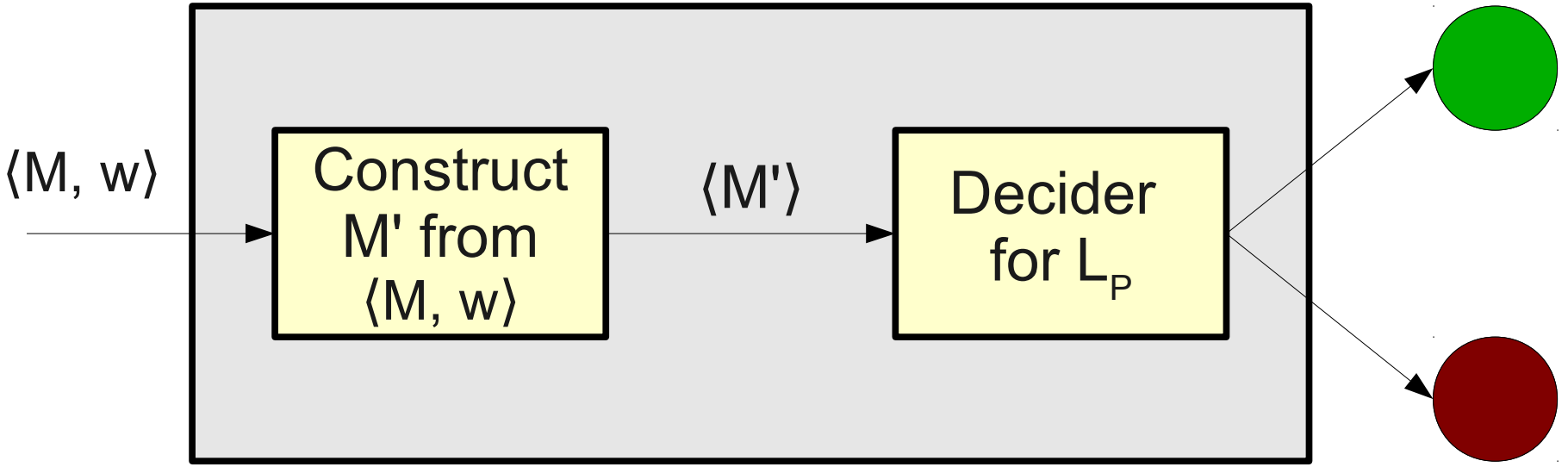
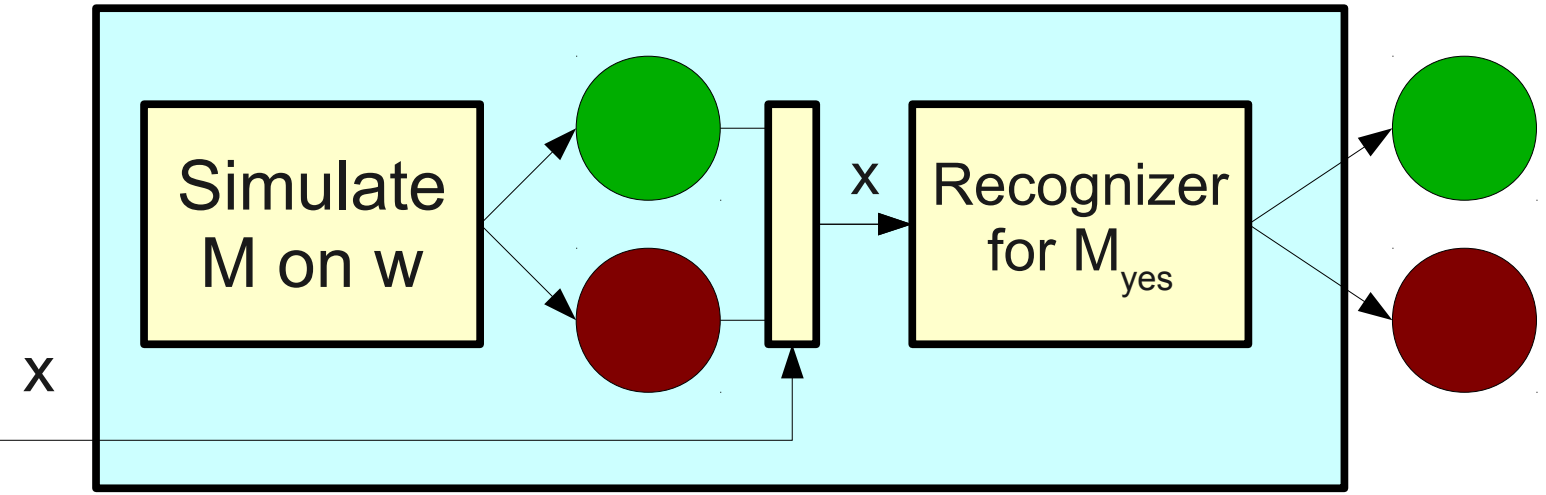
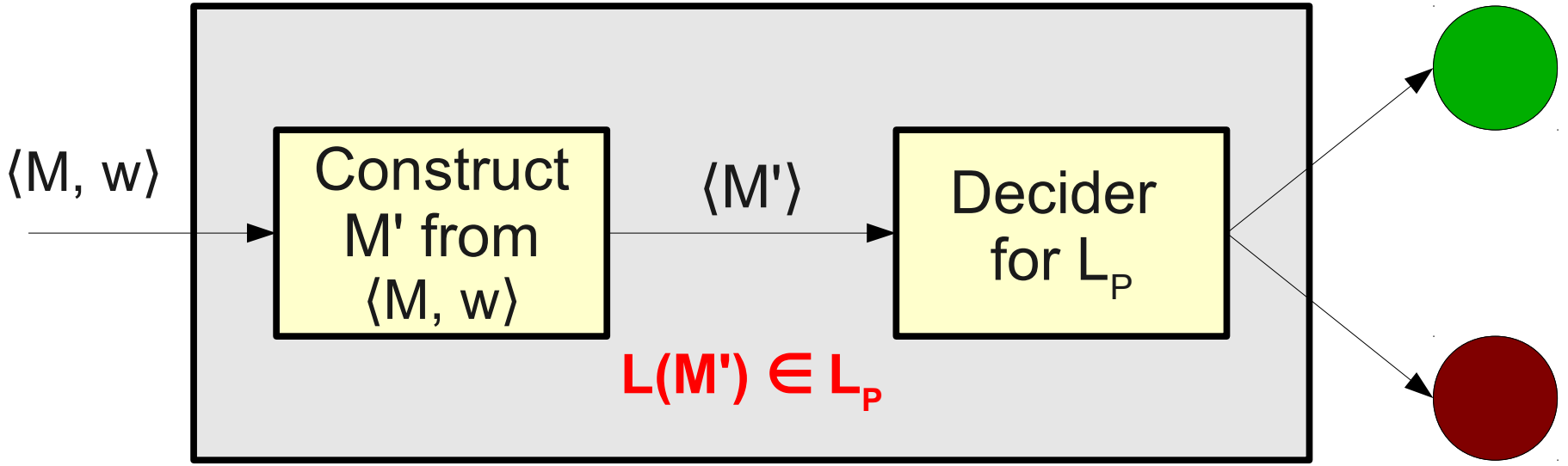
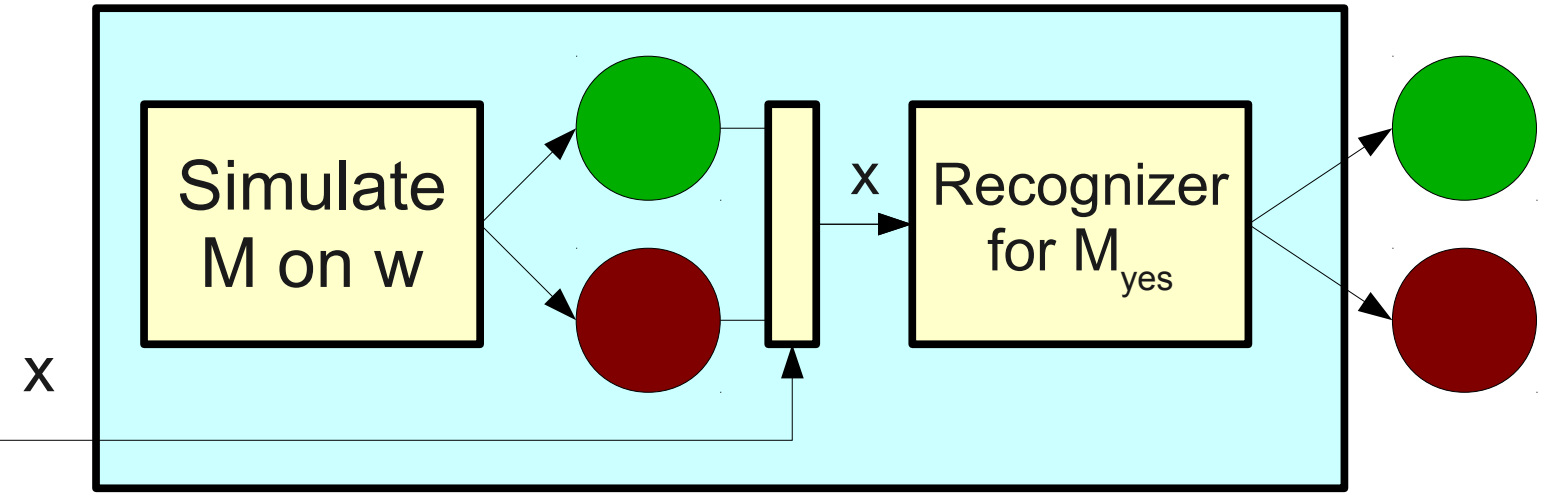




