Get Ready...

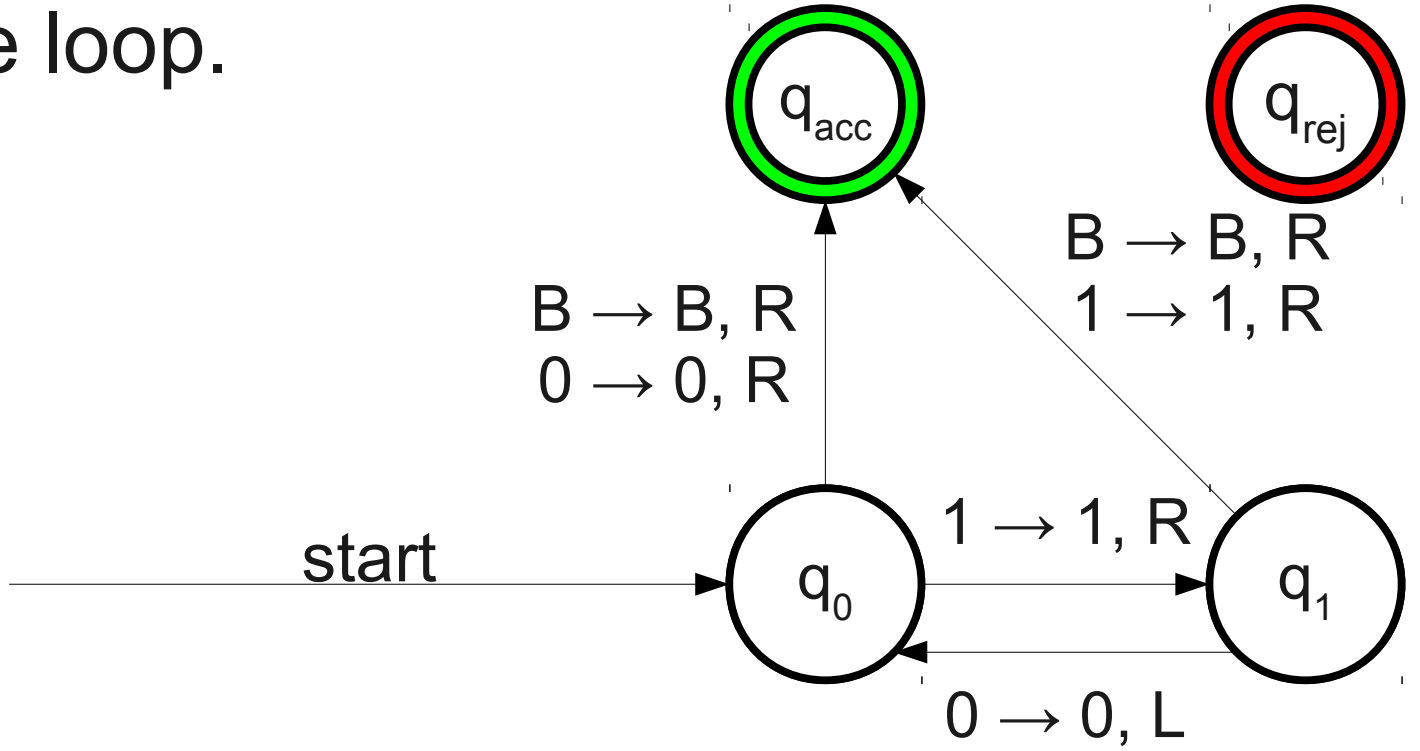
Announcements

- Problem Set 7 due Friday at 2:15 PM.
  - Stop by OH with questions!
  - Email **cs103@cs.stanford.edu** with questions!
  - Can submit using a late day before this Saturday at 2:15PM.

The **Limits** of Turing Machines

An Infinite Loop

- Unlike the other automata we've seen so far, Turing machines can sometimes enter an infinite loop.



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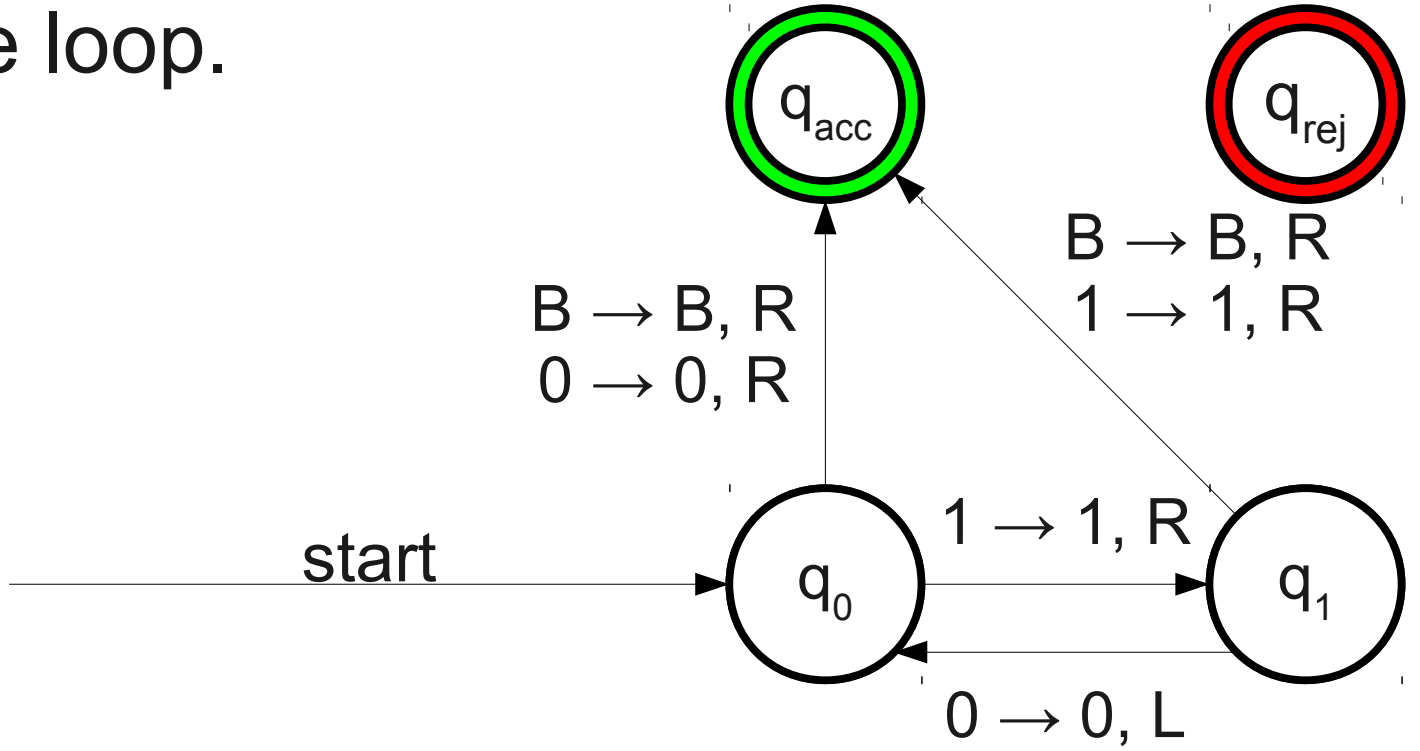

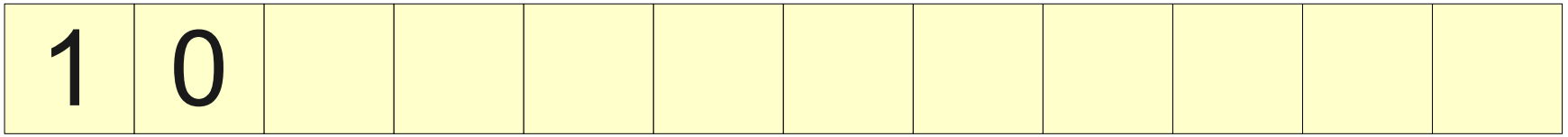

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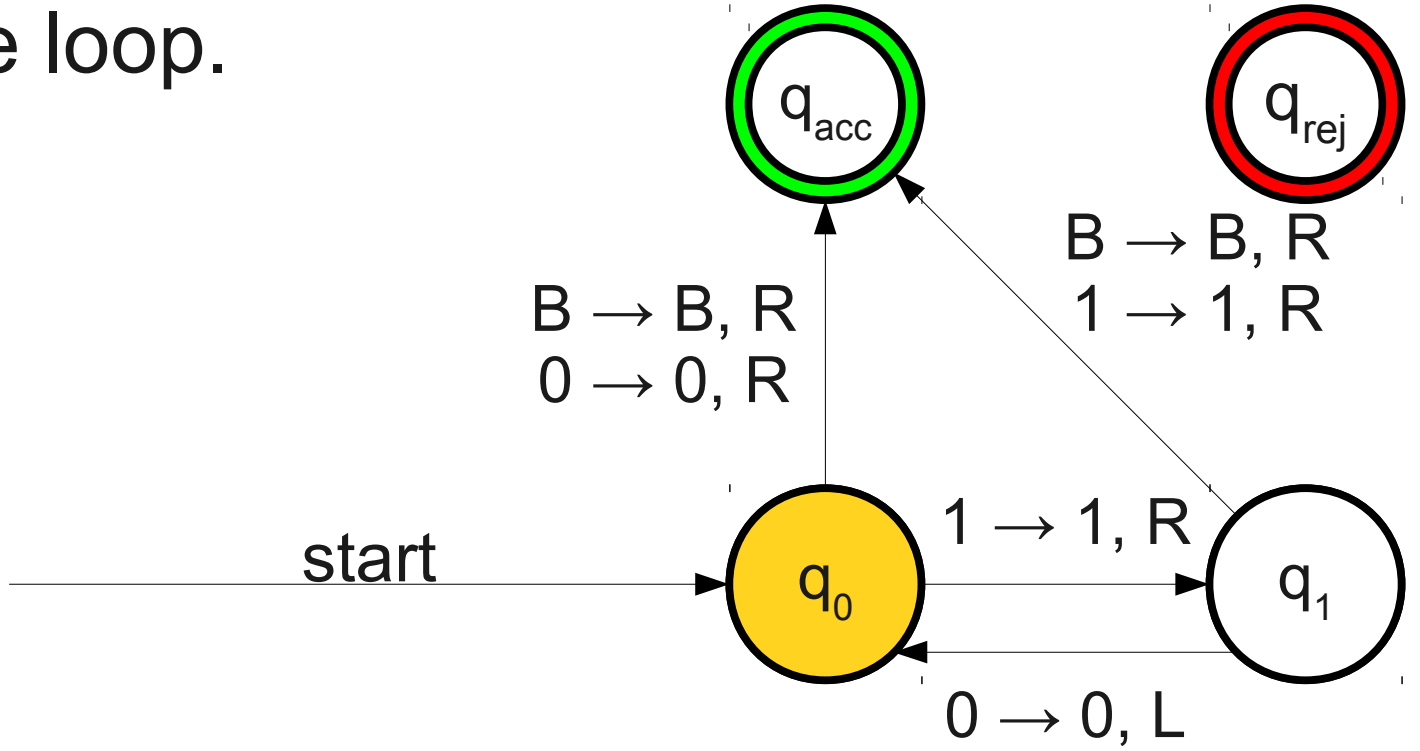

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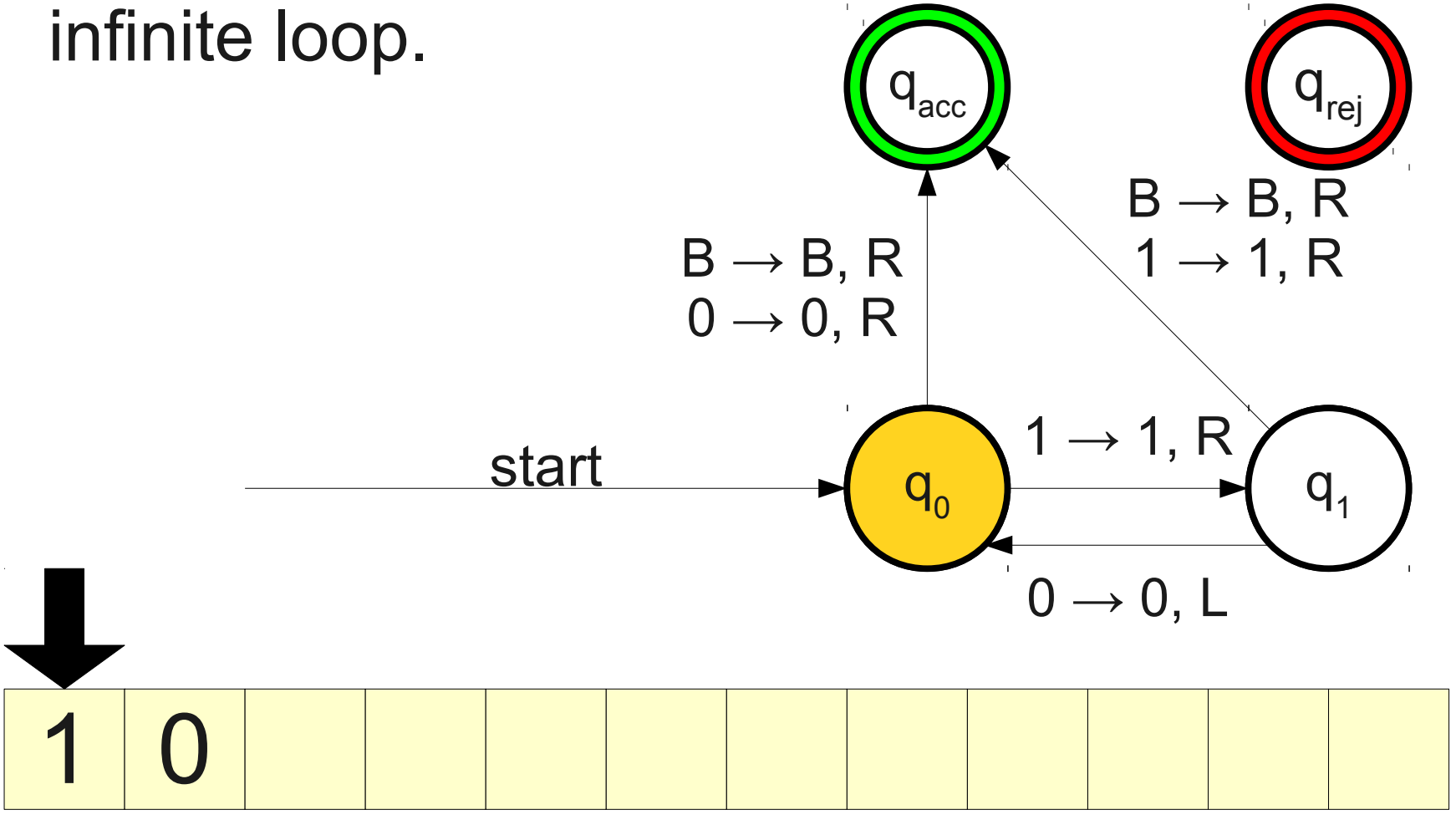

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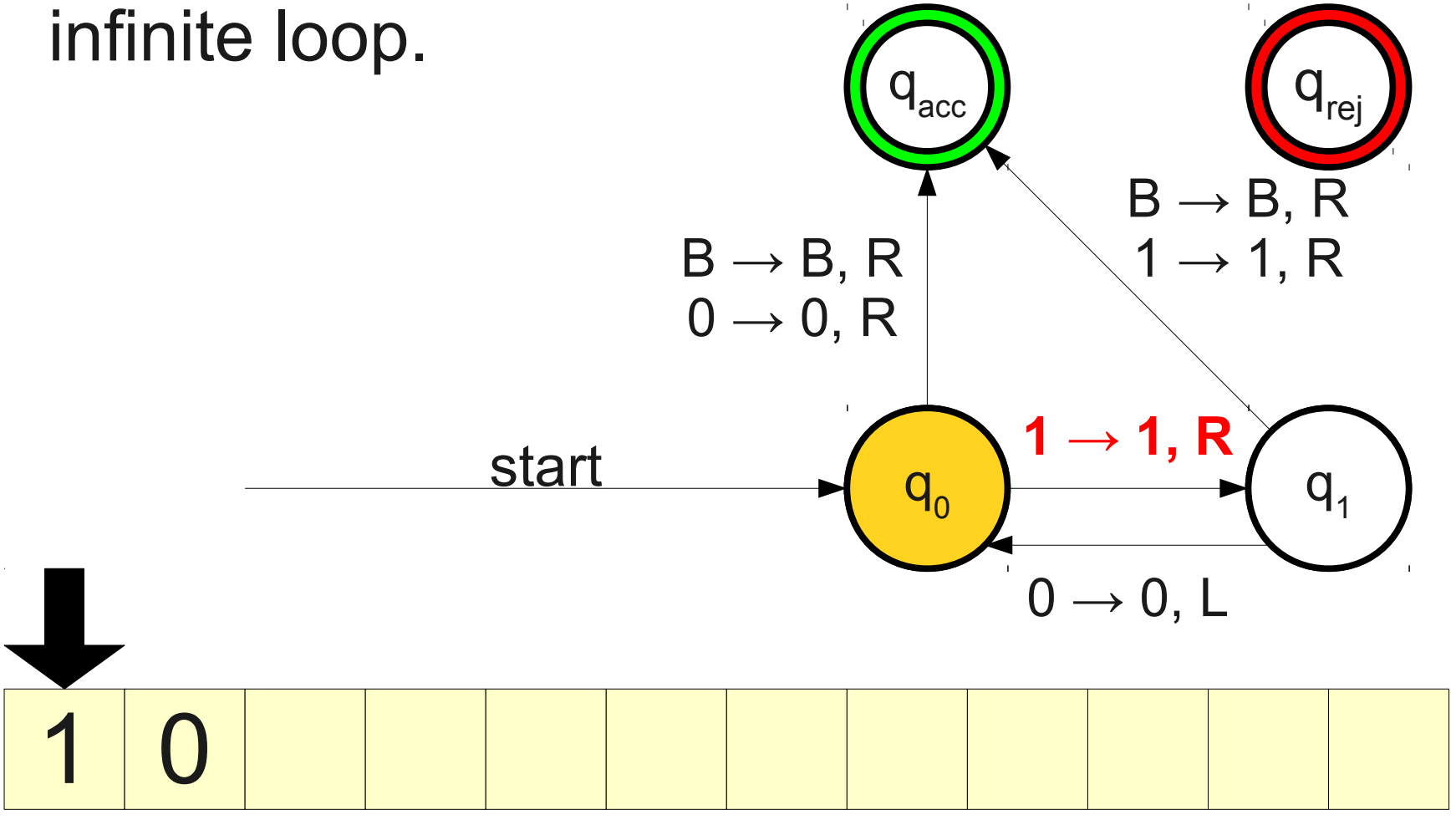

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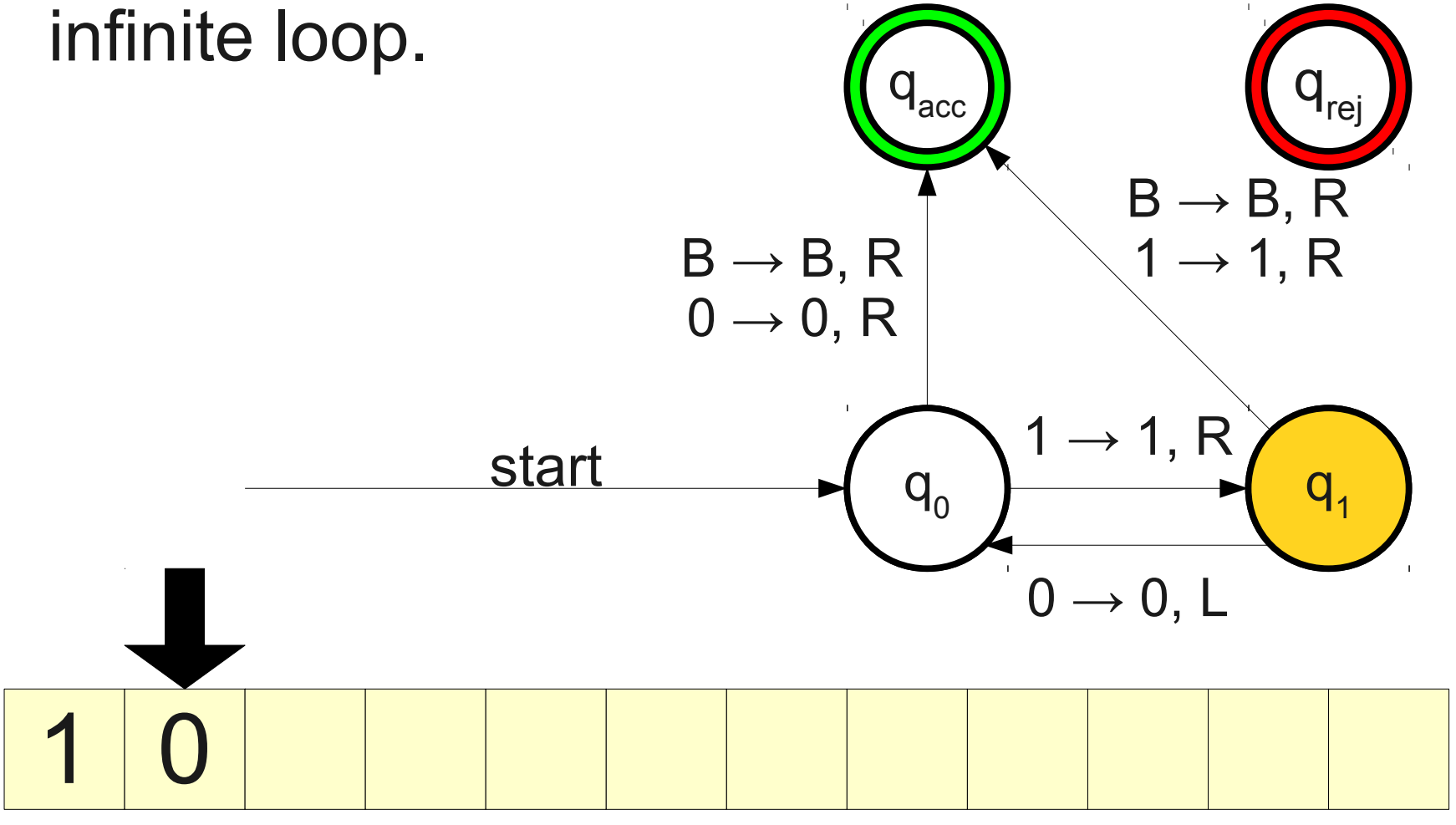