What happens if M halts on w?

The Complete Construction





What happens if M halts on w?

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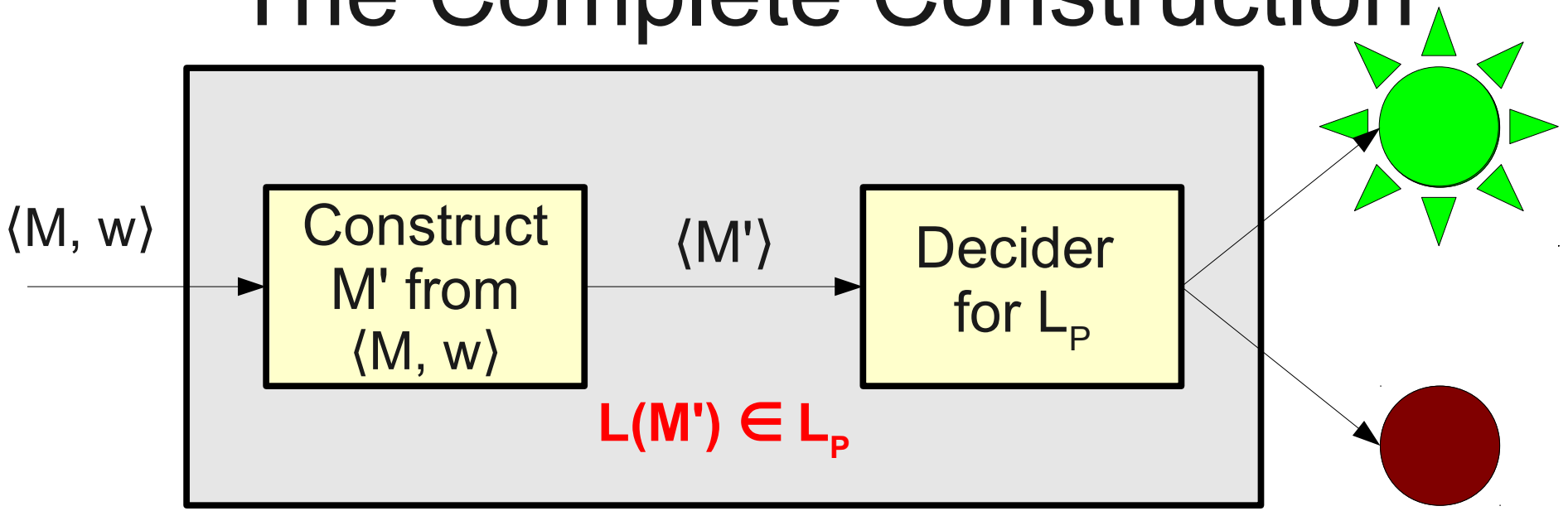

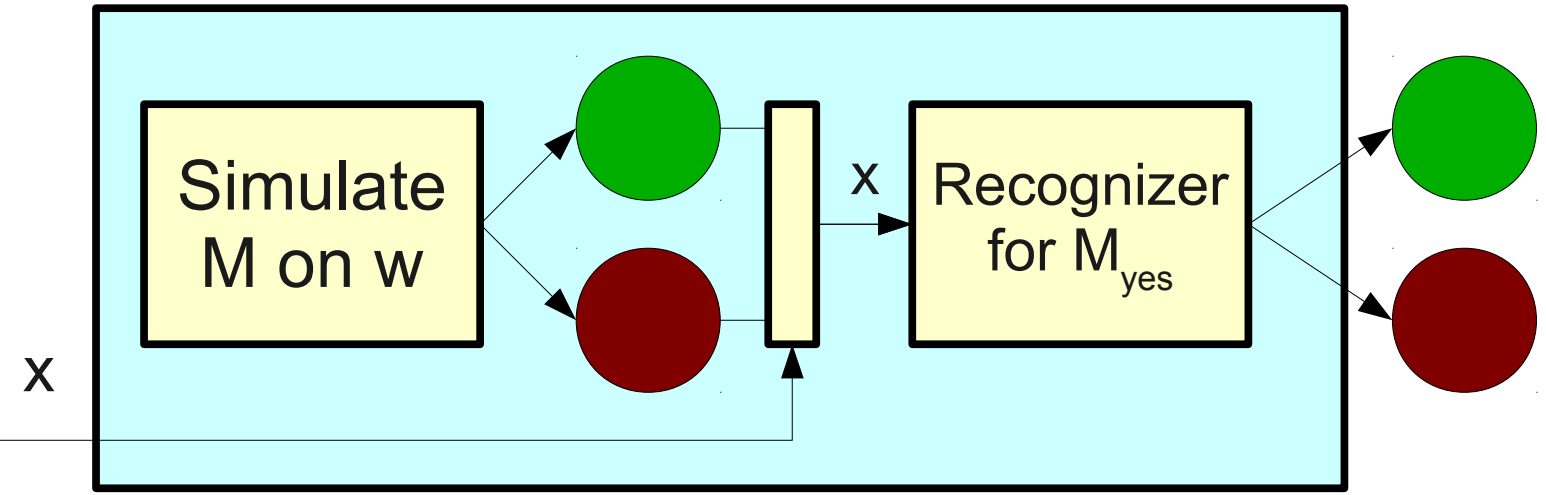

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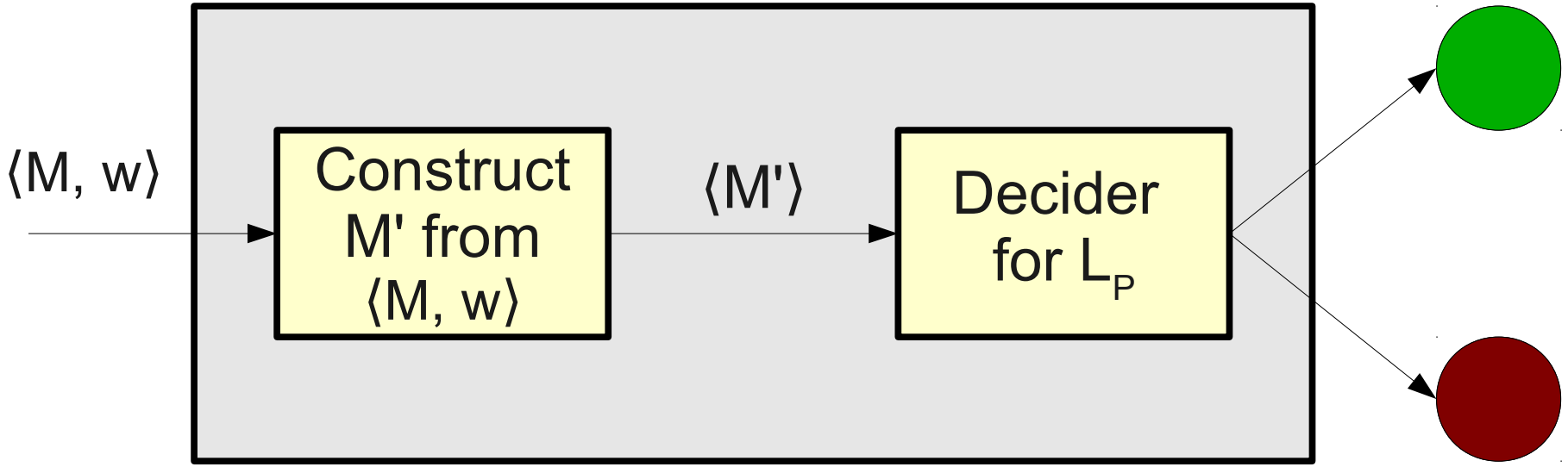