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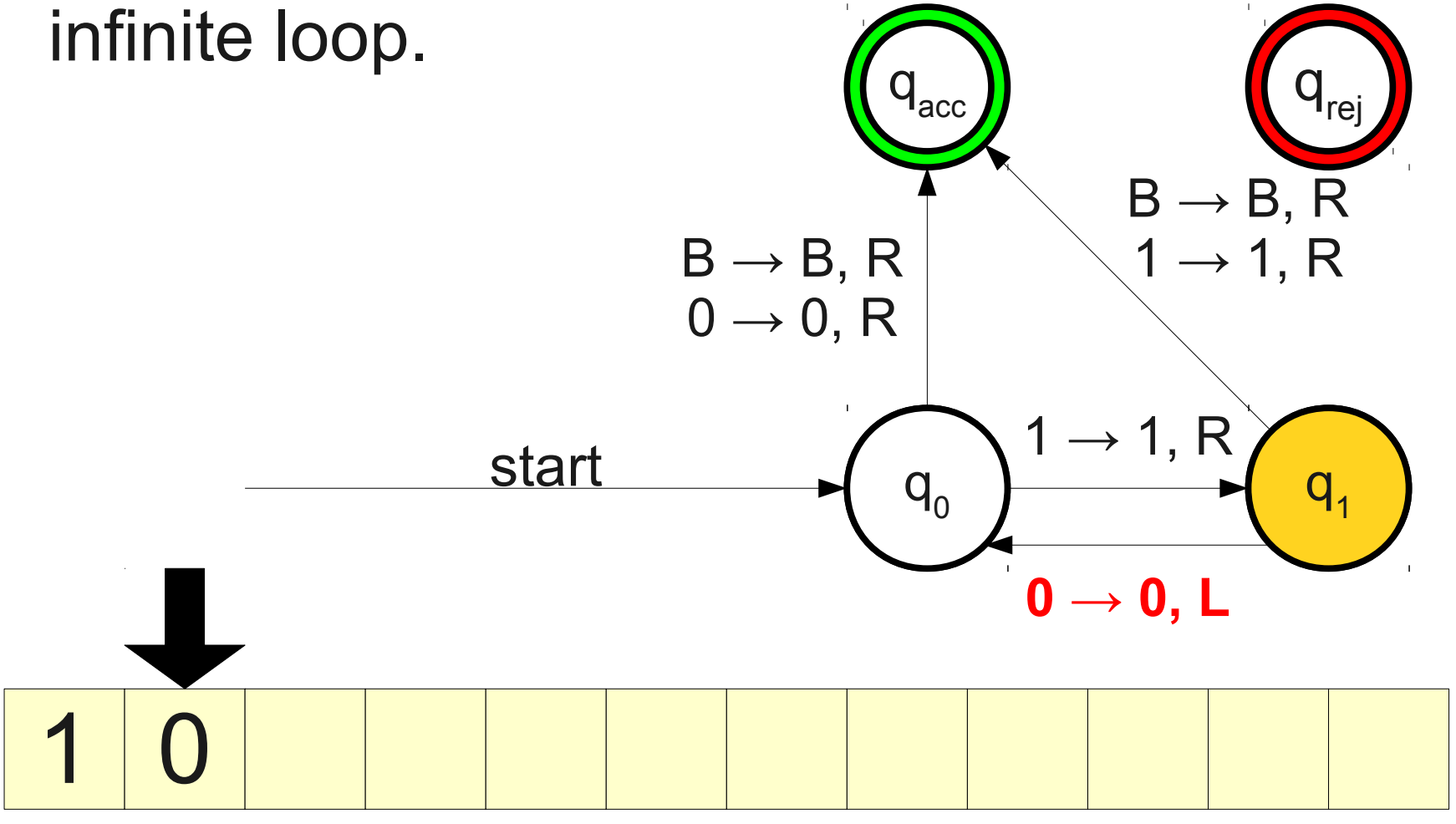

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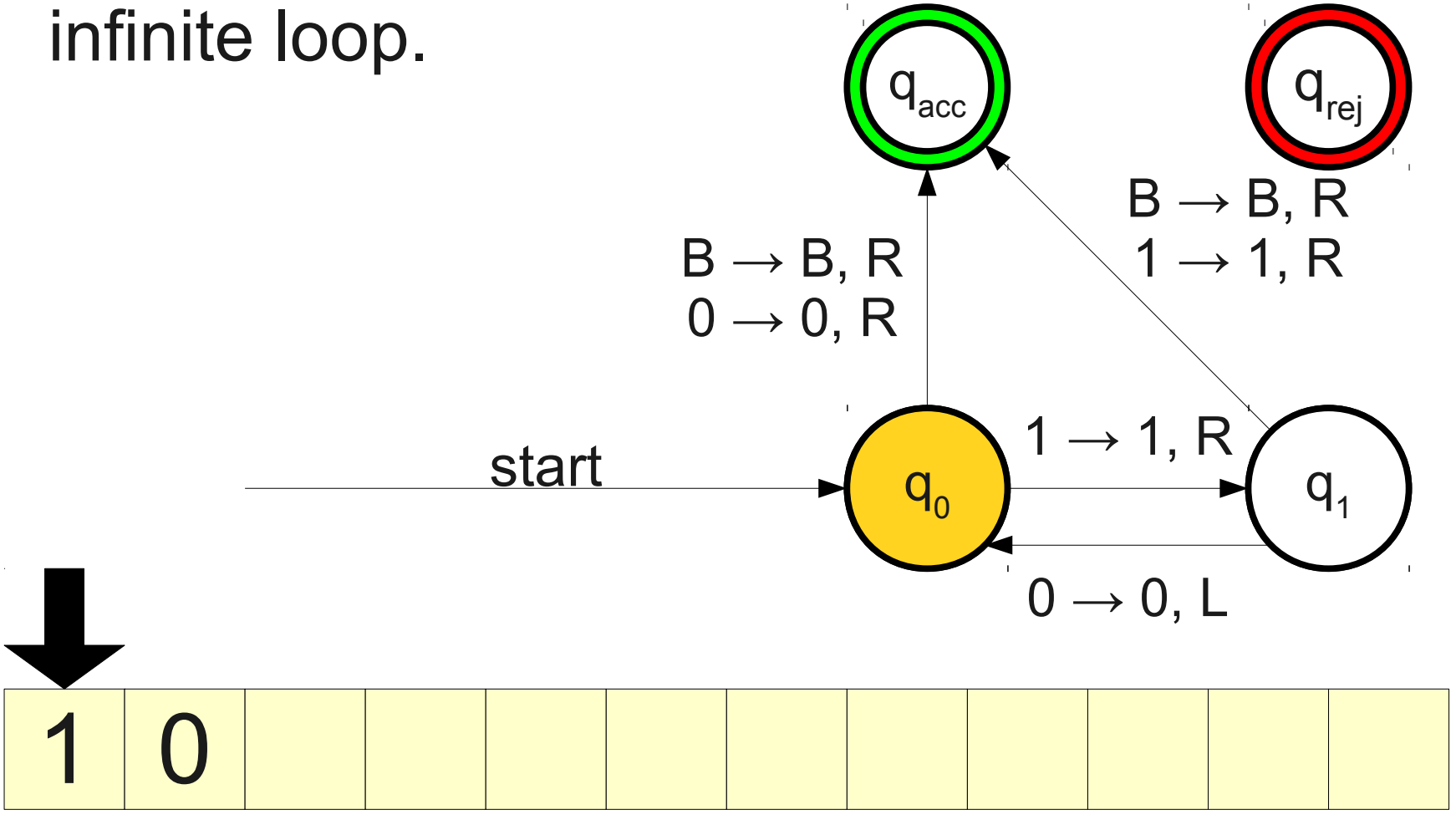

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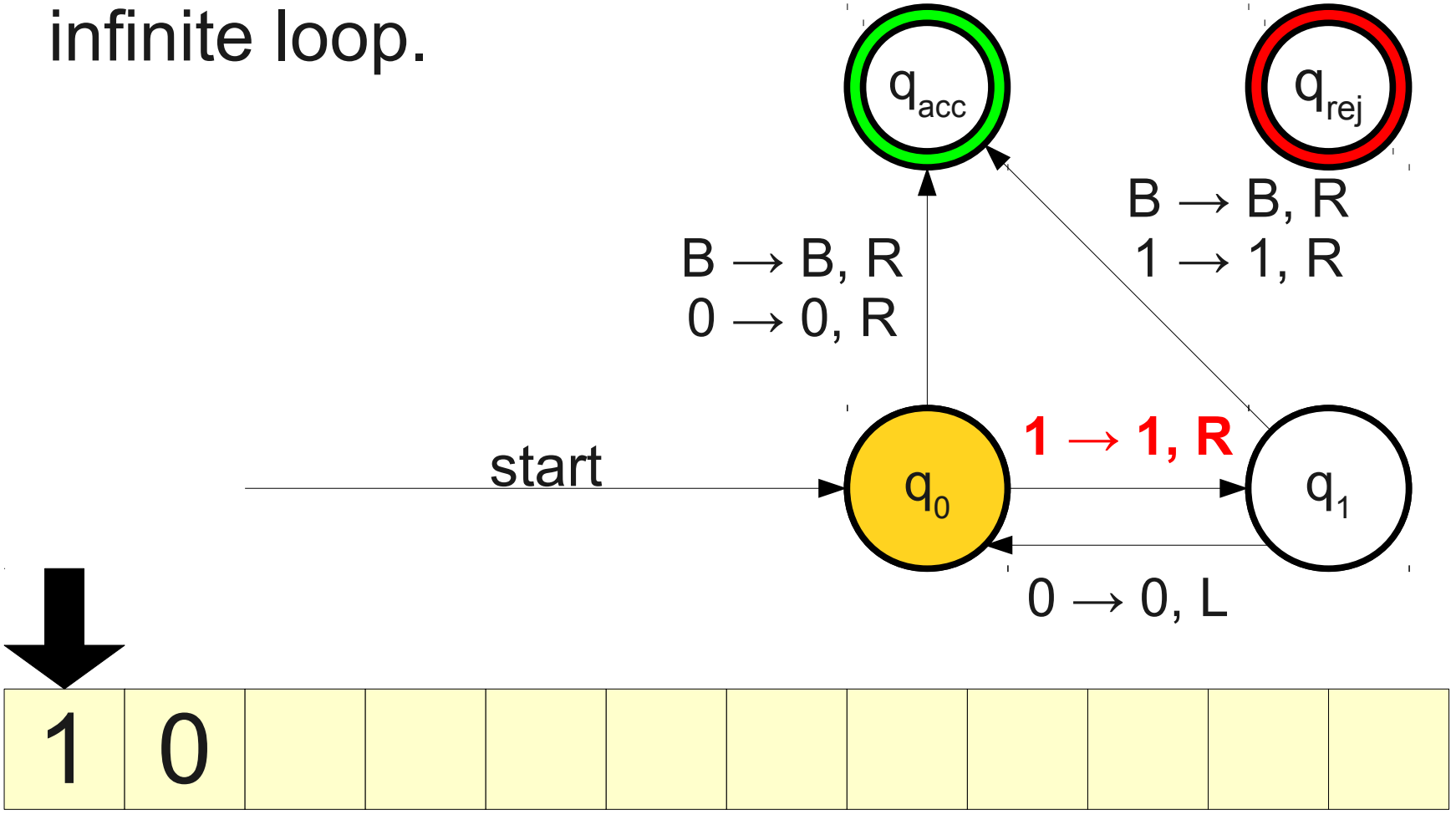

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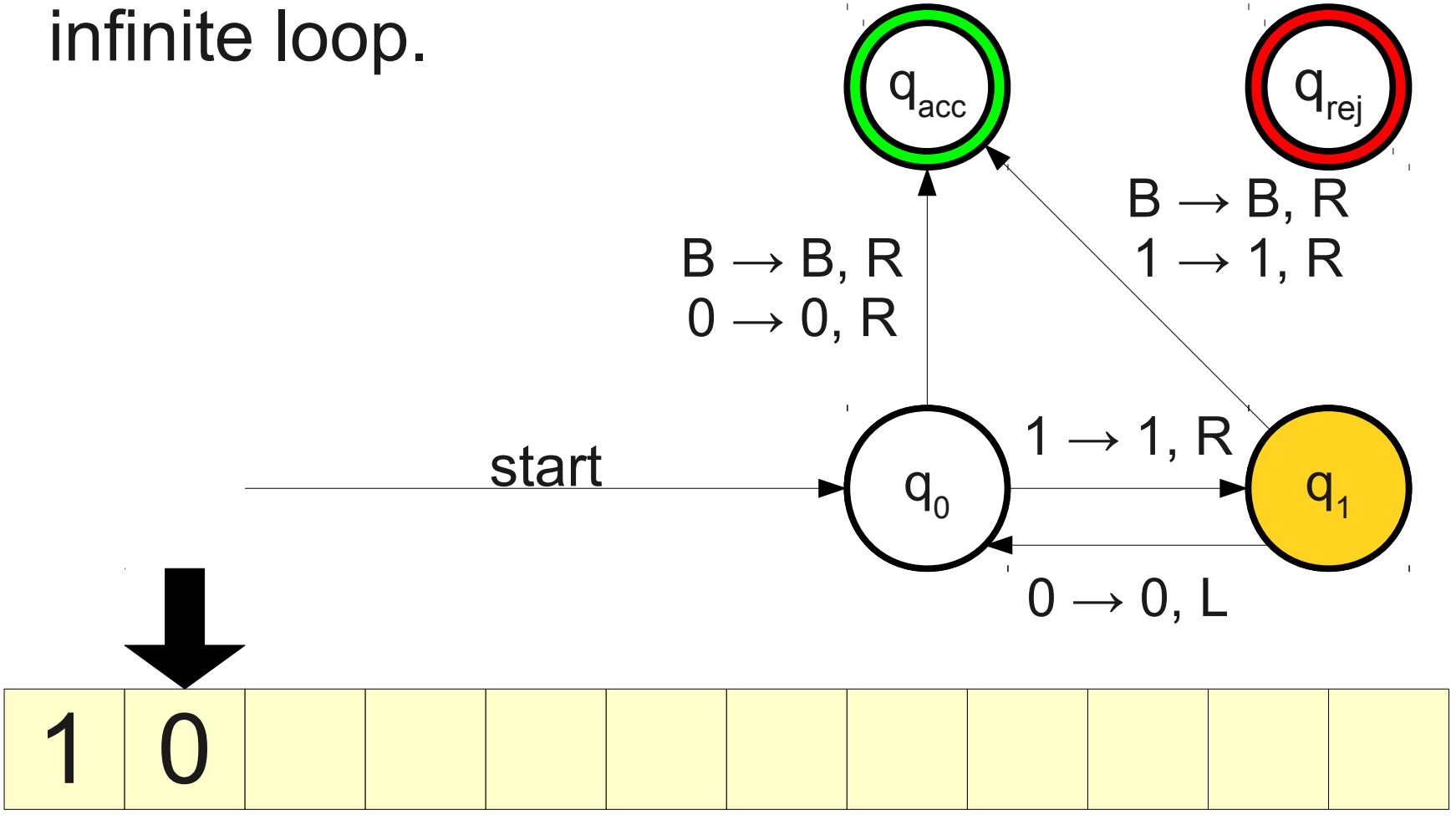

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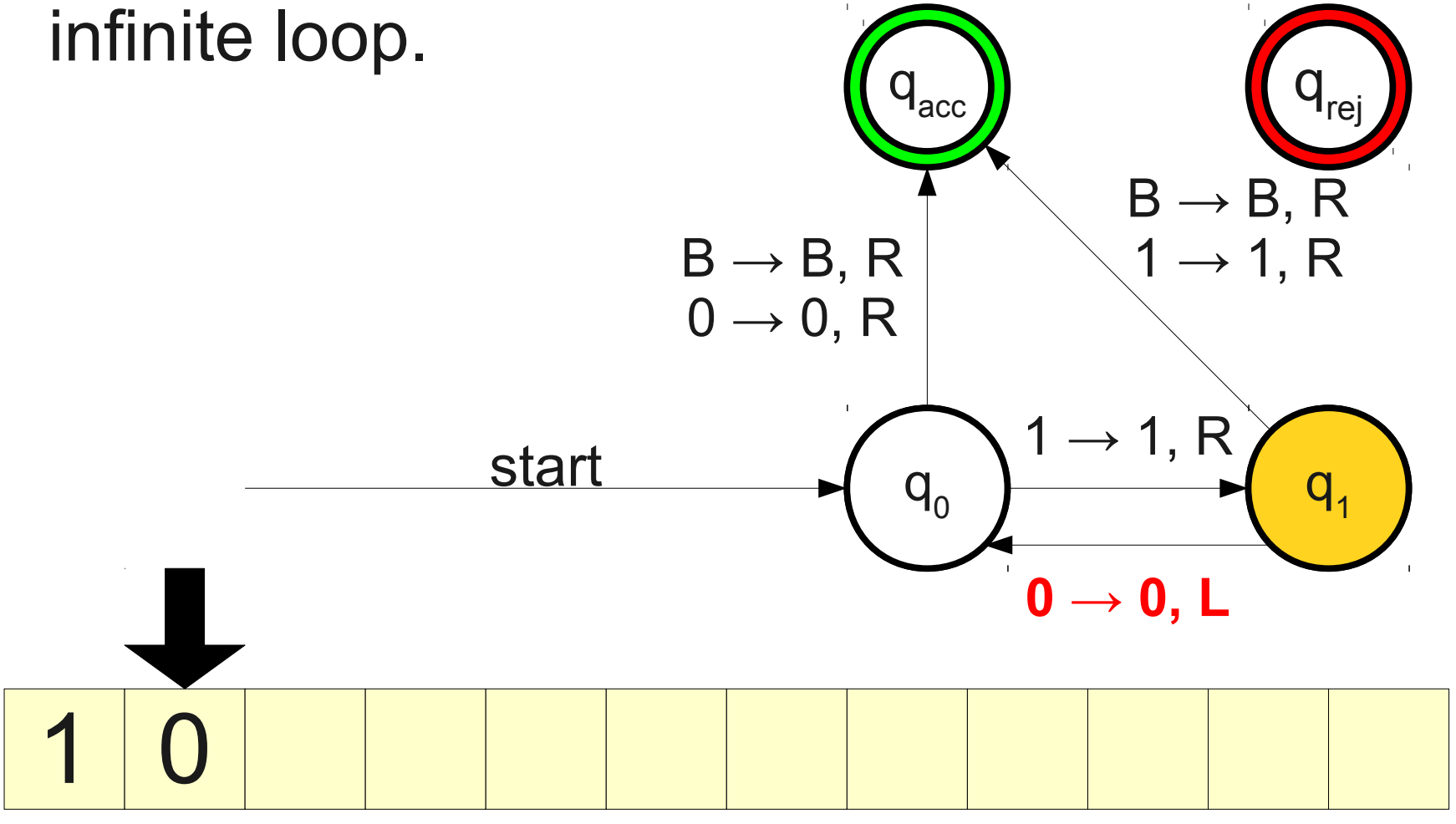

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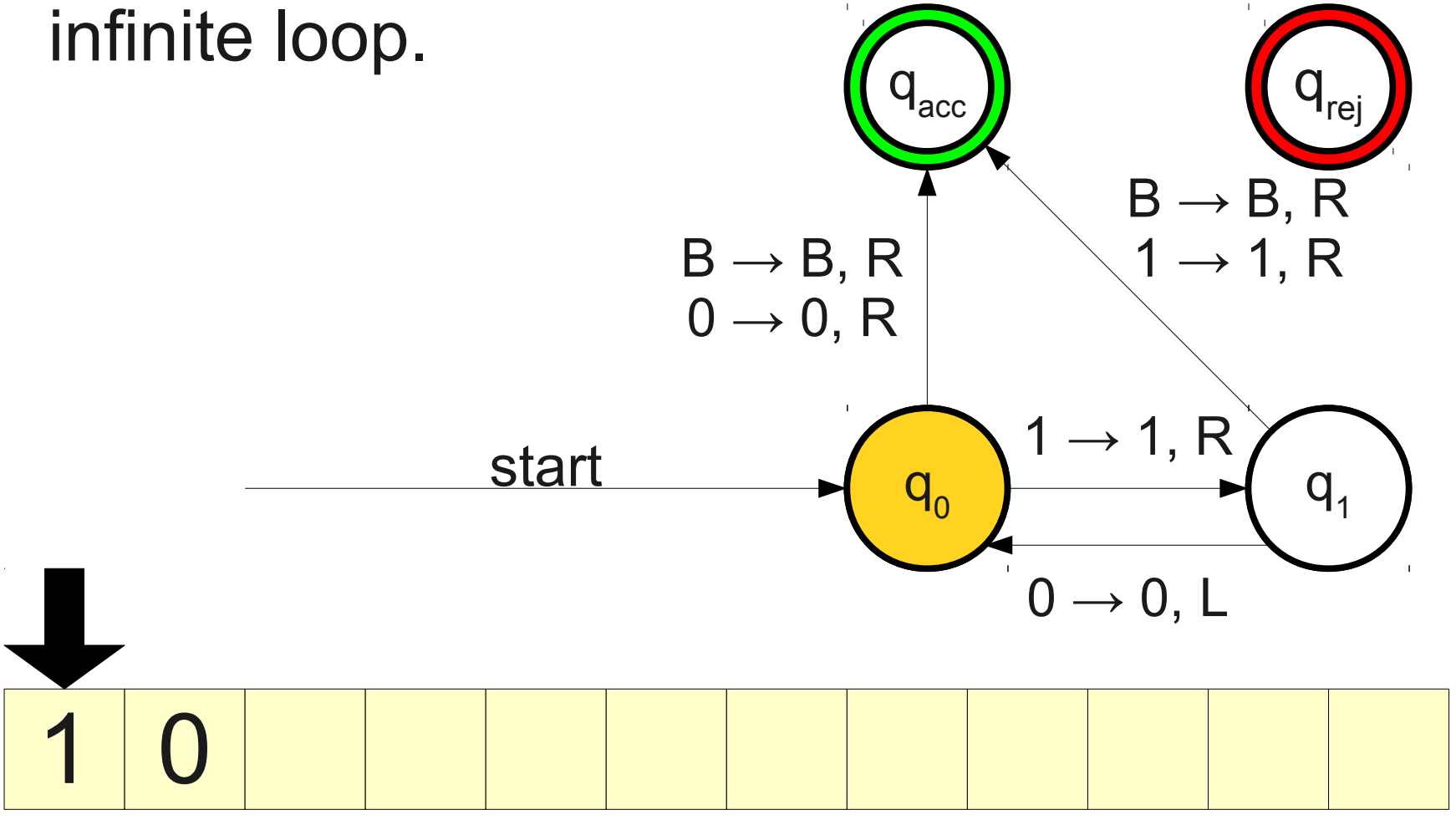