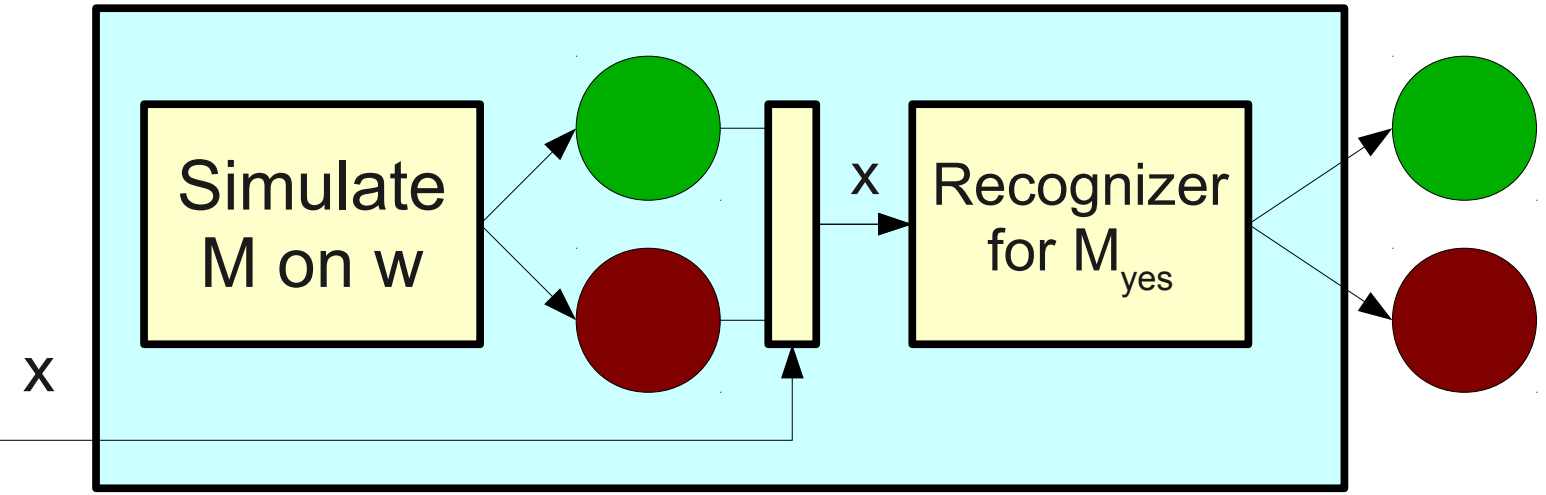

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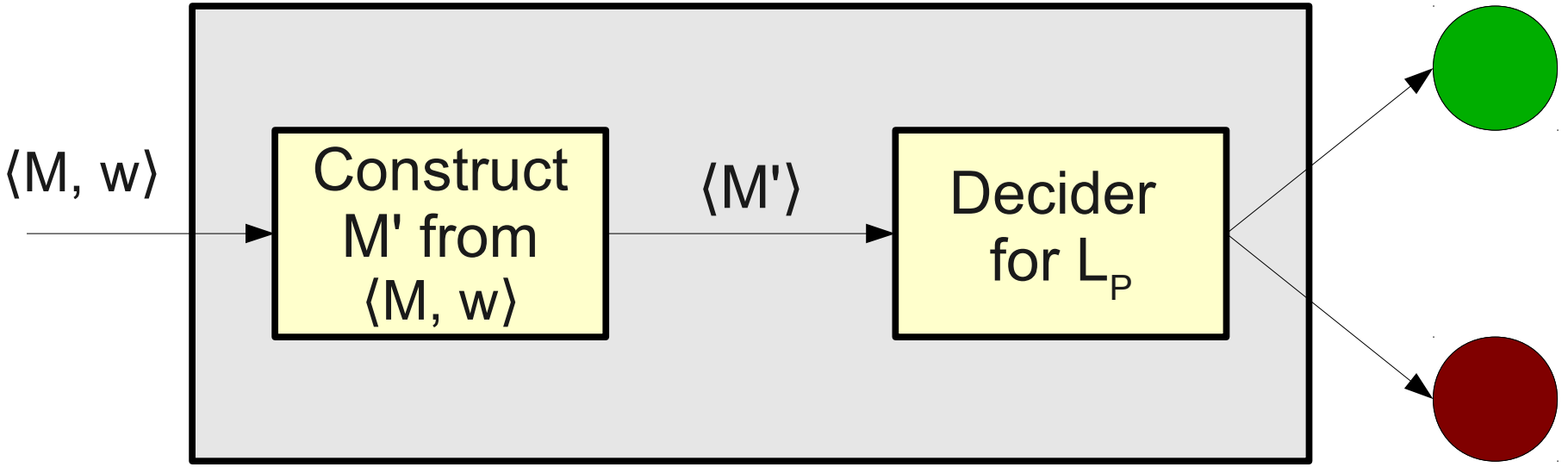