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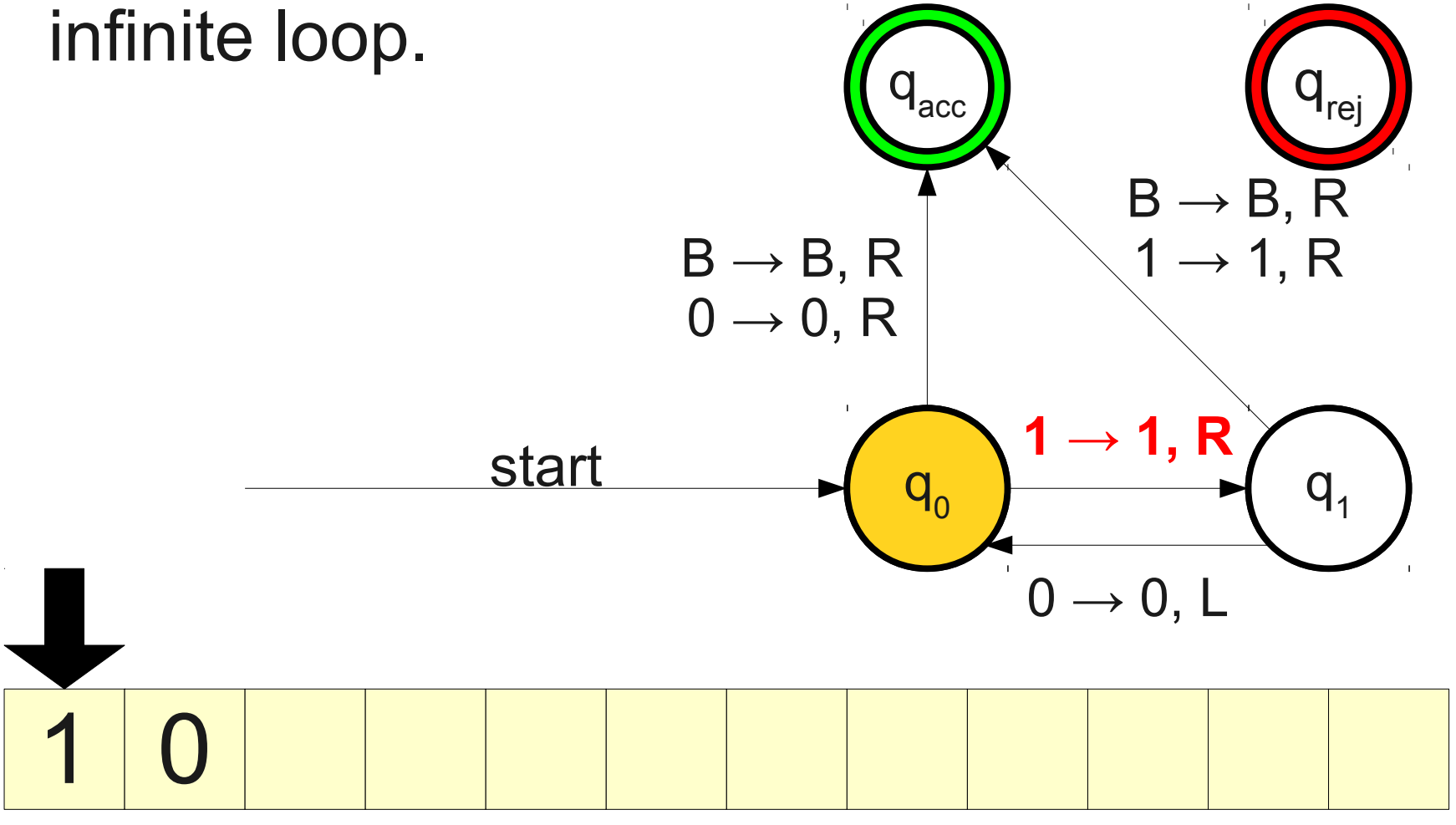

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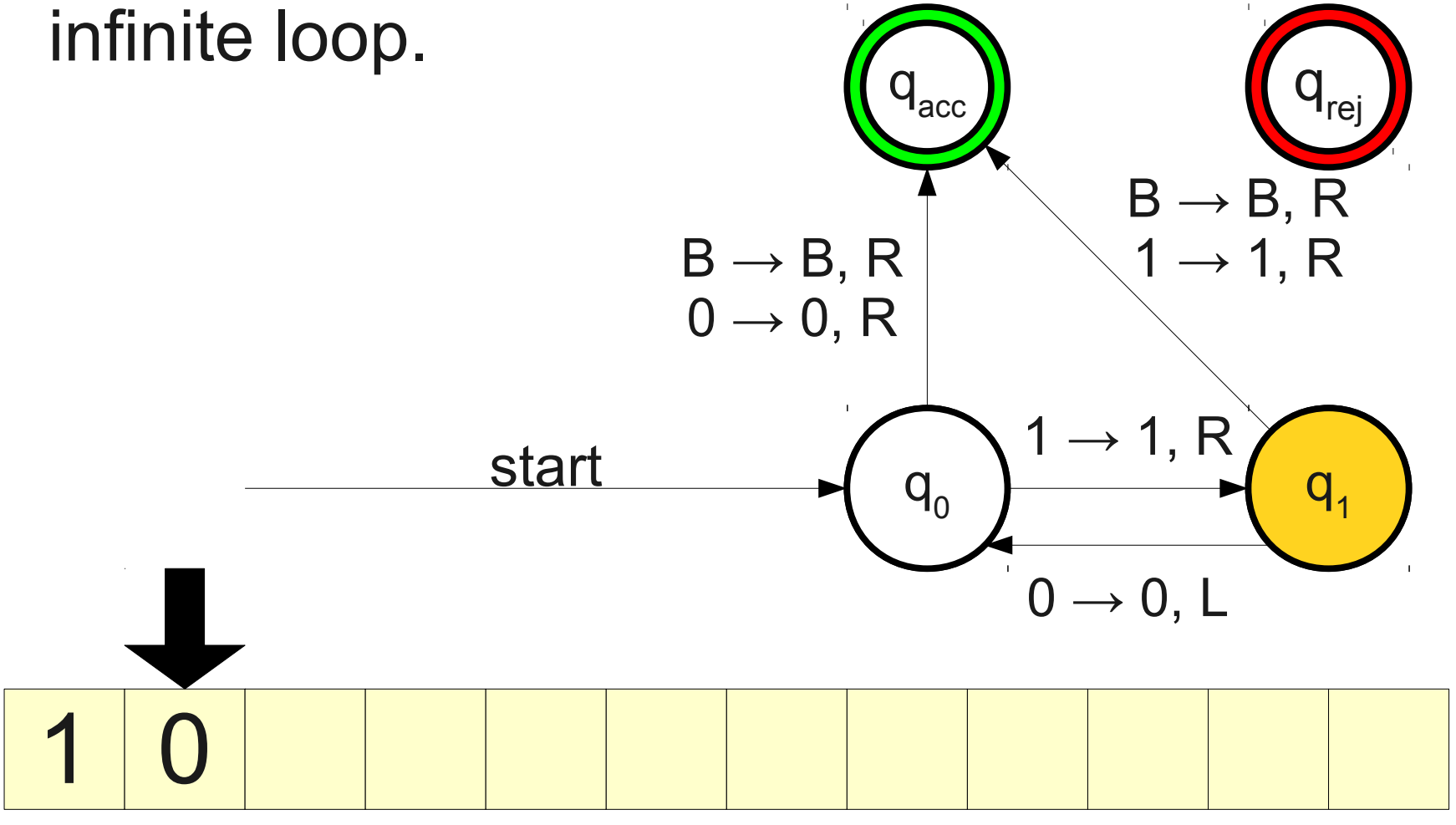

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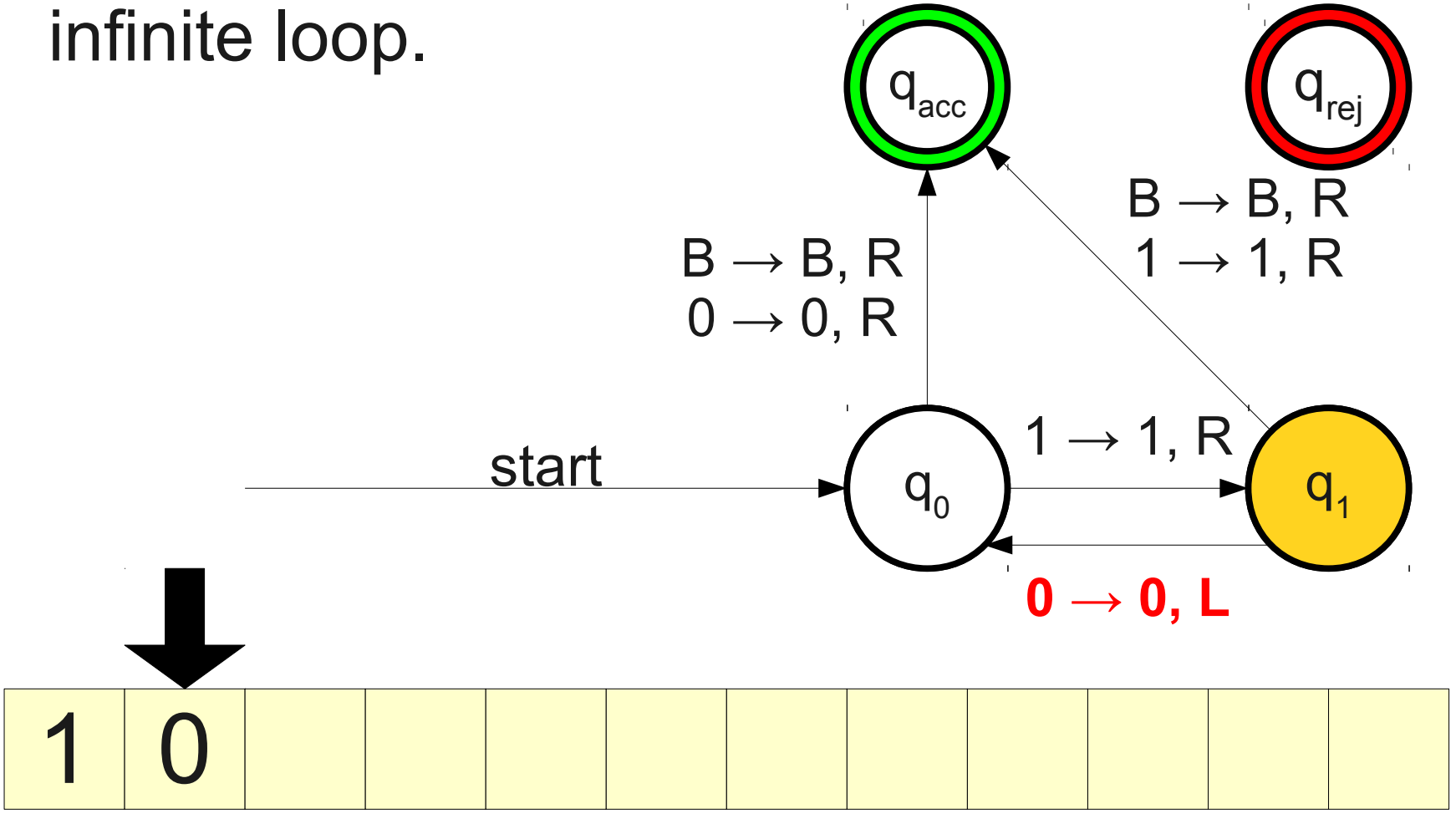

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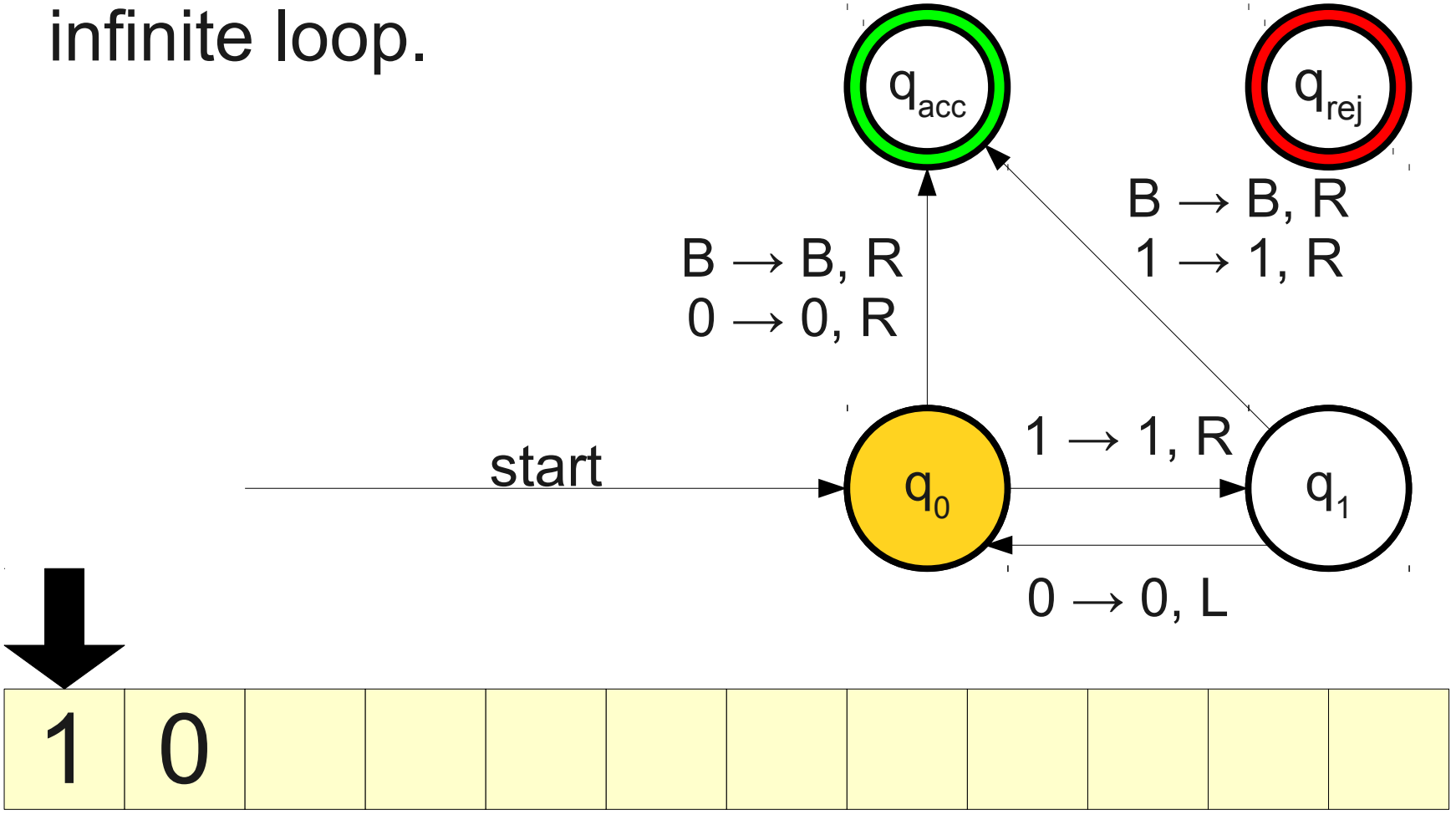

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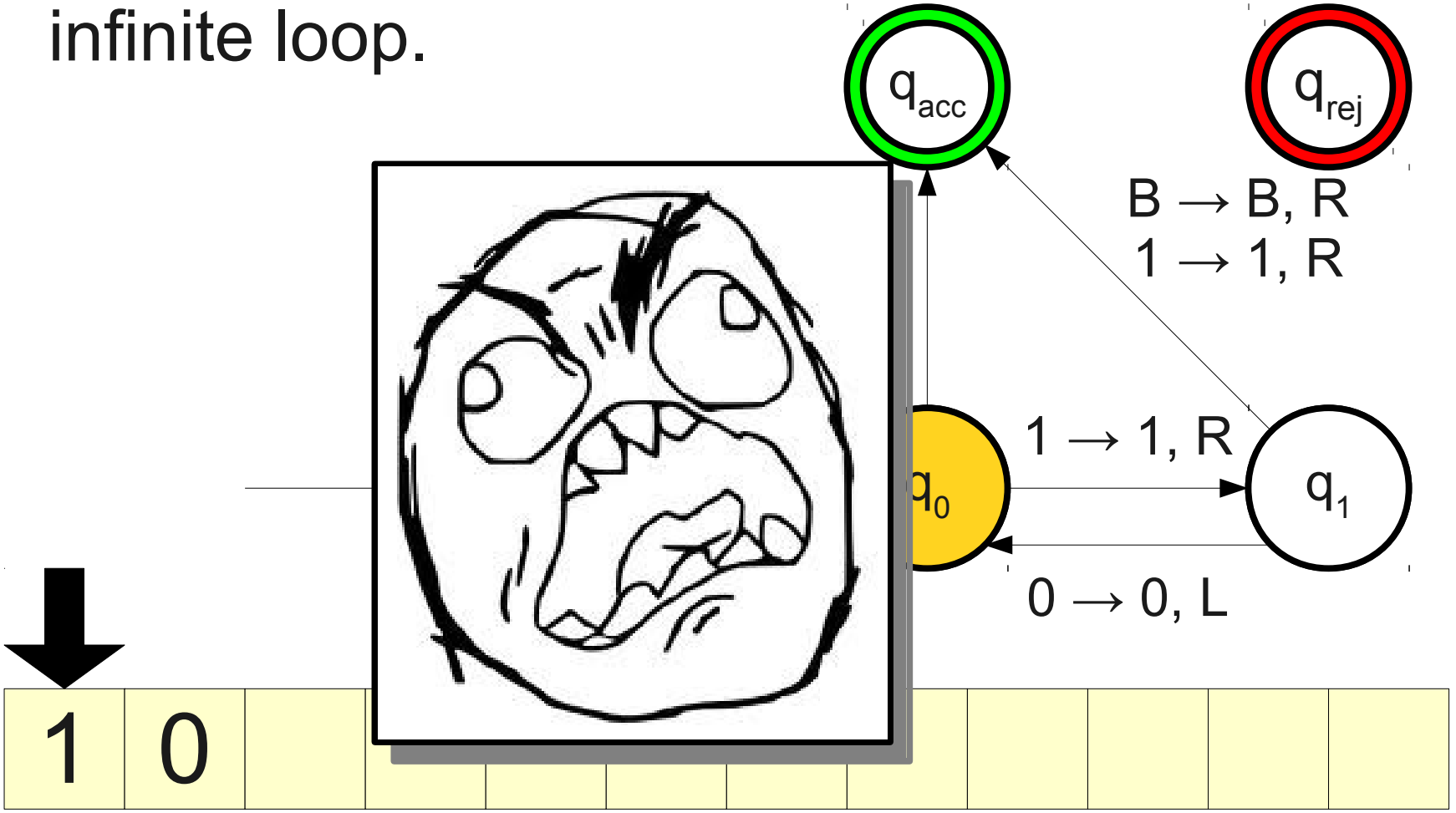
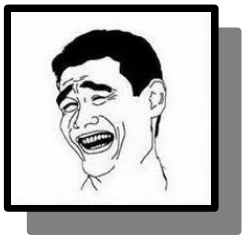
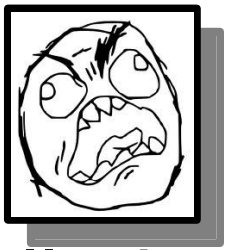


Decidability and Recognizability

- When a TM M is run on a string *w*, it can do one of three things:
  - **accept** by entering q_{acc},
  - **reject** by entering q_{rej}, or
  - **loop forever**, never accepting or rejecting.





- A TM that always accepts or rejects is called a **decider**.

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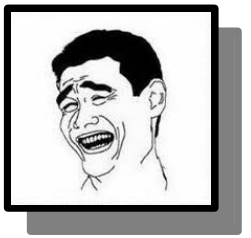

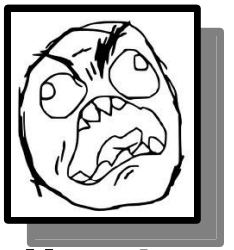

- A TM that always accepts or rejects is called a **decider**.

An Unrecognizable Language

Computing over Strings

- The decision problems we have considered so far are all **language questions** of deciding whether a string is contained within some language.
- Strings can be used to encode other objects:
  - Numbers
  - Paths in graphs
  - Objects being distributed
- Can we encode **Turing machines** as strings?