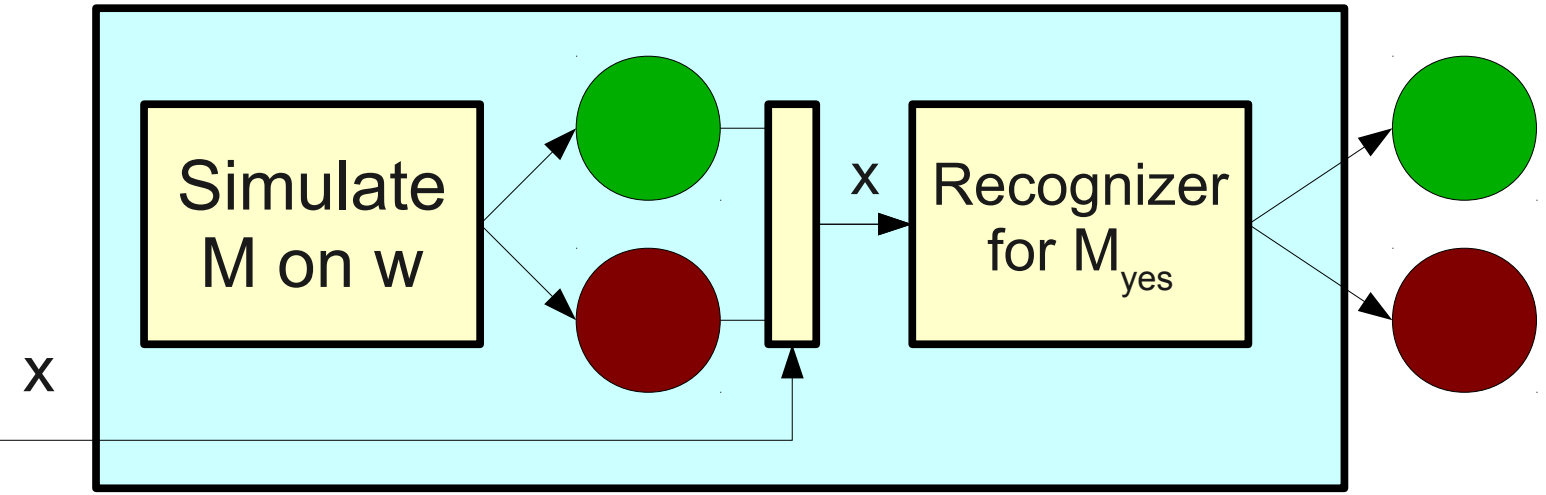
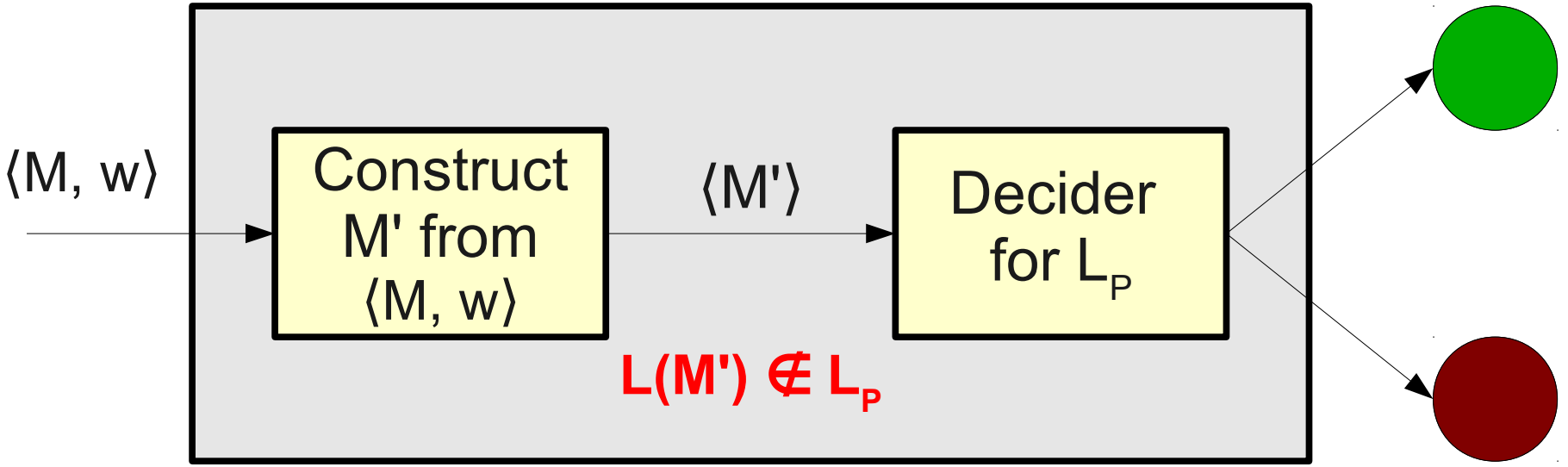
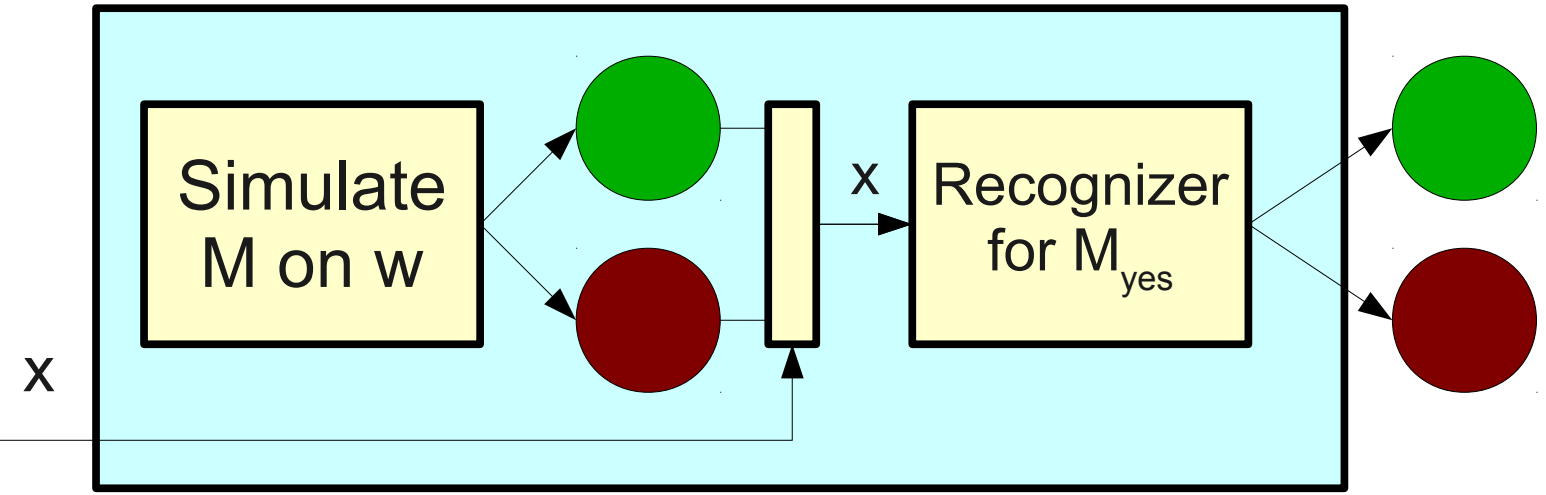




What happens if M loops on w?

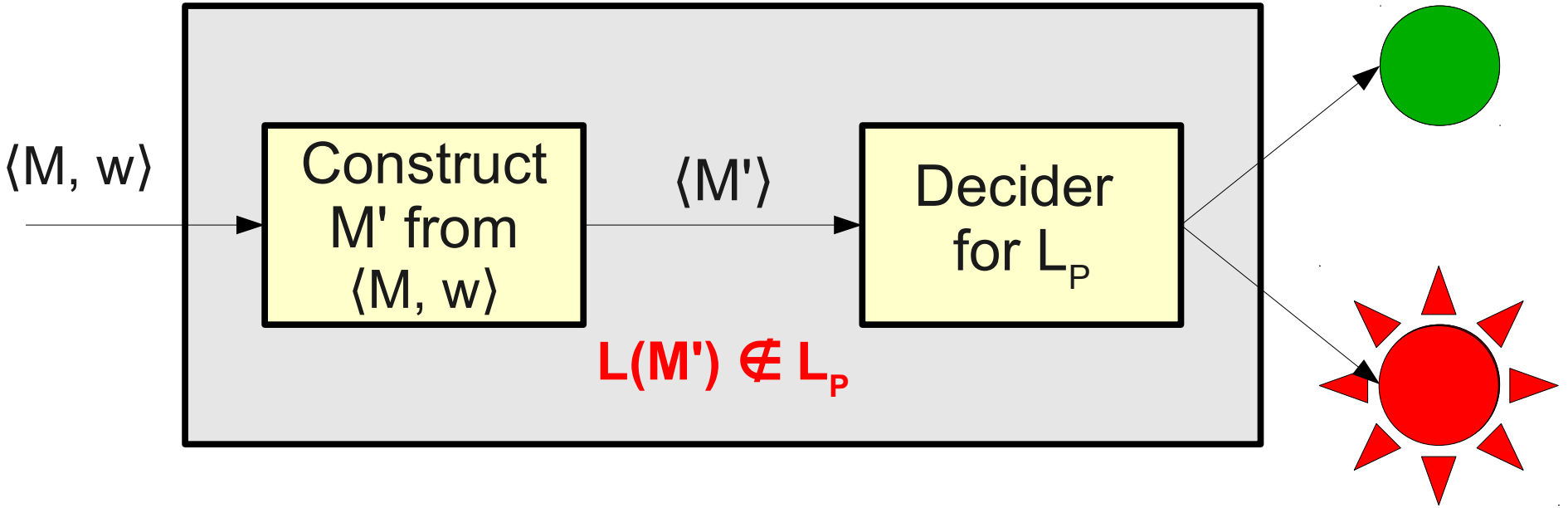
The Complete Construction

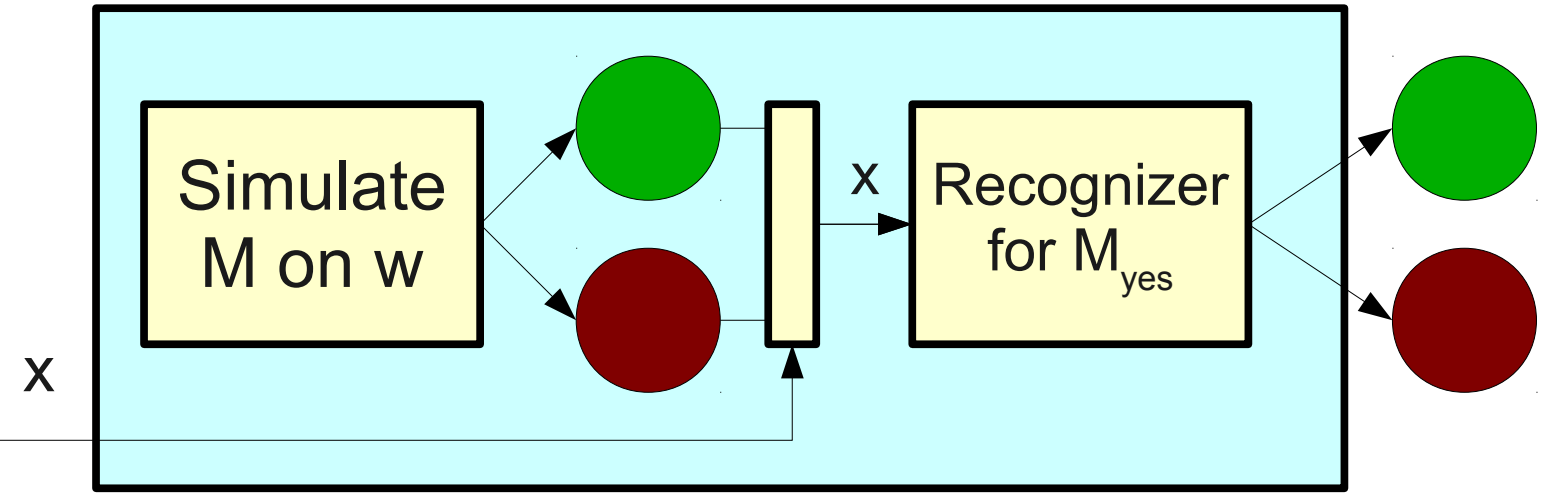




What happens if M loops on w?