Encoding a Turing Machine

- To encode a Turing machine, we can provide an encoding of
  - its set of states,
  - its alphabet,
  - its tape alphabet,
  - its transition table,
  - its start state,
  - its accepting state,
  - its rejecting state,
  - the blank symbol

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Let

$\Sigma = \{ s_1, s_2, s_3, \ldots, s_n \}$

Let

$\Gamma = \{ s_1, s_2, \ldots, s_n, s_{n+1}, \ldots, s_{n+m} \}$

Such that $B = s_{n+m}$

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  - **its set of states,**
  - its transition table,
  - **its start state,**
  - **its accepting state,**
  - **its rejecting state,**

We can number the states

$Q = \{q_0, q_1, q_2, \ldots, q_n\}$

Such that

the start state is q_0,
the accepting state is q_{n-1},
and the rejecting state is q_n

Encoding a Turing Machine

- To encode a Turing machine, we can provide an encoding of
  - its transition table

Encoding a Turing Machine

- To encode a Turing machine, we can provide an encoding of
  - its transition table

Encoding the Transition Table

	1			B		
q_0	B	R	q_1	B	R	q_{acc}
q_1	B	R	q_0	B	R	q_{rej}

Encoding the Transition Table

	1			B		
q_0	B	R	q_1	B	R	q_{acc}
q_1	B	R	q_0	B	R	q_{rej}

$(q_0, 1, B, R, q_1)(q_0, B, B, R, q_{acc})$

Encoding the Transition Table

	1			B		
q_0	B	R	q_1	B	R	q_{acc}
q_1	B	R	q_0	B	R	q_{rej}

$(q_0, 1, B, R, q_1)(q_0, B, B, R, q_{acc})$

$(q_1, 1, B, R, q_o)(q_1, B, B, R, q_{rej})$

Encoding the Transition Table

	1			B		
q_0	B	R	q_1	B	R	q_{acc}
q_1	B	R	q_0	B	R	q_{rej}

$(q_0, 1, B, R, q_1)(q_0, B, B, R, q_{acc})$

$(q_1, 1, B, R, q_o)(q_1, B, B, R, q_{rej})$

$(q_0, 1, B, R, q_1)(q_0, B, B, R, q_{acc})(q_1, 1, B, R, q_0)(q_1, B, B, R, q_{rej})$

Encoding Turing Machines

- Using some (scary!) mathematical hackery, we can convert this encoding of a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}, B)$ into a string in Σ^* for **any** choice of Σ.
  - Details are tricky. Talk to me after lecture if you want to see how to do this.
- If M is a TM, let $\langle M \rangle$ be the encoding of M using its own alphabet.
  - More generally, if O is some object, then $\langle O \rangle$ is a string encoding of that object.
- **This allows TMs to compute over TMs.**

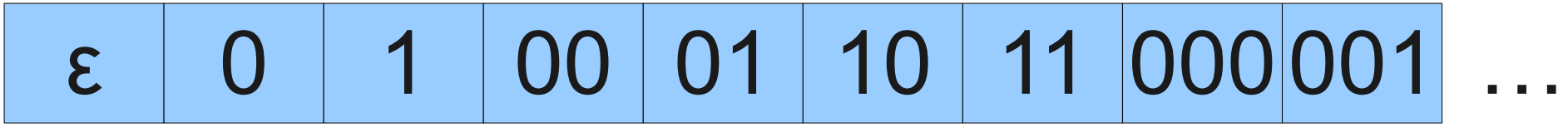
Enumerating Strings

- We can list off all strings over some alphabet in order by sorting by length, then alphabetically.
- For example:

  ε, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, …

- Some strings encode TMs; others do not.
- Every TM has an encoding.

Enumerating Turing Machines



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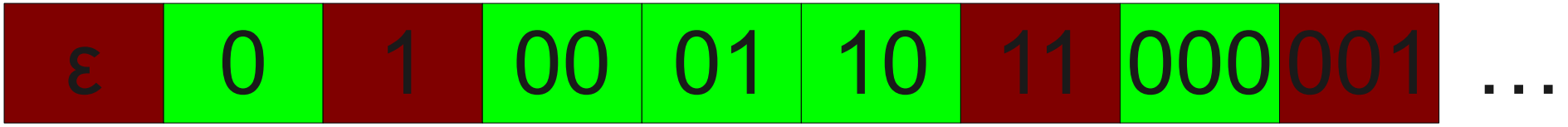

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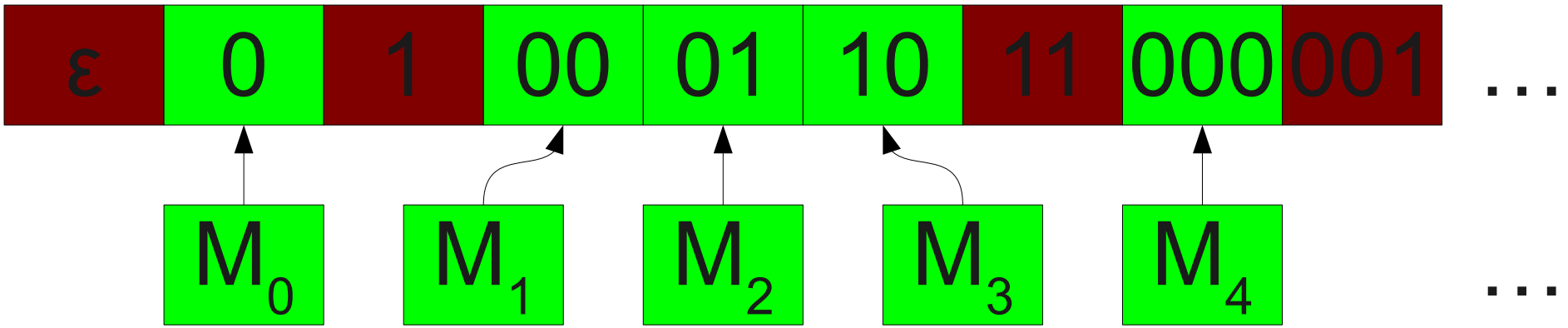

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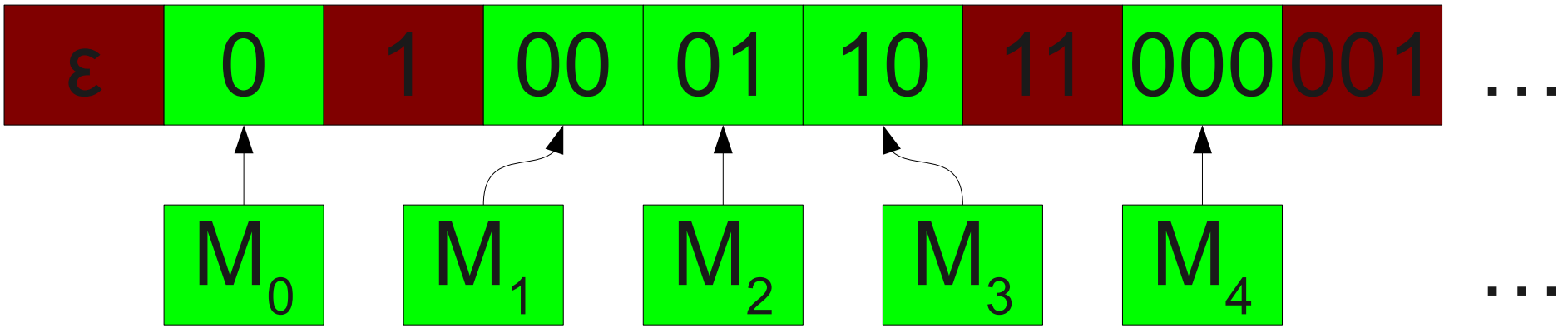


Every Turing machine appears somewhere in this sequence.

Enumerating Strings

- We can list off all strings over some alphabet in order by sorting by length, then alphabetically.
- For example:

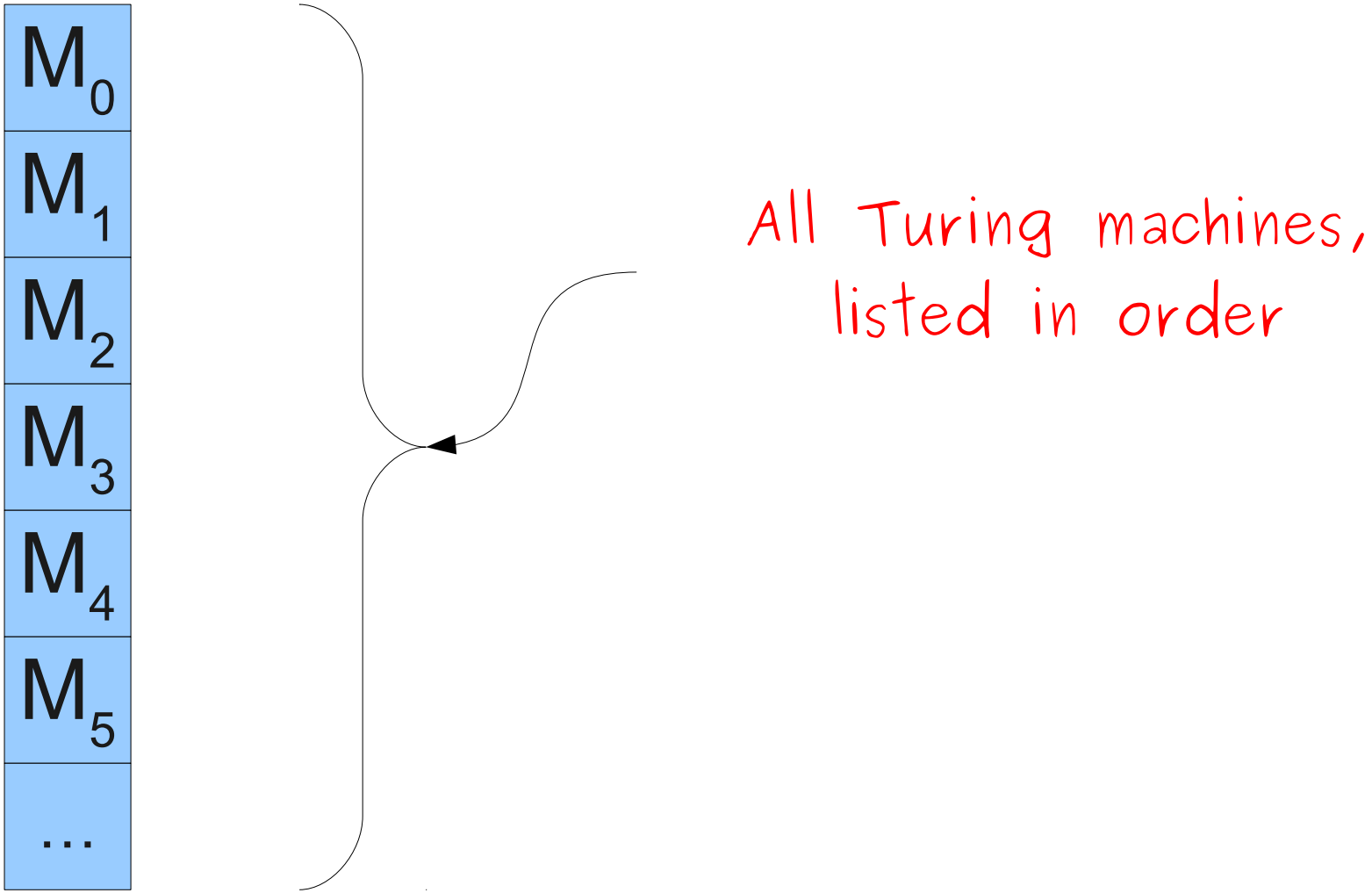
  ε, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, …

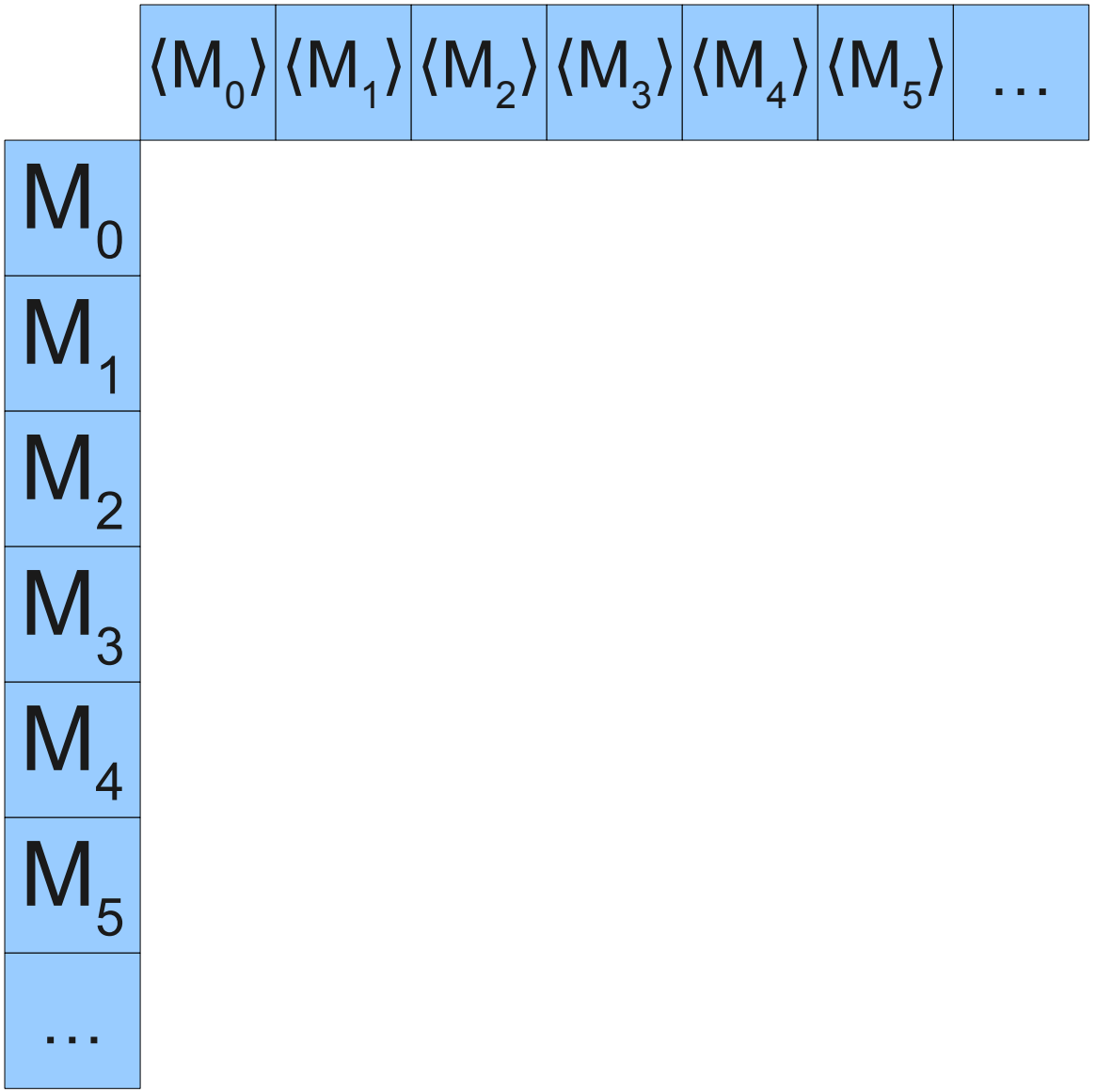
- Some strings encode TMs; others do not.
- Every TM has an encoding.
- We can list off all TMs as M_0, M_1, M_2, … as follows:
  - M_0 is the TM encoded by first string that encodes a TM.
  - M_{n+1} is the TM encoded by the first string after the one encoding M_n that encodes a TM.

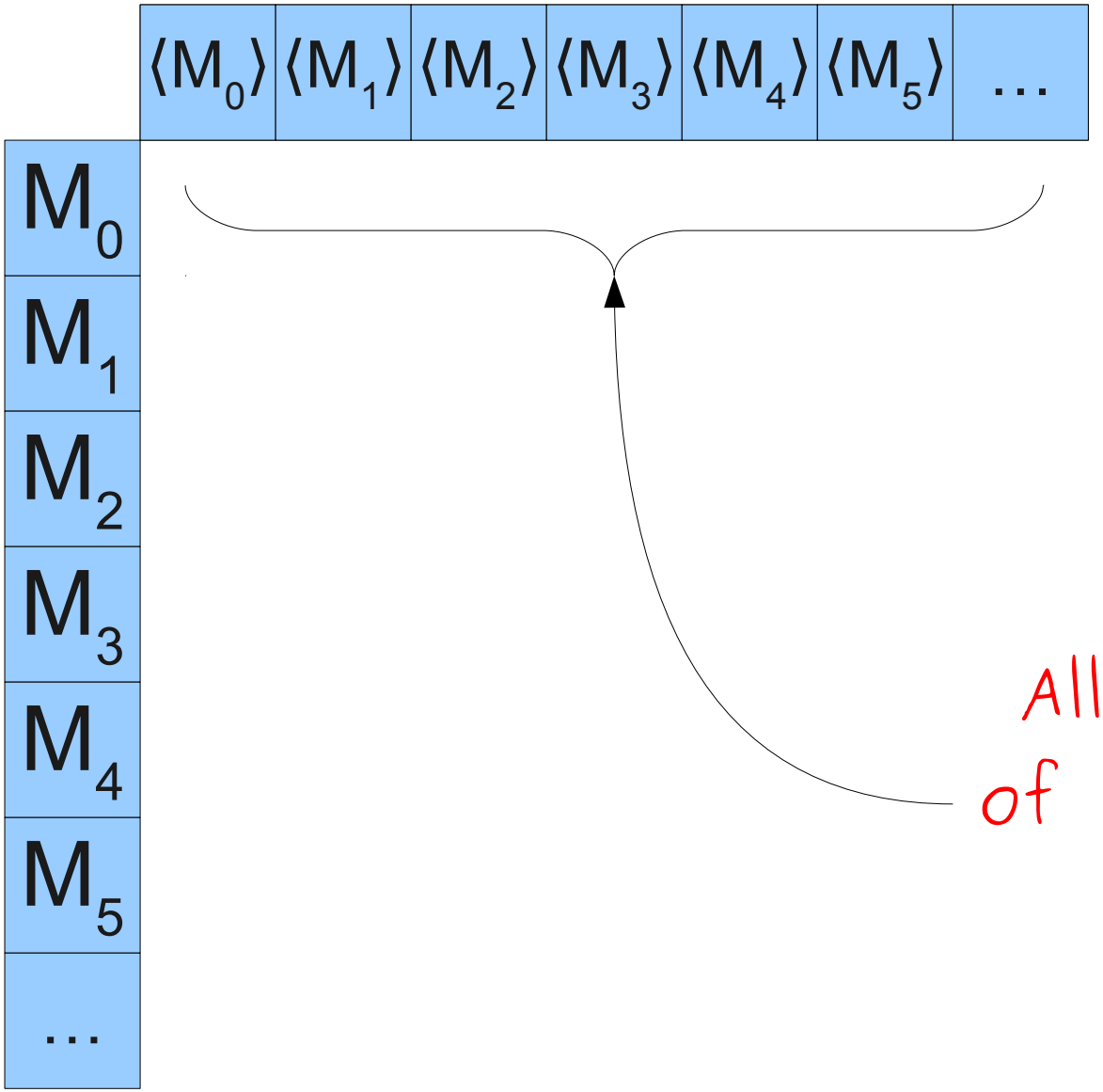
A Critical Observation

Every TM M either accepts ⟨M⟩ or it does not
(it either rejects or loops infinitely).


M_0
M_1
M_2
M_3
M_4
M_5
...


M_0
M_1
M_2
M_3
M_4
M_5
...
All Turing machines,
listed in order


⟨M0⟩
⟨M1⟩
⟨M2⟩
⟨M3⟩
⟨M4⟩
⟨M5⟩
…
M0
M1
M2
M3
M4
M5
…



All descriptions of TMs, listed in order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1							
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

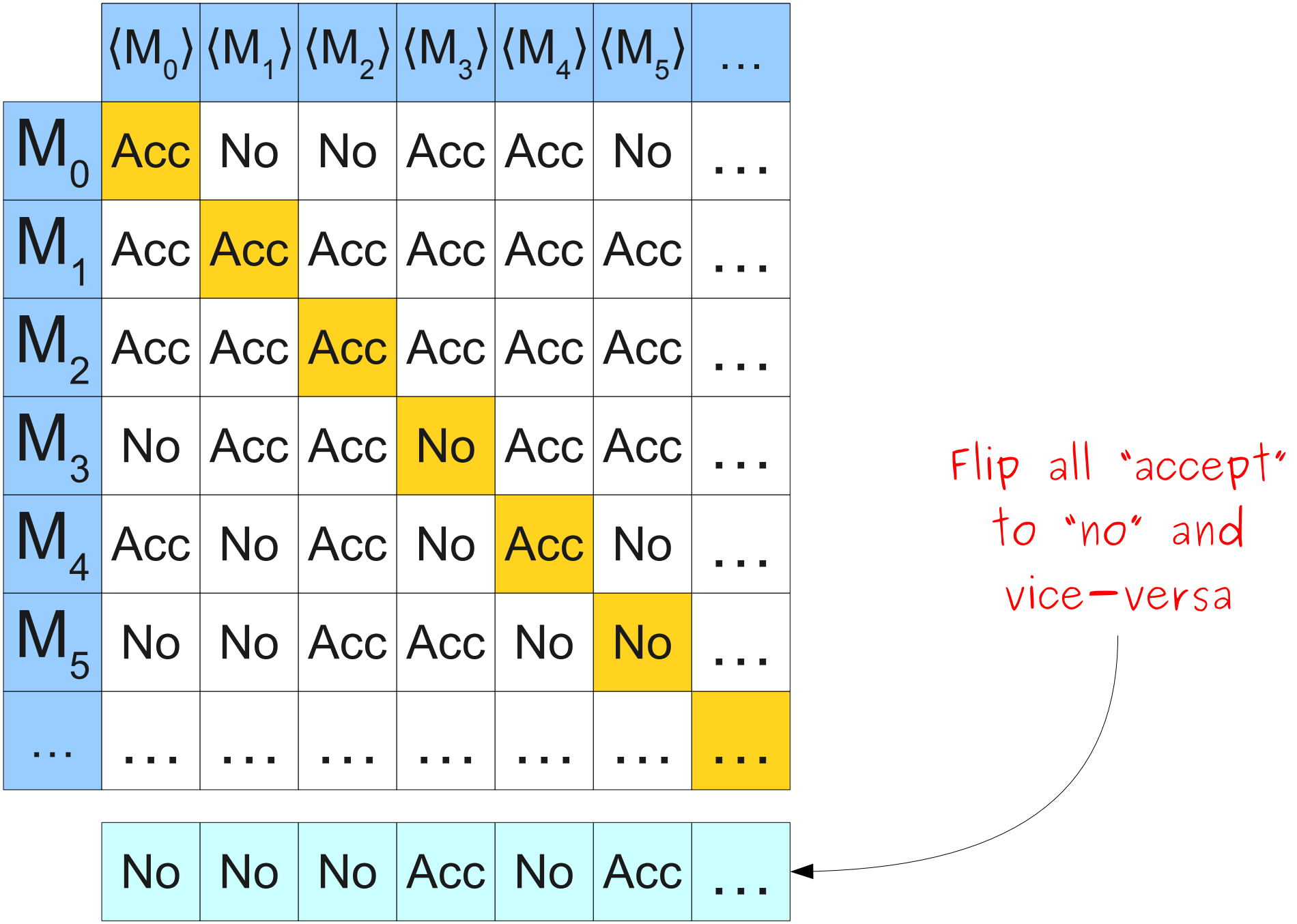
	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

Acc	Acc	Acc	No	Acc	No	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	No	No	No	Acc	No	Acc	…


⟨M0⟩ ⟨M1⟩ ⟨M2⟩ ⟨M3⟩ ⟨M4⟩ ⟨M5⟩ …
M0 Acc No No Acc Acc No …
M1 Acc Acc Acc Acc Acc Acc …
M2 Acc Acc Acc Acc Acc Acc …
M3 No Acc Acc No Acc Acc …
M4 Acc No Acc No Acc No …
M5 No No Acc Acc No No …
… … … … … … … …
No No No Acc No Acc …
Flip all "accept" to "no" and vice-versa

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

What TM has this behavior?

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
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M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
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M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
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M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
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M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
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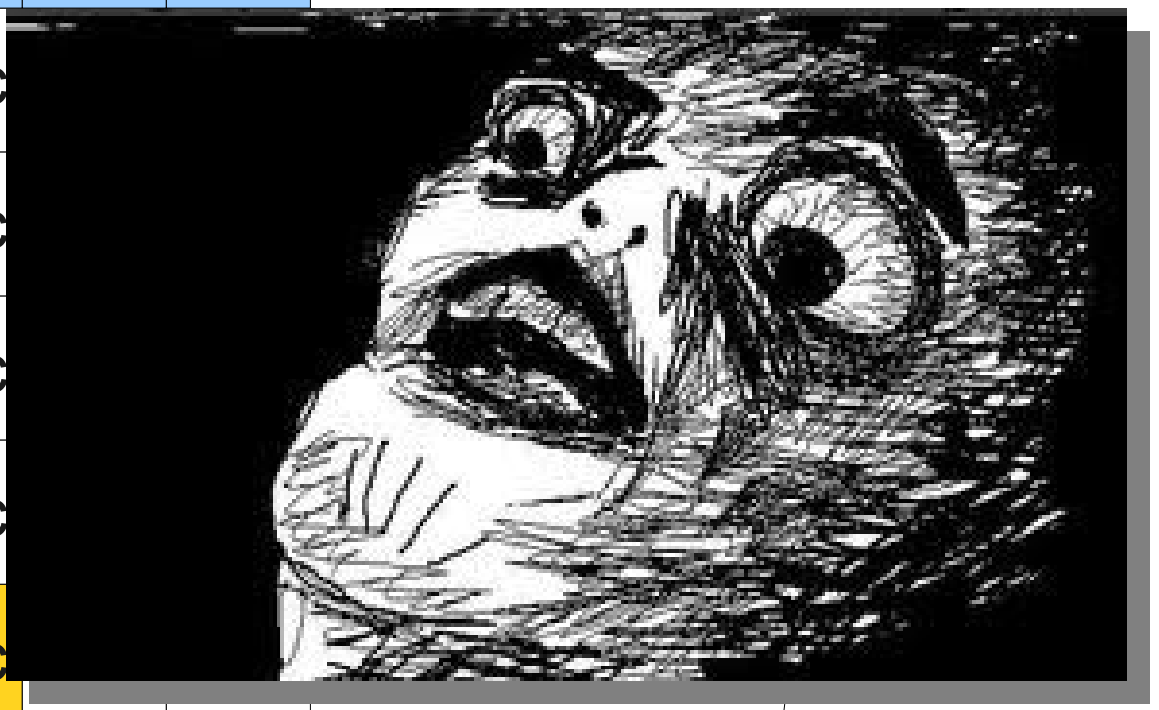
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…
	No	No	No	Acc	No	Acc	…

No TM has this behavior!

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc		
M_1	Acc	Acc	Acc	Acc	Acc		
M_2	Acc	Acc	Acc	Acc	Acc		
M_3	No	Acc	Acc	No	Acc		
M_4	Acc	No	Acc	No	Acc		
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…



No	No	No	Acc	No	Acc	…

A Deviously Tricky Problem

- The **diagonalization language** L_D is defined as

$$L_D = \{ w \mid w = \langle M \rangle \text{ for some TM } M, \text{ and } M \text{ does not accept } \langle M \rangle \}$$

- We can alternatively write this as

$$L_D = \{ \langle M \rangle \mid M \text{ does not accept } \langle M \rangle \}$$

- That is, L_D is the set of descriptions of Turing machines that do not accept themselves.
- Is L_D Turing-recognizable?

$$L_D = \{ \langle M \rangle \mid M \text{ does not accept } \langle M \rangle \}$$

Theorem: L_D is not Turing-recognizable.

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Proof: By contradiction; assume that L_D is Turing-recognizable.

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Theorem: L_D is not Turing-recognizable.

Proof: By contradiction; assume that L_D is Turing-recognizable. Then there must be a TM H such that $L(H) = L_D$, so H accepts *w* iff it is contained in L_D.

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Case 1: H accepts $\langle H \rangle$. Then $\langle H \rangle \in L_D$, so H does not accept $\langle H \rangle$. This is impossible, since H does indeed accept $\langle H \rangle$.

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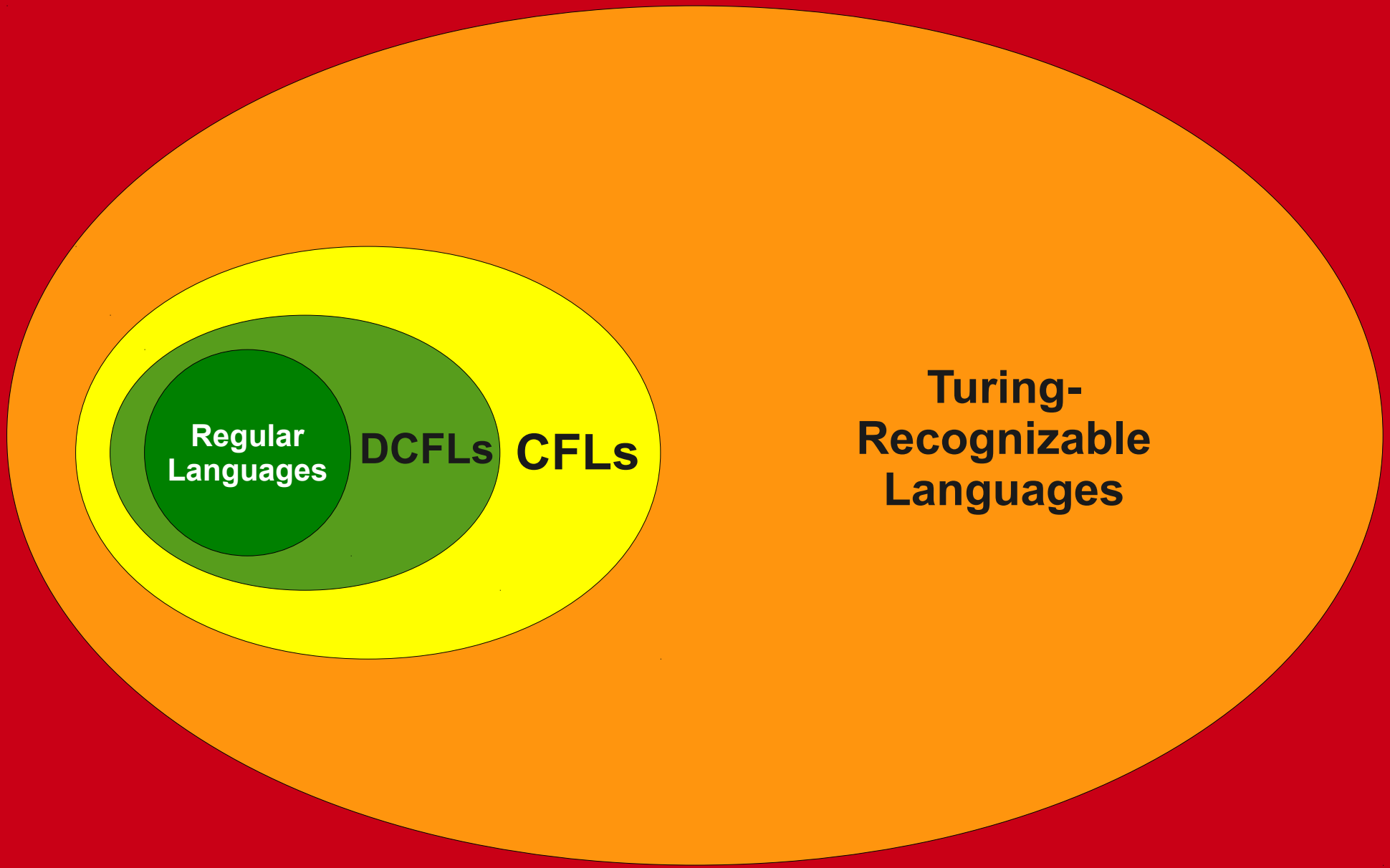
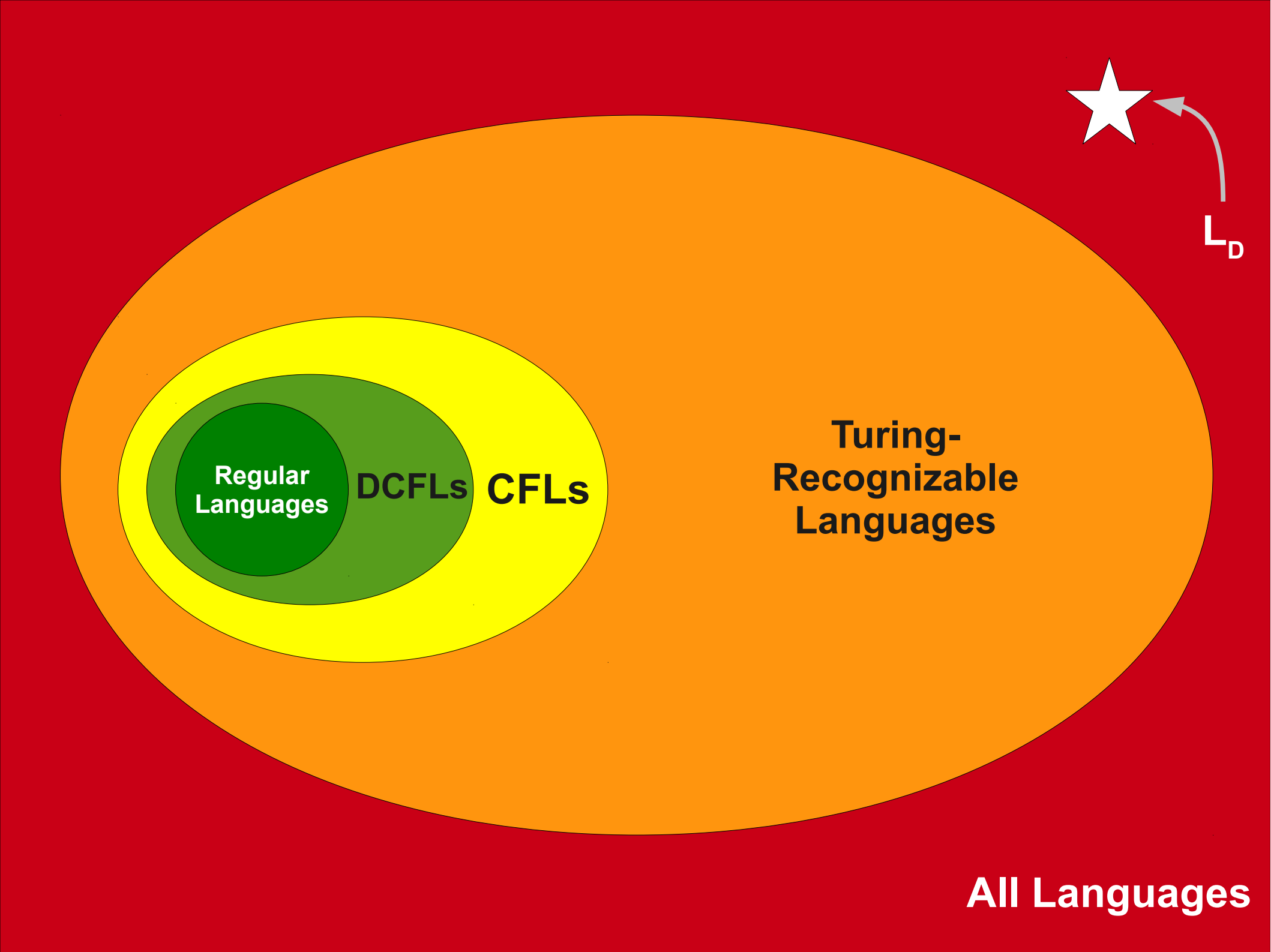

Regular Languages
DCFLs
CFLs
Turing-Recognizable Languages
All Languages


L_D
Regular Languages
DCFLs
CFLs
Turing-Recognizable Languages
All Languages

What Just Happened?

- What is it about L_D that makes it impossible to solve with a Turing machine?
- **Indirect self-reference.**
- Because TMs can be encoded as strings, TMs that compute over other TMs can be forced to compute some property of themselves **without realizing it**.
- The language L_D self-destructs given a Turing machine that recognizes L_D by stating “this machine accepts itself if and only if it does not accept itself.”

Diagonalization Revisited

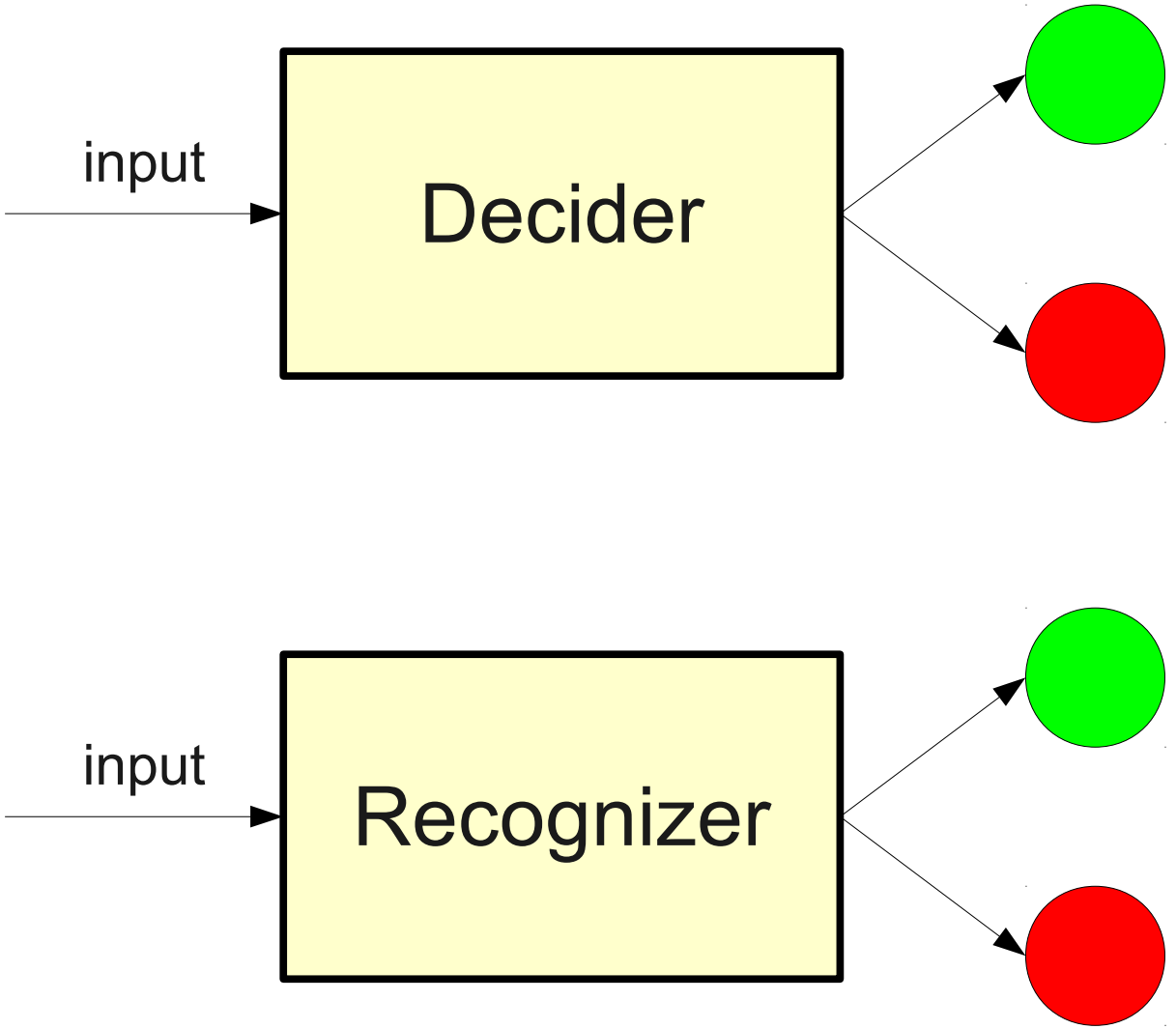
- At our first lecture, we saw Cantor's diagonal argument to show that |S| < |P(S)| for any choice of S.
- The proof that L_D is undecidable is almost **exactly** the same proof.
  - The language of a TM may contain descriptions of TMs.
  - We can draw a matrix showing which TMs are accepted by which TMs.
  - The complemented diagonal cannot be accepted by any TM, because each TM disagrees with itself.
  - So there is some language not recognized by any TM.

An Undecidable Problem

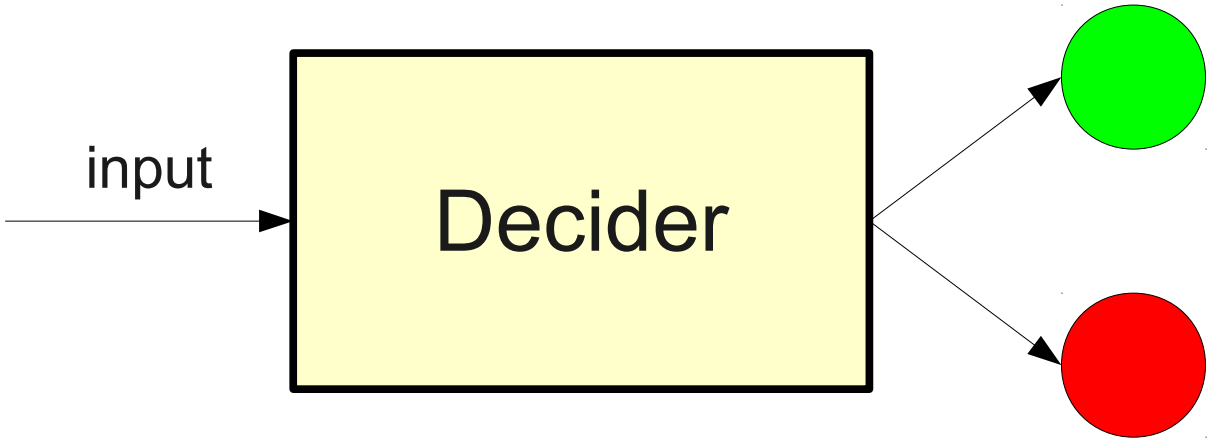
Decidability and Recognizability

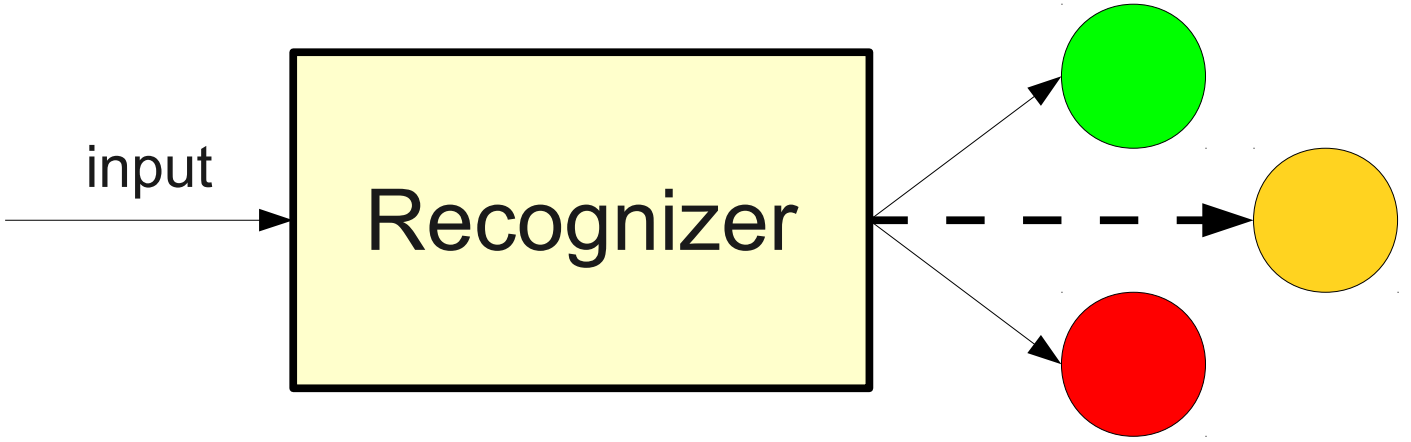
- L is **Turing-recognizable** if there is a Turing machine M that accepts every string in L.
  - It might loop forever on strings not in L.
- L is **Turing-decidable** (if there is a Turing machine M that accepts every string in L and rejects every string not in L.
  - It never enters an infinite loop.
  - These languages are also sometimes called **computable**, **decidable**, or **recursive**.