The Complete Construction





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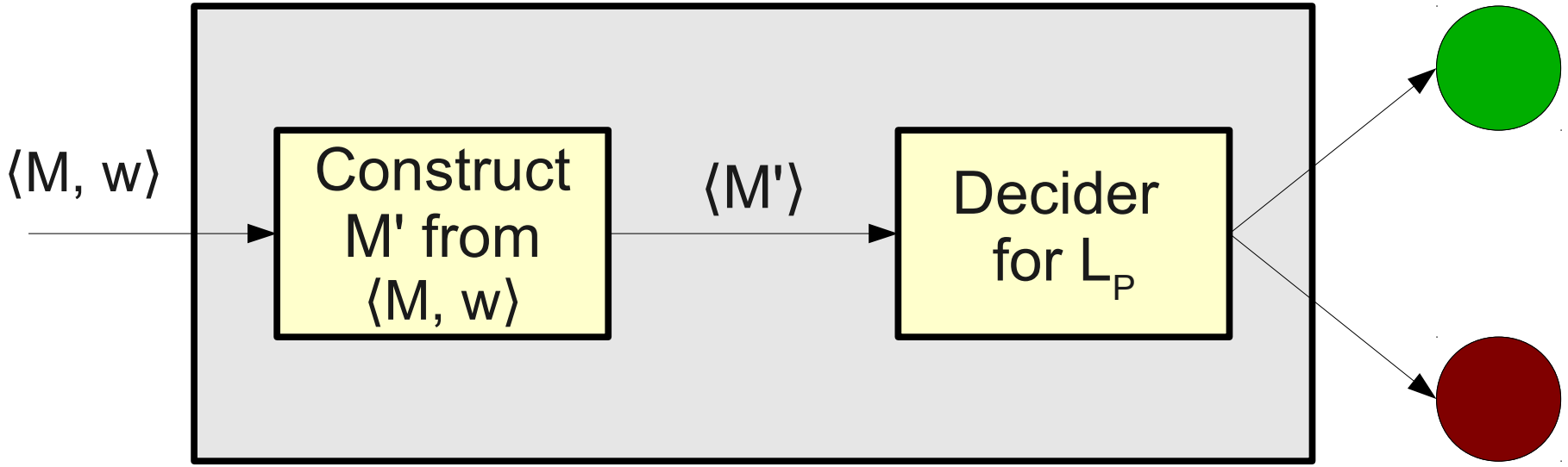

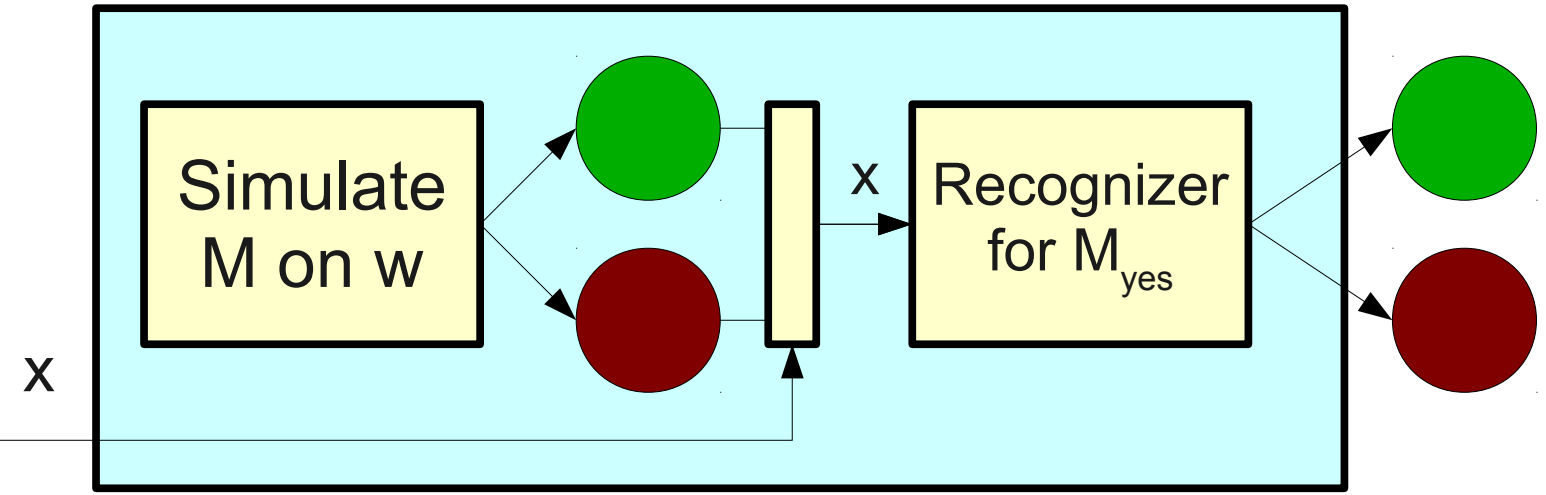

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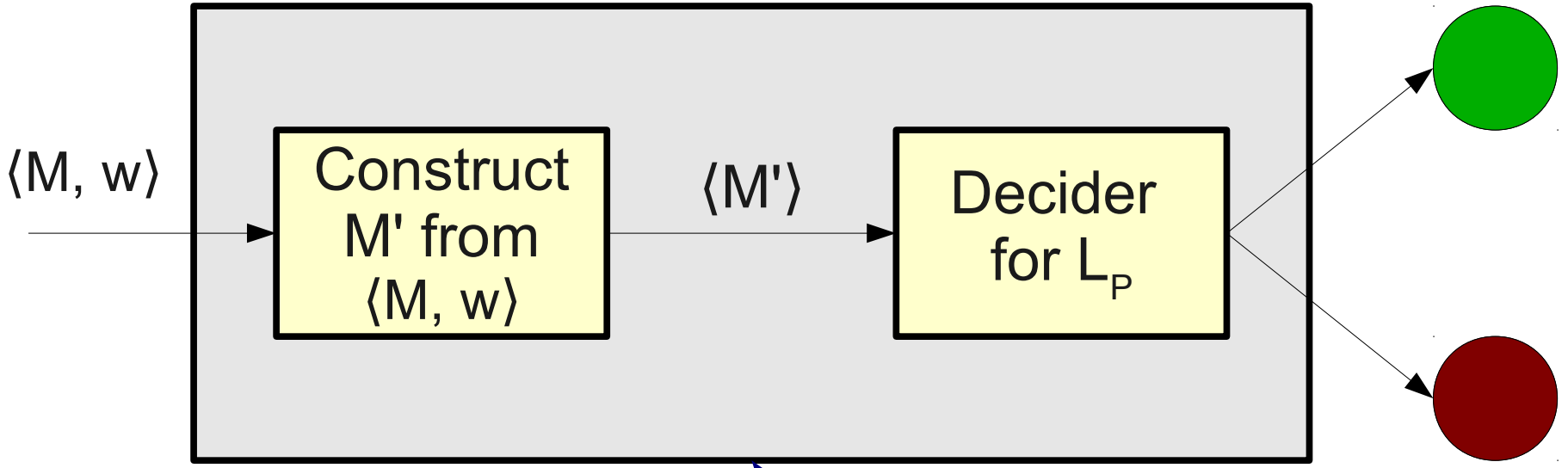