Decidability and Recognizability



Decidability and Recognizability





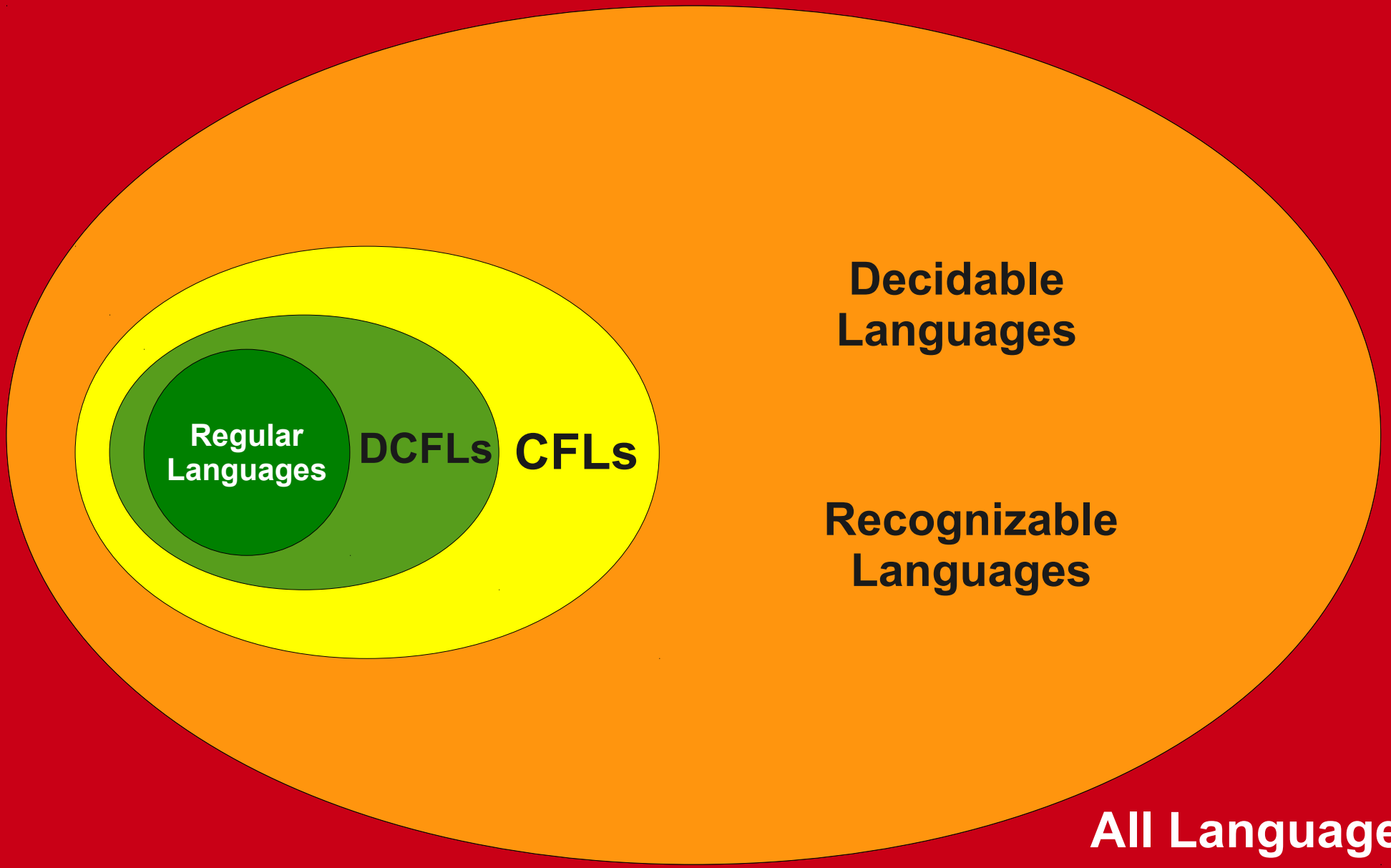
Decidable Problems

- The Church-Turing thesis asserts that any feasible model of computation is no more powerful than a Turing machine.
- The Turing-decidable languages are, therefore, problems for which there is some computer that can always produce a yes or no answer.
- A problem is decidable precisely when there is some algorithm to solve it.
- **Decidability formalizes the definition of an algorithm.**

Decidable Problems

- Any regular language is decidable.
  - Obtain a DFA for L.
  - Convert L into a TM by adding accept and reject states.
  - Make one pass over the input, then accept if in an accepting state and reject otherwise.
- Any DCFL is decidable.
  - Obtain a DPDA for L.
  - Use a multitape TM to simulate consuming the input and maintaining the stack.
- Any CFL is decidable.
  - A trickier proof; see **Earley's algorithm** for details on how to do this.

Which Picture is Correct?



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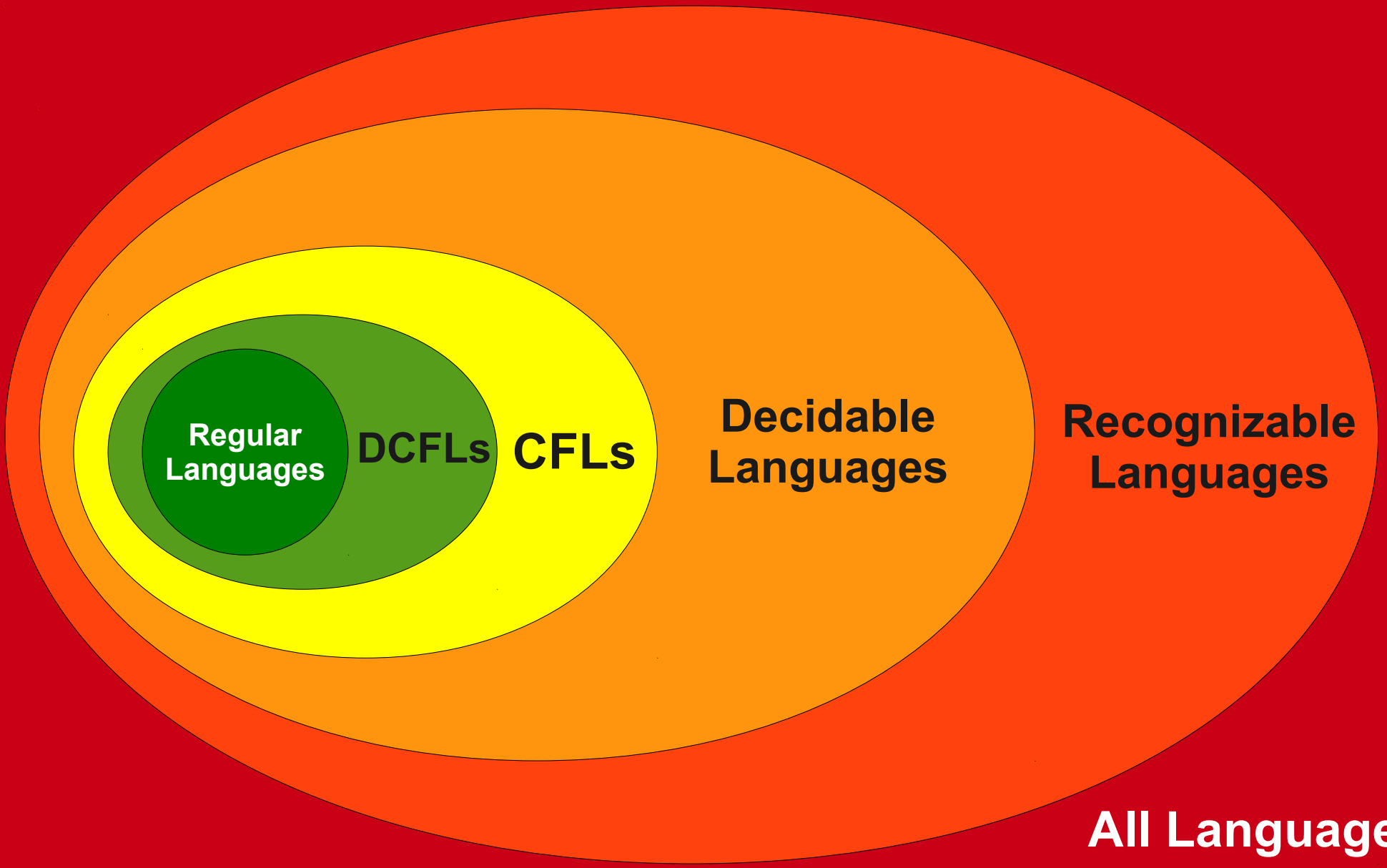
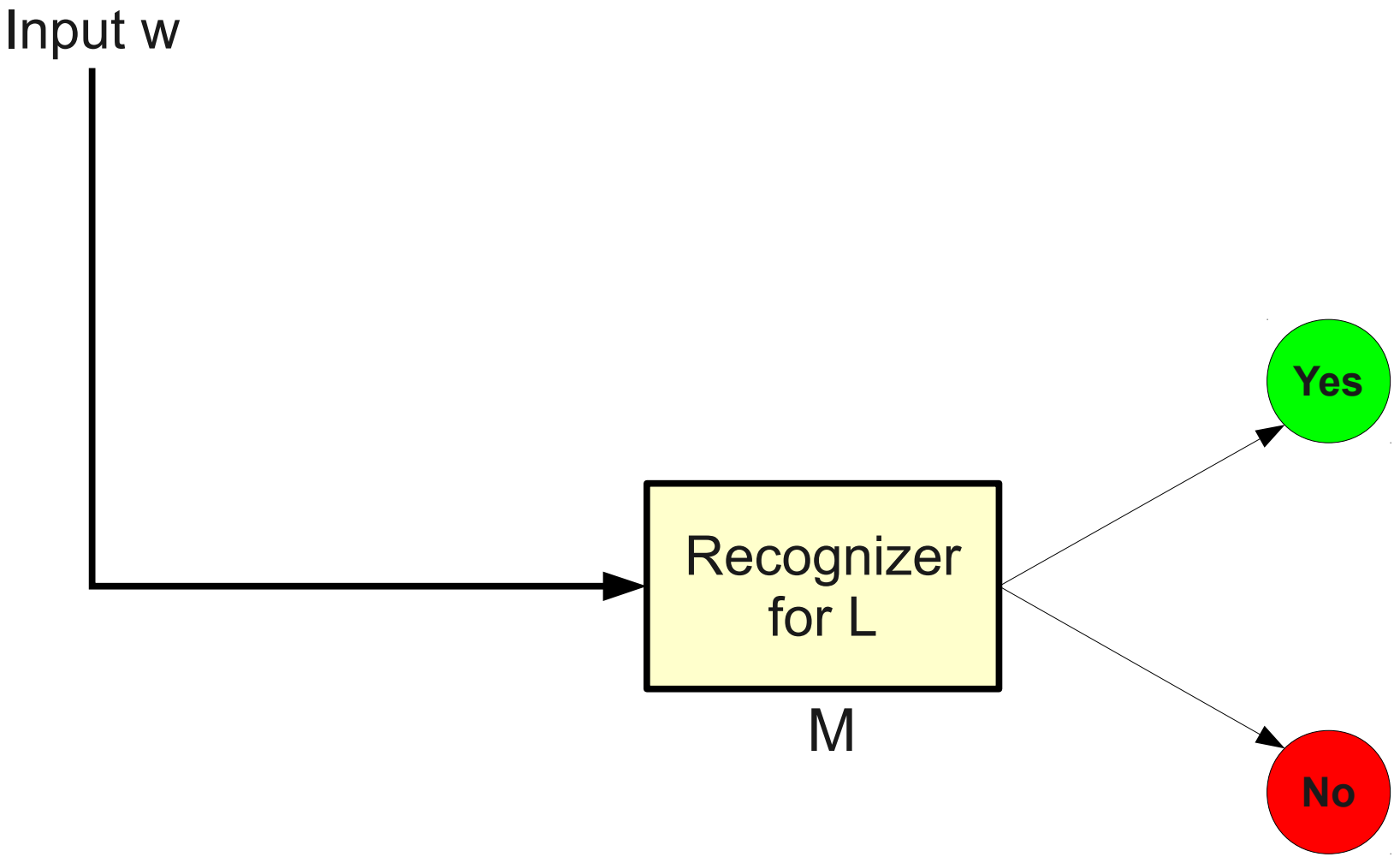


From Recognizers to Deciders

- Recall: When running a TM on an input *w*, the TM will do exactly one of the following:
  - **Accept** w,
  - **Reject** w, or
  - **Loop forever**.
- If we could detect whether the TM could loop forever, then we could build a decider for it.
  - Check whether the TM will eventually halt.
  - If so, run the TM and output its result.
  - If not, immediately reject.

From Recognizers to Deciders



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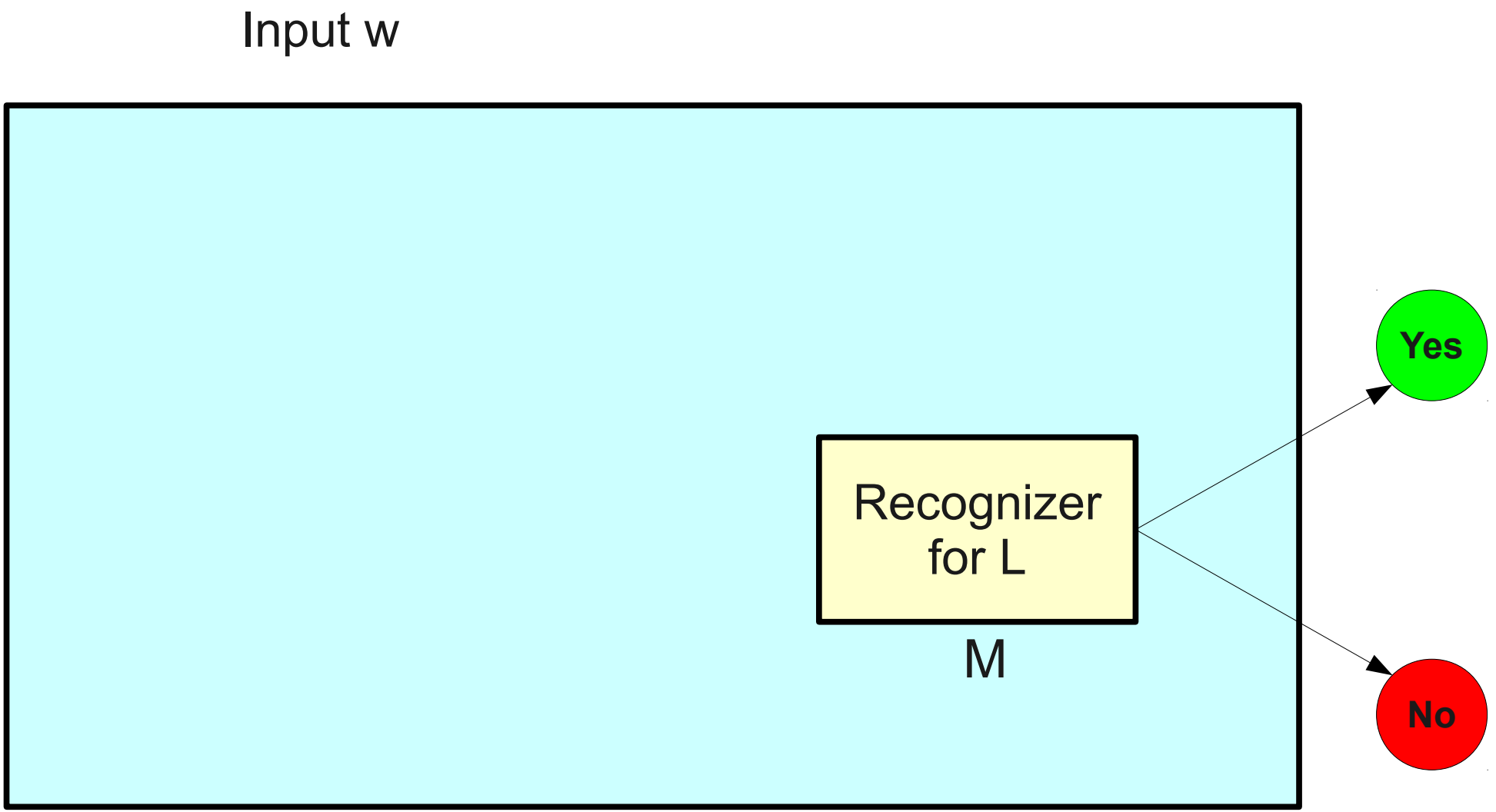

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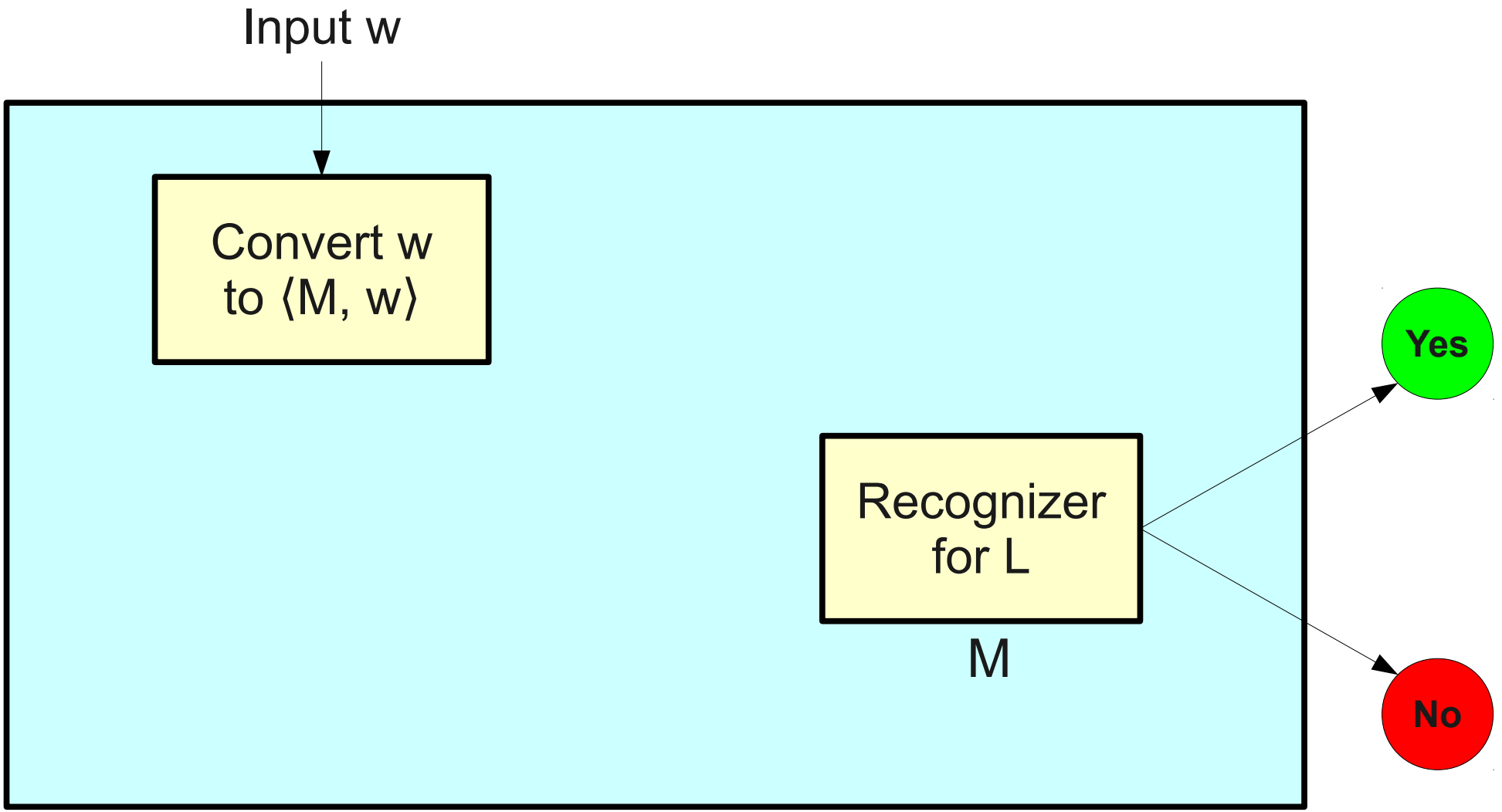
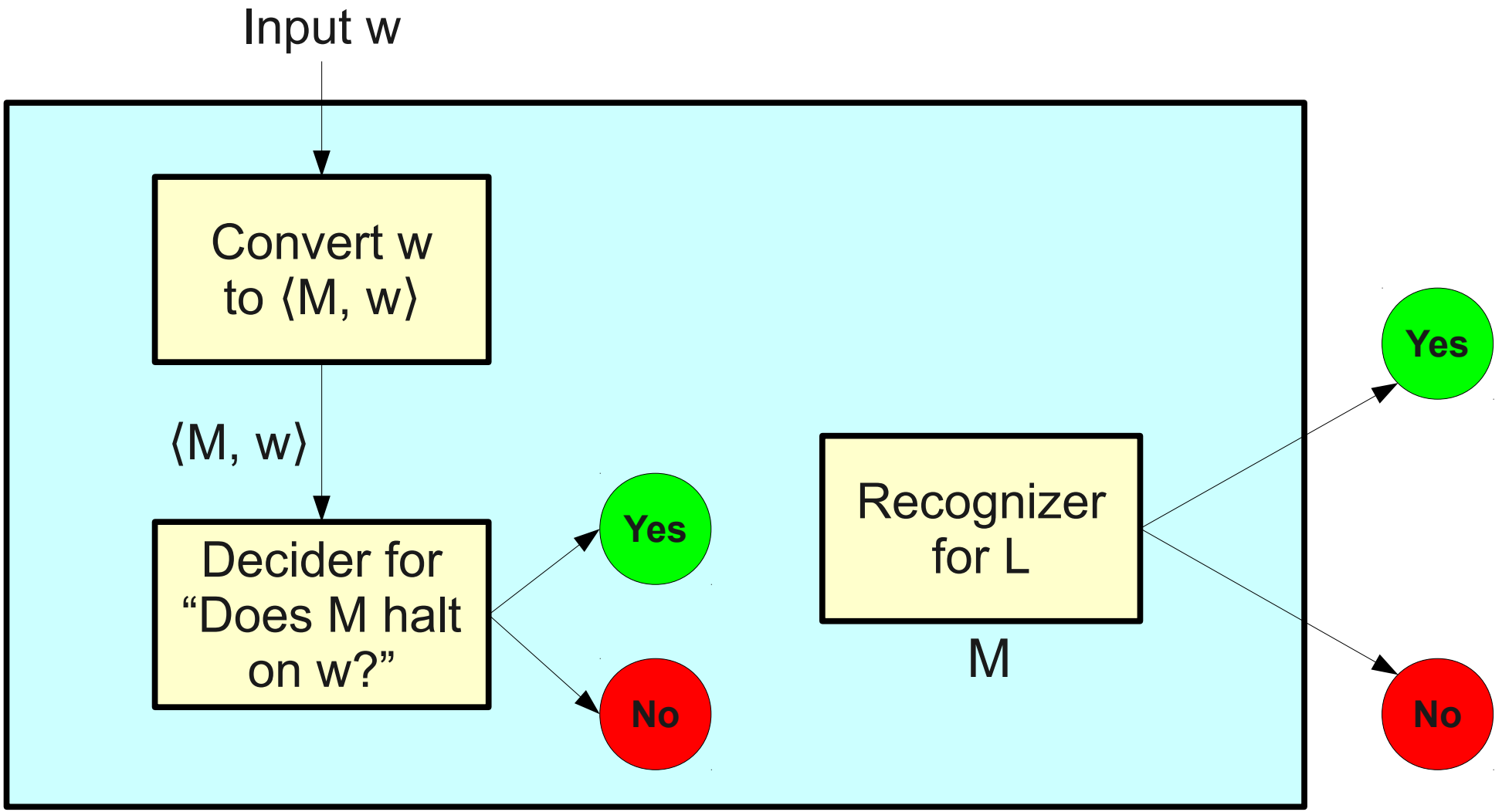
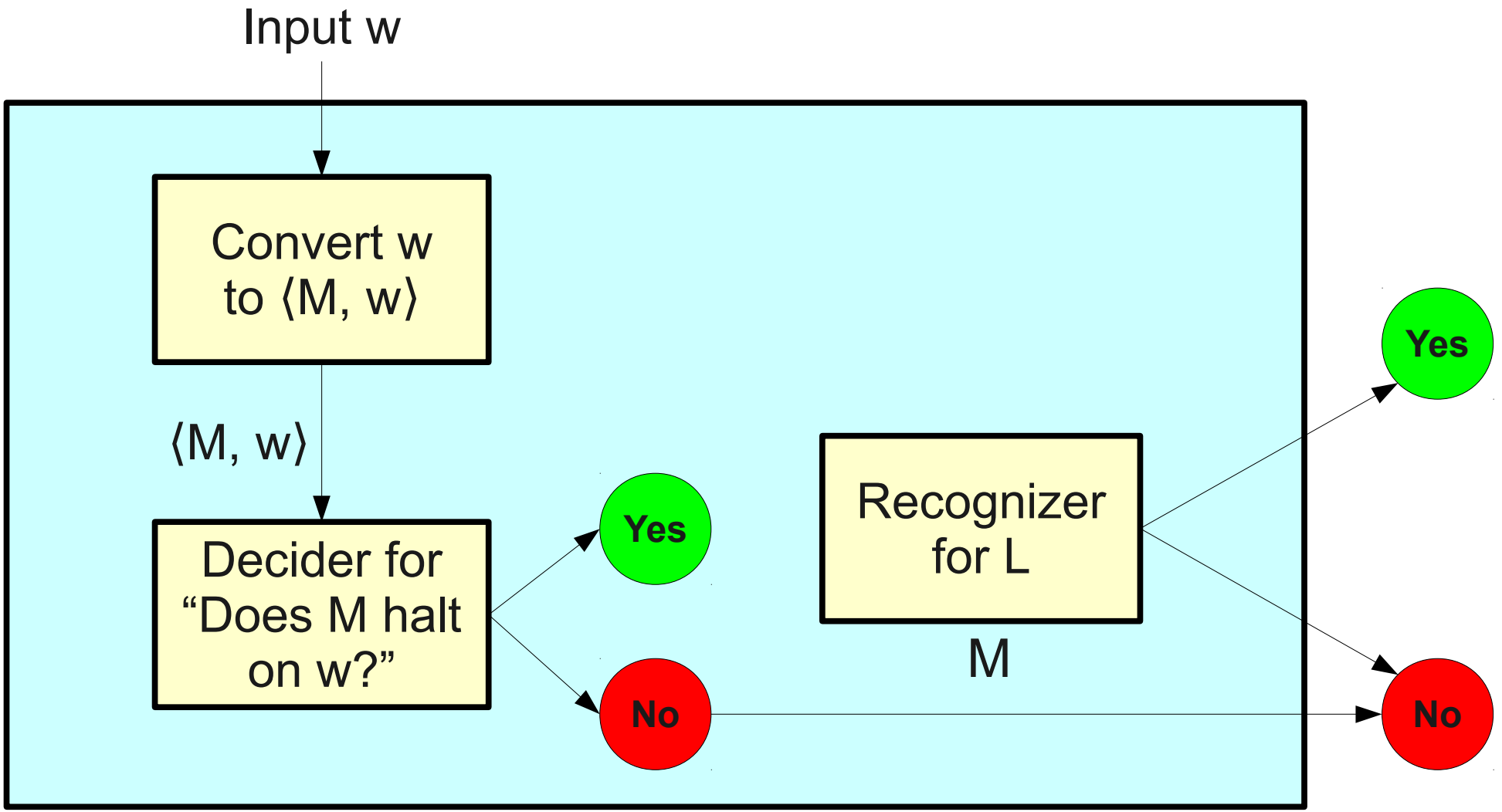


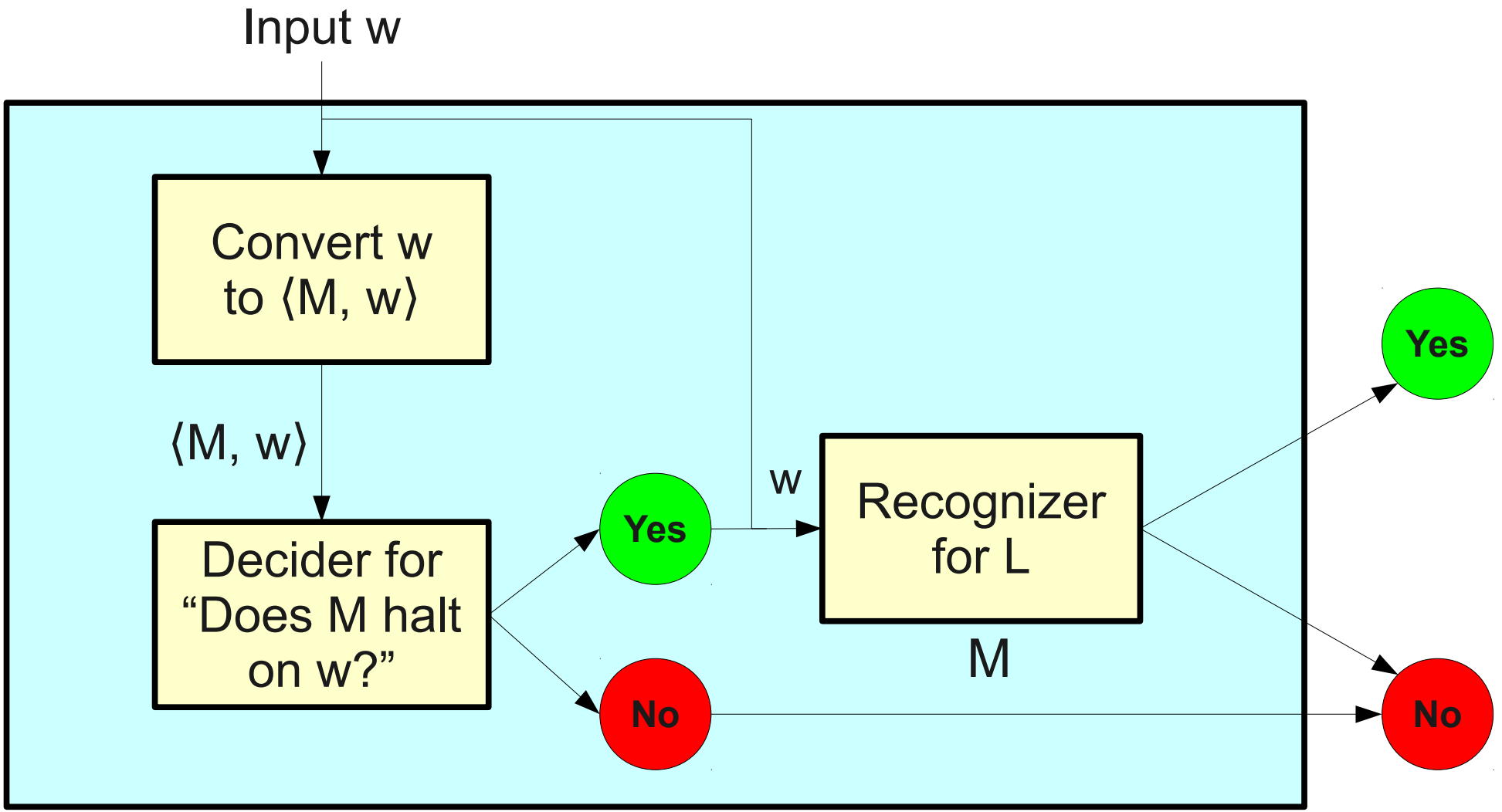
From Recognizers to Deciders



From Recognizers to Deciders



From Recognizers to Deciders



What This Would Mean

- The previous construction relies on the following language being decidable:

  $HALT = \{ \langle M, w \rangle \mid M \text{ halts when run on } w \}$

- This is called the **halting problem**.
- If this is a decidable language, this has **huge** implications.

If *HALT* is Decidable...

- Suppose we have a TM H that decides *HALT*.
- We can check whether some (computable) property P(n) is true for all natural numbers n.
  - Build a TM M that ignores its input and starts checking P(0), P(1), P(2), …
    - M terminates and rejects if for some n, P(n) is false.
    - M loops infinitely otherwise.
  - If *H* halts on ⟨M, ε⟩, then P(n) is not true for all n.
  - Otherwise, P(n) is true for all n.

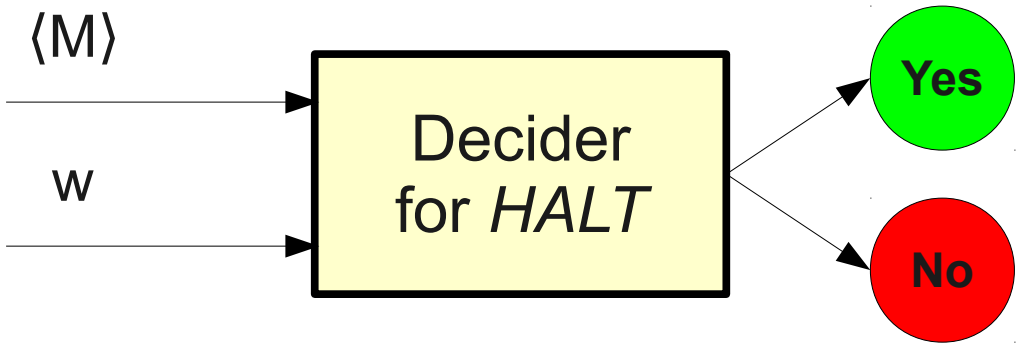
If *HALT* is Decidable...

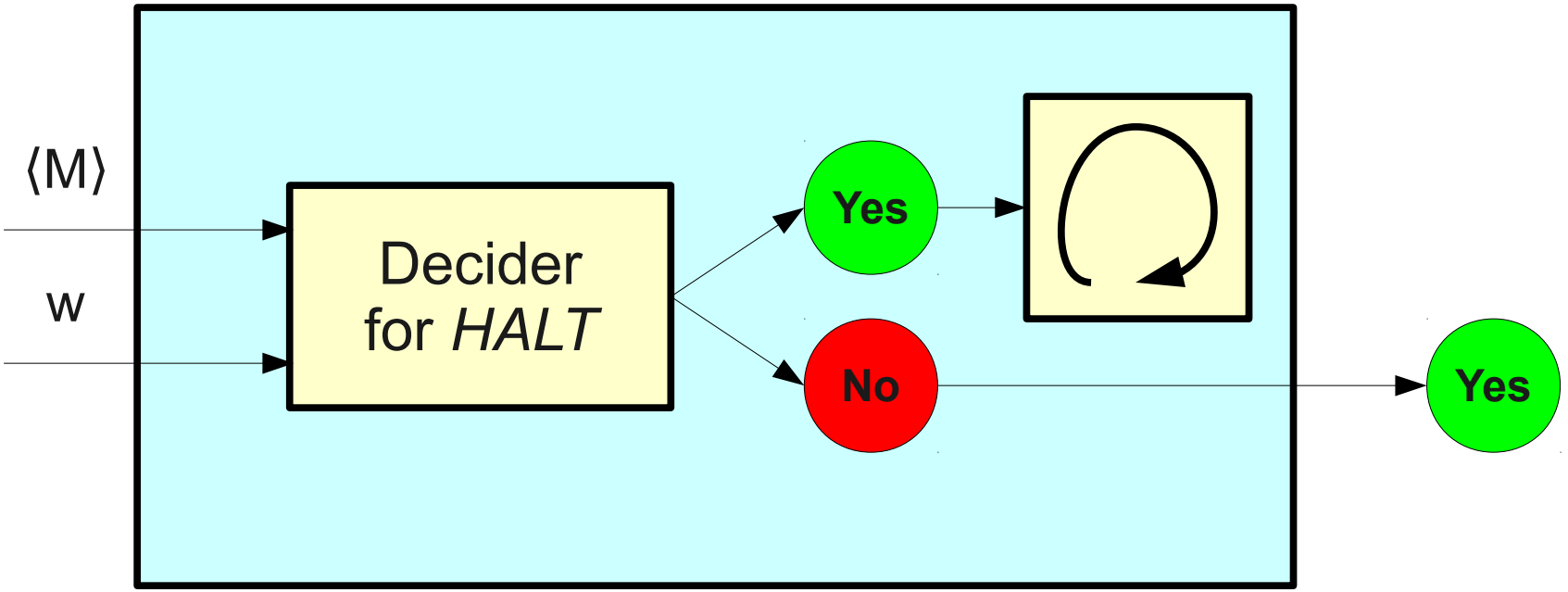
- Suppose we have a TM H that decides *HALT*.
- We can build a check whether or not a statement φ in first-order logic is true.
  - Build a TM M that begins listing off all true FOL statements one at a time.
    - Begin with a set of **axioms**.
    - Keep applying **inference rules** one at a time.
  - If φ is true, M halts and accepts.
  - If φ is false, M loops forever.
  - If H halts on ⟨M, φ⟩, then φ is true.
  - Otherwise, φ is not true.

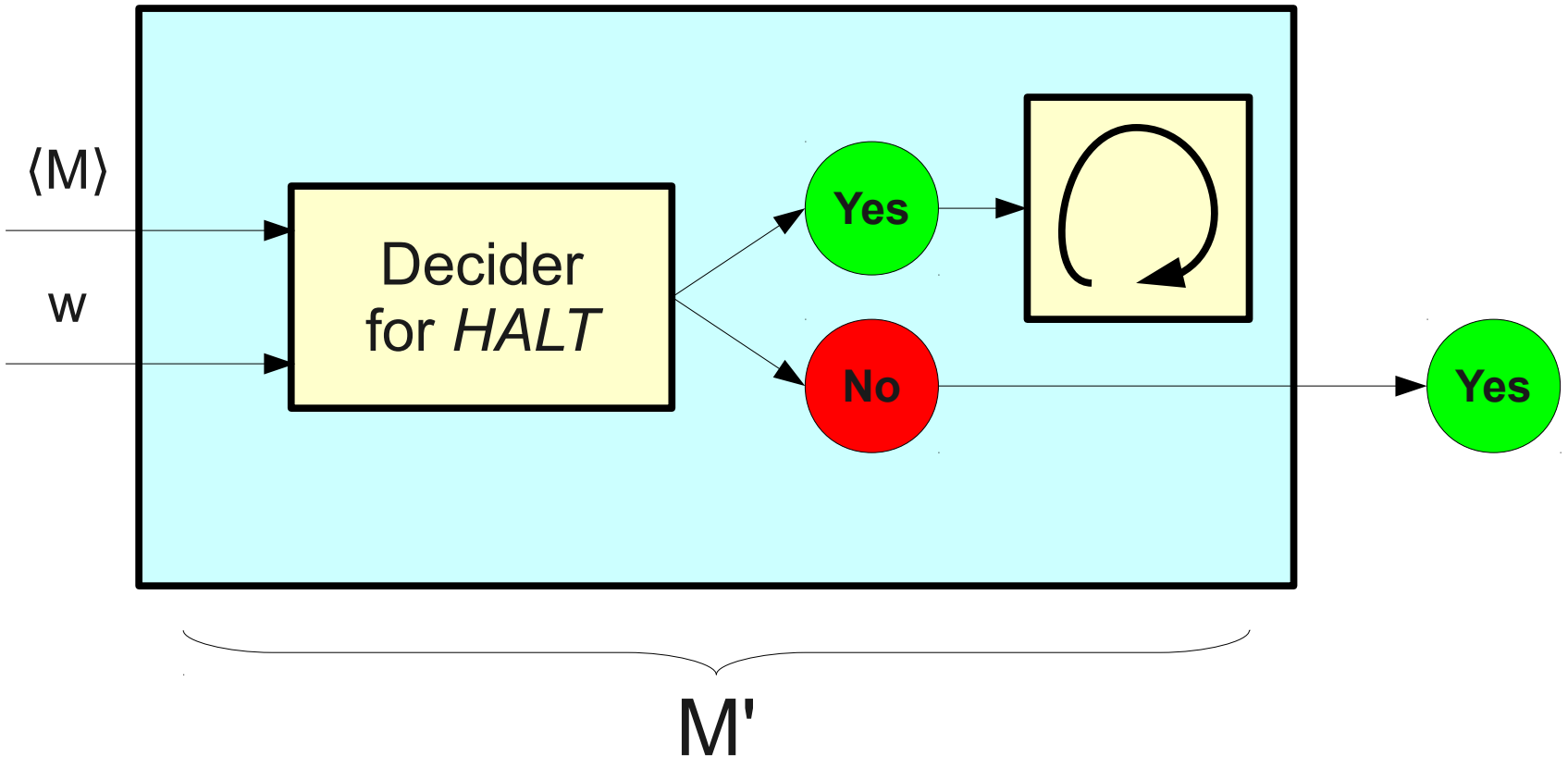
If *HALT* is decidable, then most (if not all) of mathematics could be automated.

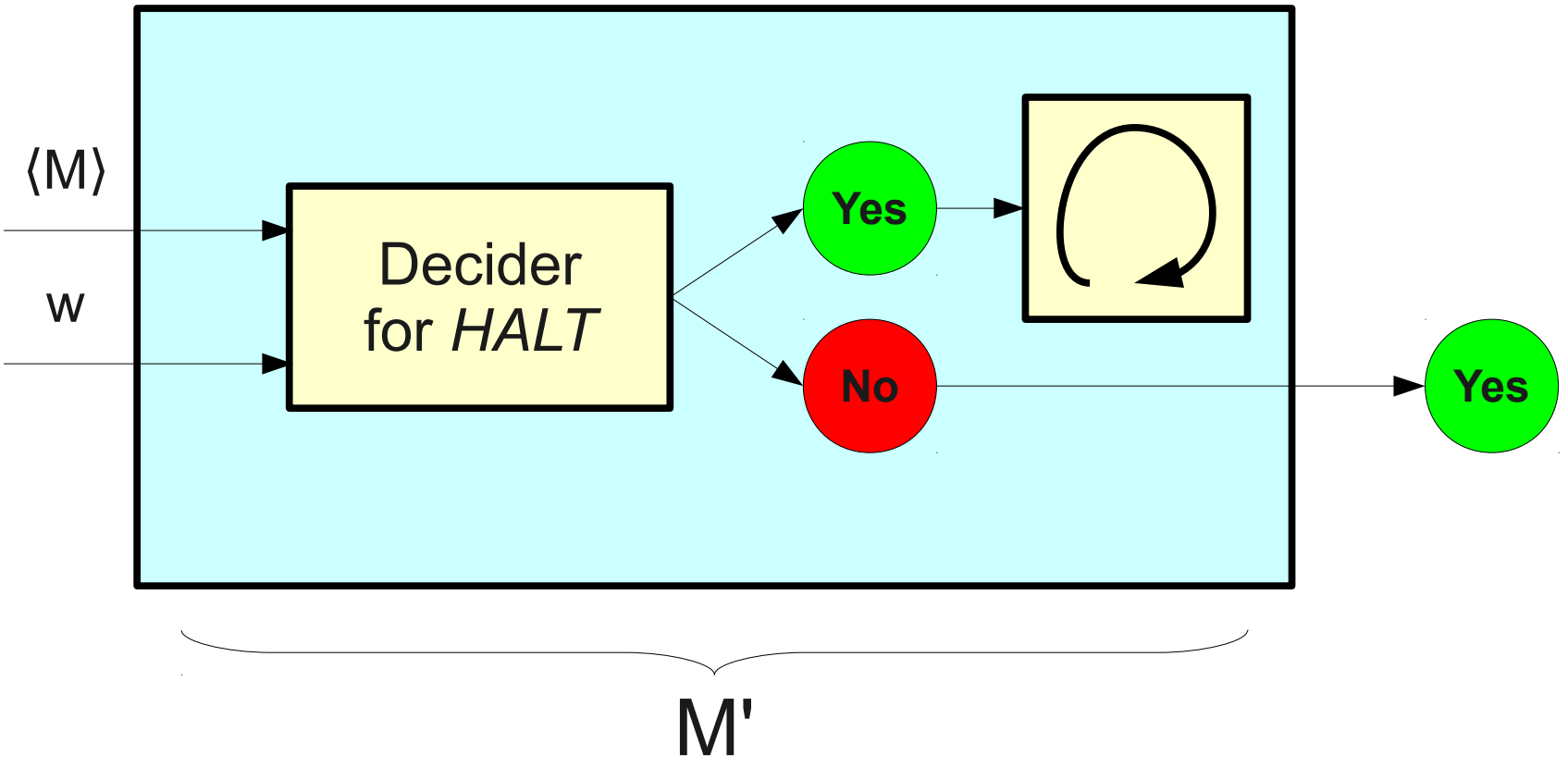
Unfortunately, *HALT* is **undecidable**.

Assume, for the sake
of contradiction, that
HALT is decidable.