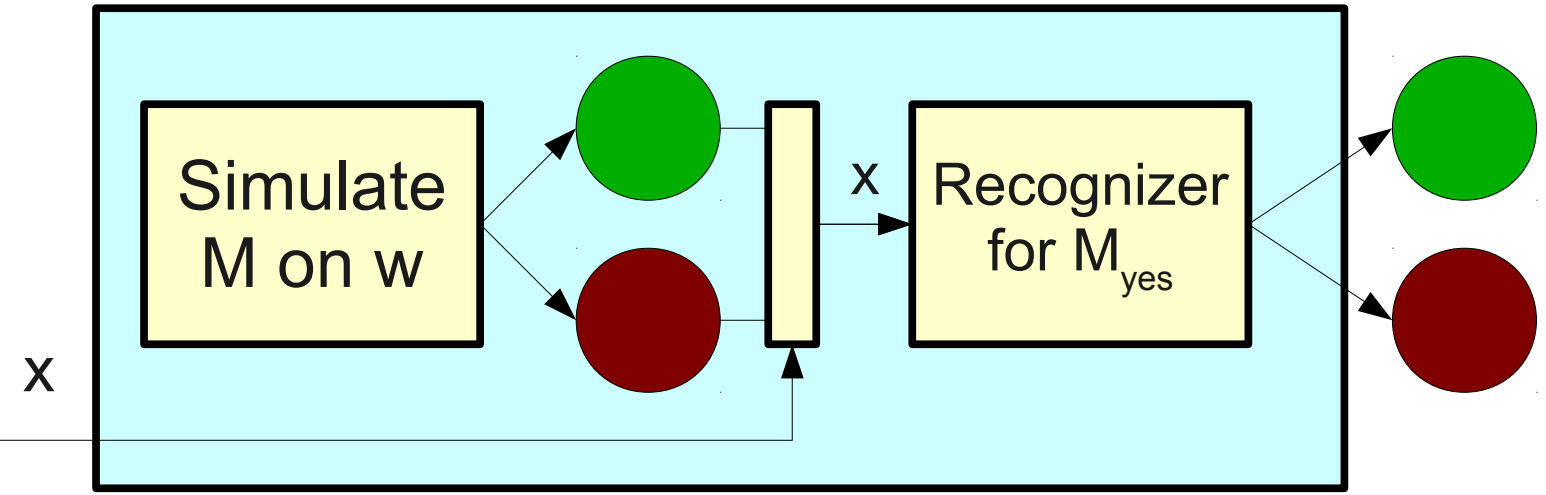

This machine decides HALT!

Theorem (Rice): Any nontrivial property of the RE languages is undecidable.
Proof: Let $M_{\emptyset}$ be a TM that accepts the empty language. Then either $\langle M_{\emptyset} \rangle \notin L_P$ or $\langle M_{\emptyset} \rangle \in L_P$. In the former case, we prove that L_P is undecidable directly. In the latter case, we prove that $\overline{L}_P$ is undecidable; this means that L_P is undecidable as well. So assume without loss of generality that $\langle M_{\emptyset} \rangle \notin L_P$. Assume for the sake of contradiction that L_P is a decidable property of the RE languages.

Since L_P is a nontrivial property of the RE languages and $\langle M_{\emptyset} \rangle \notin L_P$, there must be some TM M_{yes} such that $\langle M_{yes} \rangle \in L_P$. Moreover, $L(M_{yes}) \neq L(M_{\emptyset})$, because otherwise if $L(M_{yes}) = L(M_{\emptyset})$, we would have that either $\langle M_{\emptyset} \rangle \in L_P$ and $\langle M_{yes} \rangle \in L_P$ or $\langle M_{\emptyset} \rangle \notin L_P$ and $\langle M_{yes} \rangle \notin L_P$, both of which are false.

Because L_P is decidable, let D be a decider for L_P. Then consider the following TM:

H = "On input $\langle M, w \rangle$:
    Construct the TM M' = "On input x:
        Run M on w, and if M halts, run M_{yes} on x.
        Accept if M_{yes} accepts, reject if M_{yes} rejects."
    Run D on $\langle M' \rangle$ and accept if D accepts and reject if D rejects."

We claim that H is a decider for *HALT*. If we can show this, then we have reached a contradiction because *HALT* is undecidable. Thus our assumption was wrong, so no nontrivial property of the RE languages is decidable.

To see that H is a decider, note that since D is a decider, after we construct M' and run D on $\langle M' \rangle$, D always halts, so H always halts. Thus H halts on all inputs.

To see that L(H) = *HALT*, note that if M loops on w, then L(M') = Ø, and so D rejects $\langle M' \rangle$ because $L(M') = \emptyset = L(M_{\emptyset})$ and $\langle M_{\emptyset} \rangle \notin L_P$. Otherwise, if M halts on w, then M' accepts x iff M_{yes} accepts x, so $L(M') = L(M_{yes})$, and since $\langle M_{yes} \rangle \in L_P$, D accepts $\langle M' \rangle$. Thus H accepts $\langle M, w \rangle$ iff D accepts $\langle M' \rangle$ iff M halts on w iff $\langle M, w \rangle \in$ *HALT*, so L(H) = *HALT*. ■

Summary

- Reductions from the halting problem can be used to prove that certain languages are undecidable.
- Rice's theorem can be used to immediately prove that large classes of languages are also undecidable.

Next Time

- **Complexity Theory**
  - How hard is it to solve certain types of problems?
  - Of the decidable problems, which ones could be computed efficiently?
- **P and NP**
  - What problems can be *solved* efficiently?
  - What problems can be *checked* efficiently?