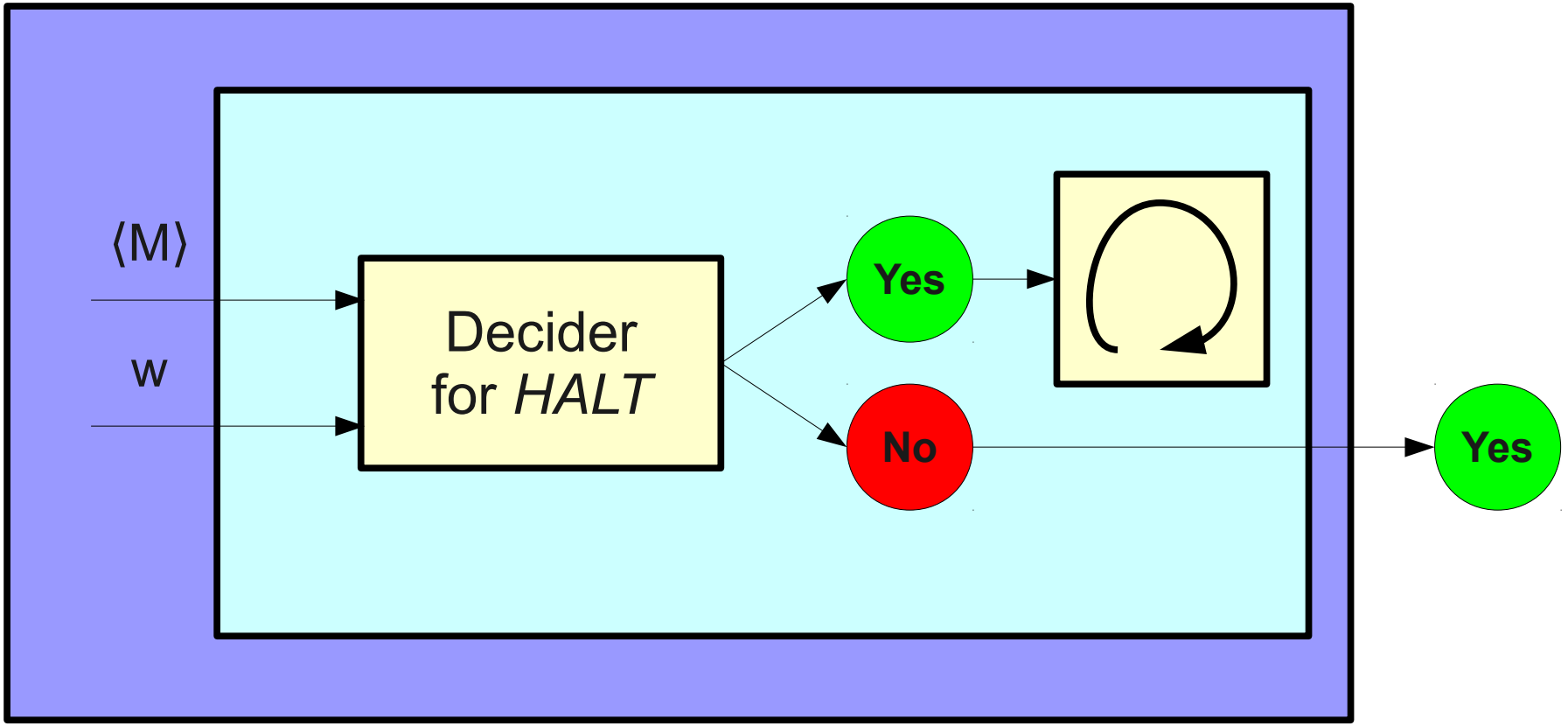

⟨M⟩
w
Decider
for *HALT*
Yes
No

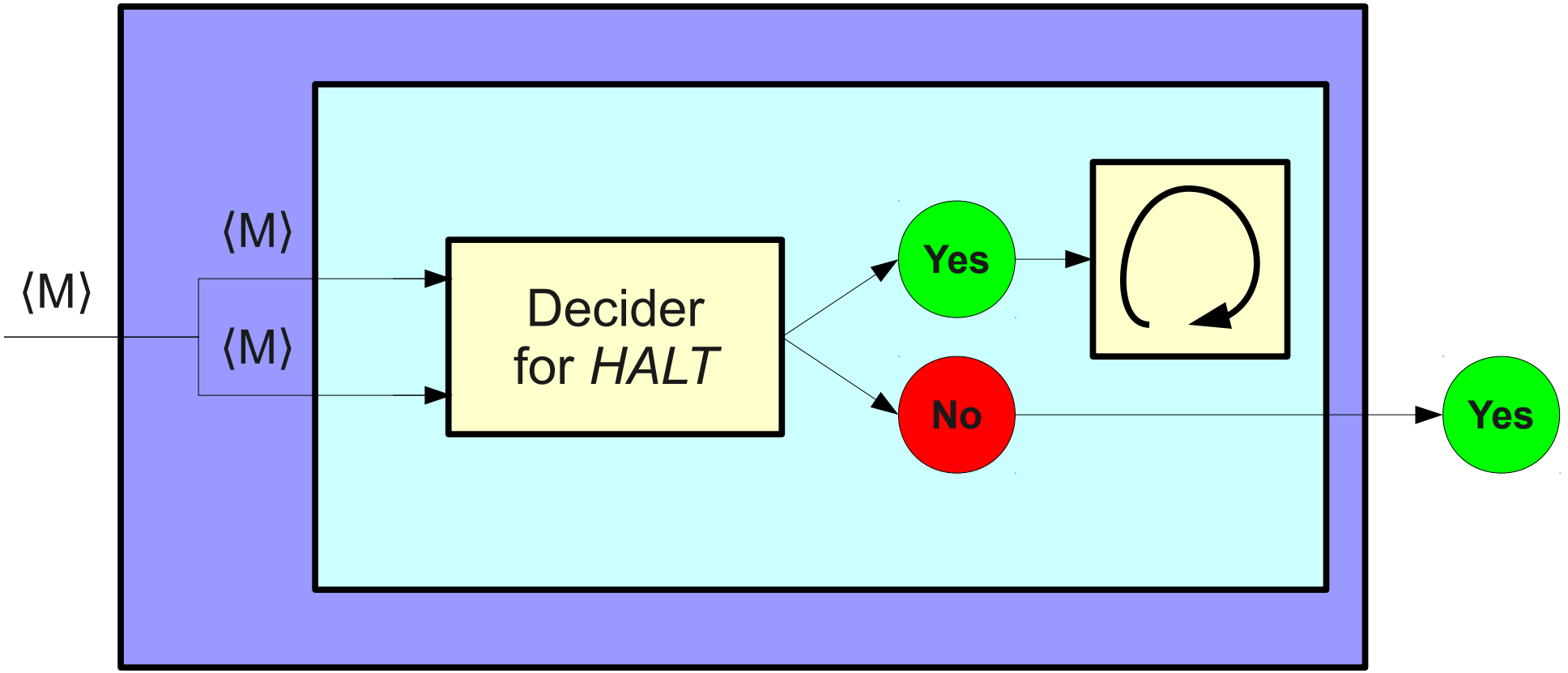

⟨M⟩
w
Decider
for *HALT*
Yes
No
Yes

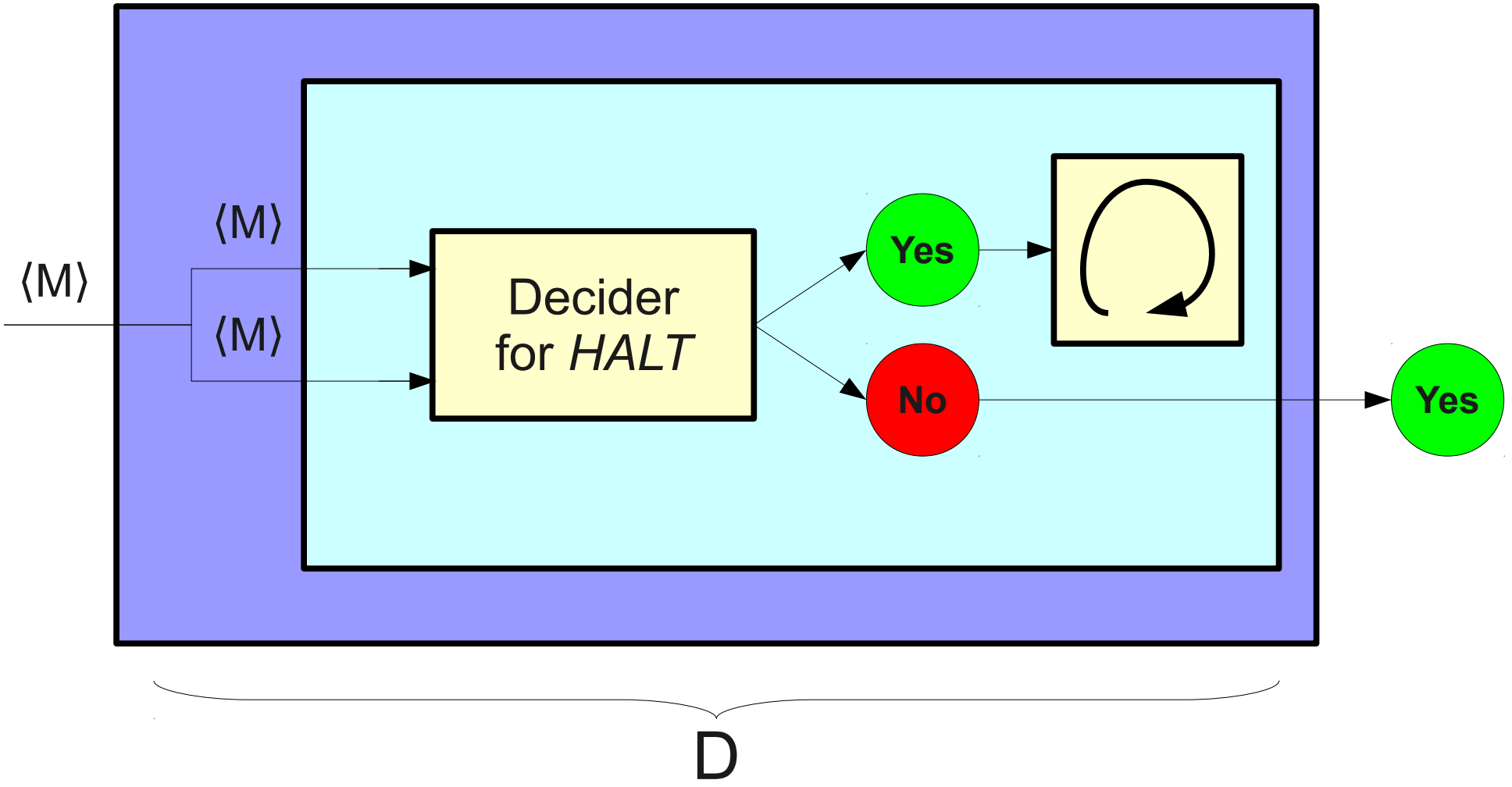

⟨M⟩
w
Decider
for *HALT*
Yes
No
Yes
M'

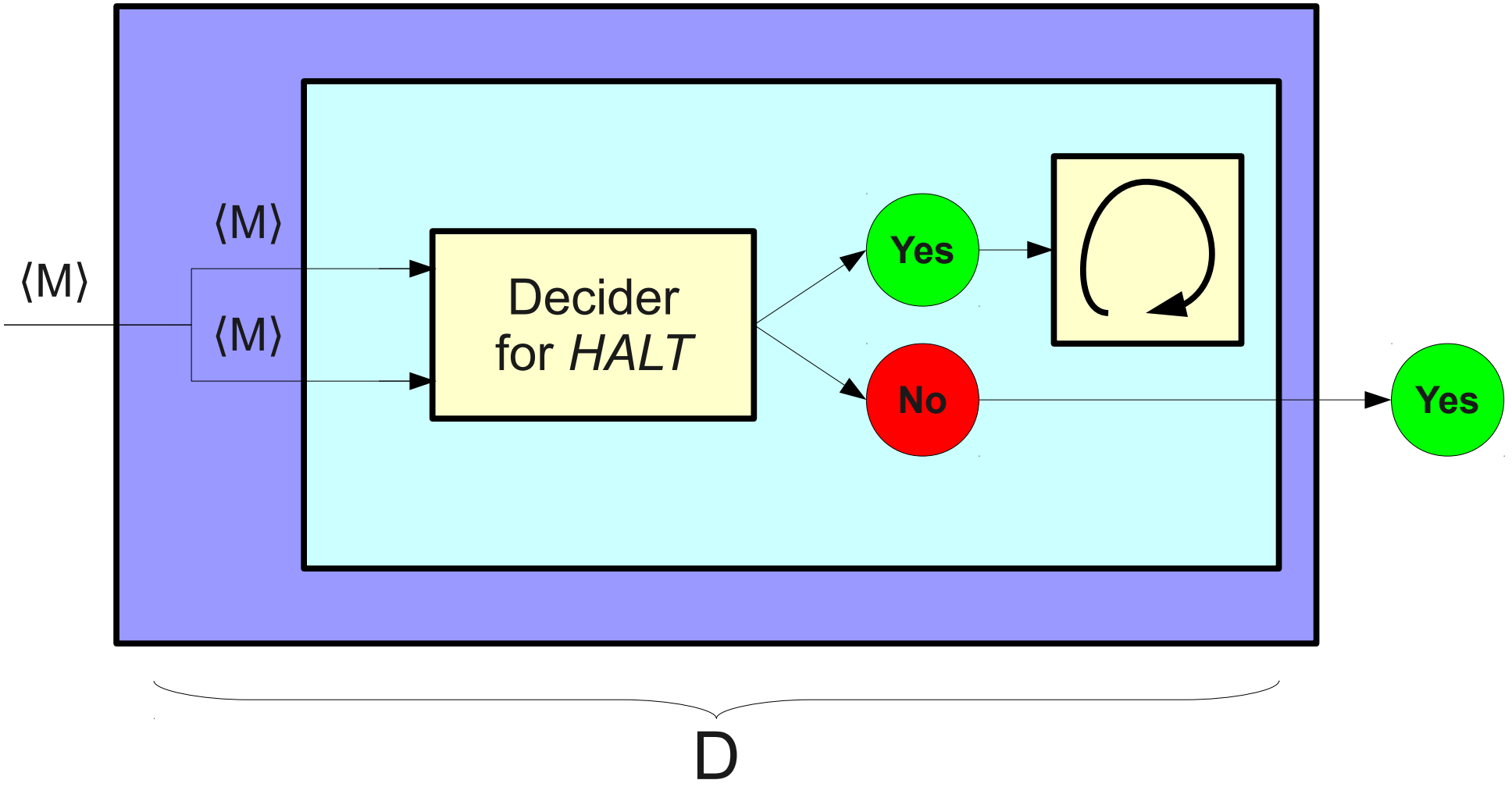


M' = “On input ⟨M, w⟩,
If M halts on w, loop infinitely.
If M loops on w, accept.

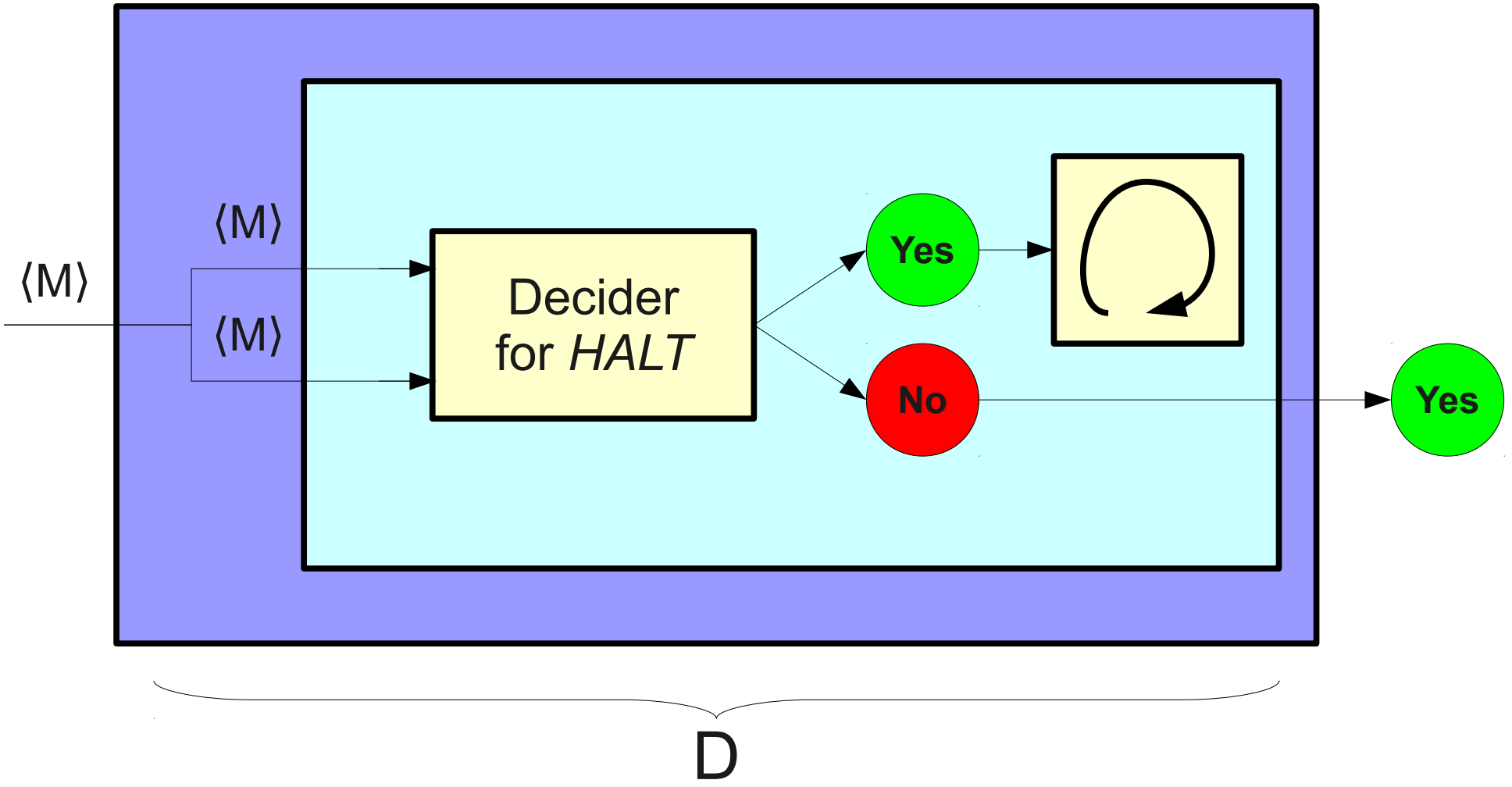

⟨M⟩
w
Decider
for *HALT*
Yes
No
Yes


⟨M⟩
⟨M⟩
⟨M⟩
Decider
for *HALT*
Yes
No
Yes

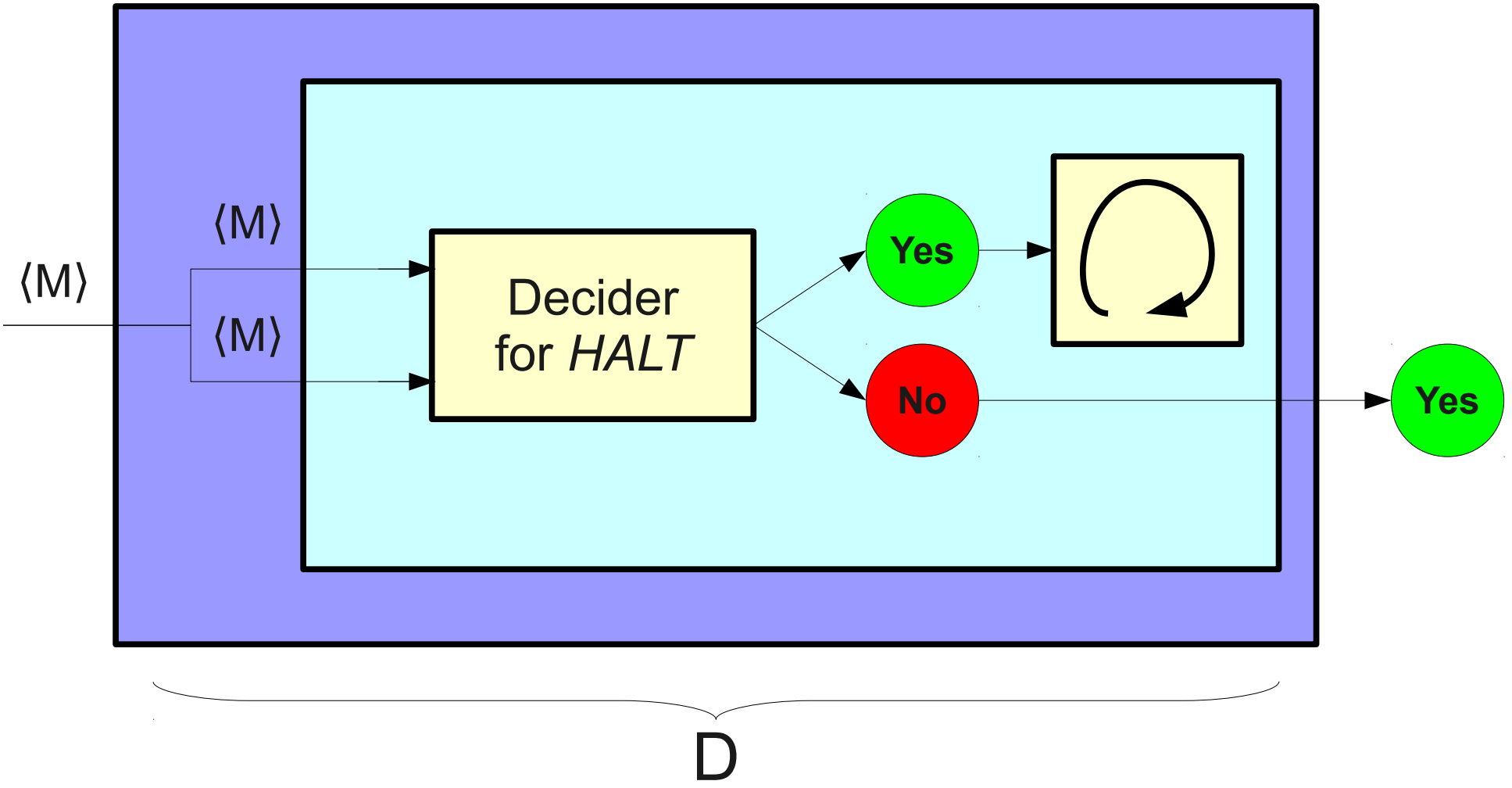

⟨M⟩
⟨M⟩
⟨M⟩
Decider
for HALT
Yes
No
Yes
D



D = “On input ⟨M⟩,



D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.

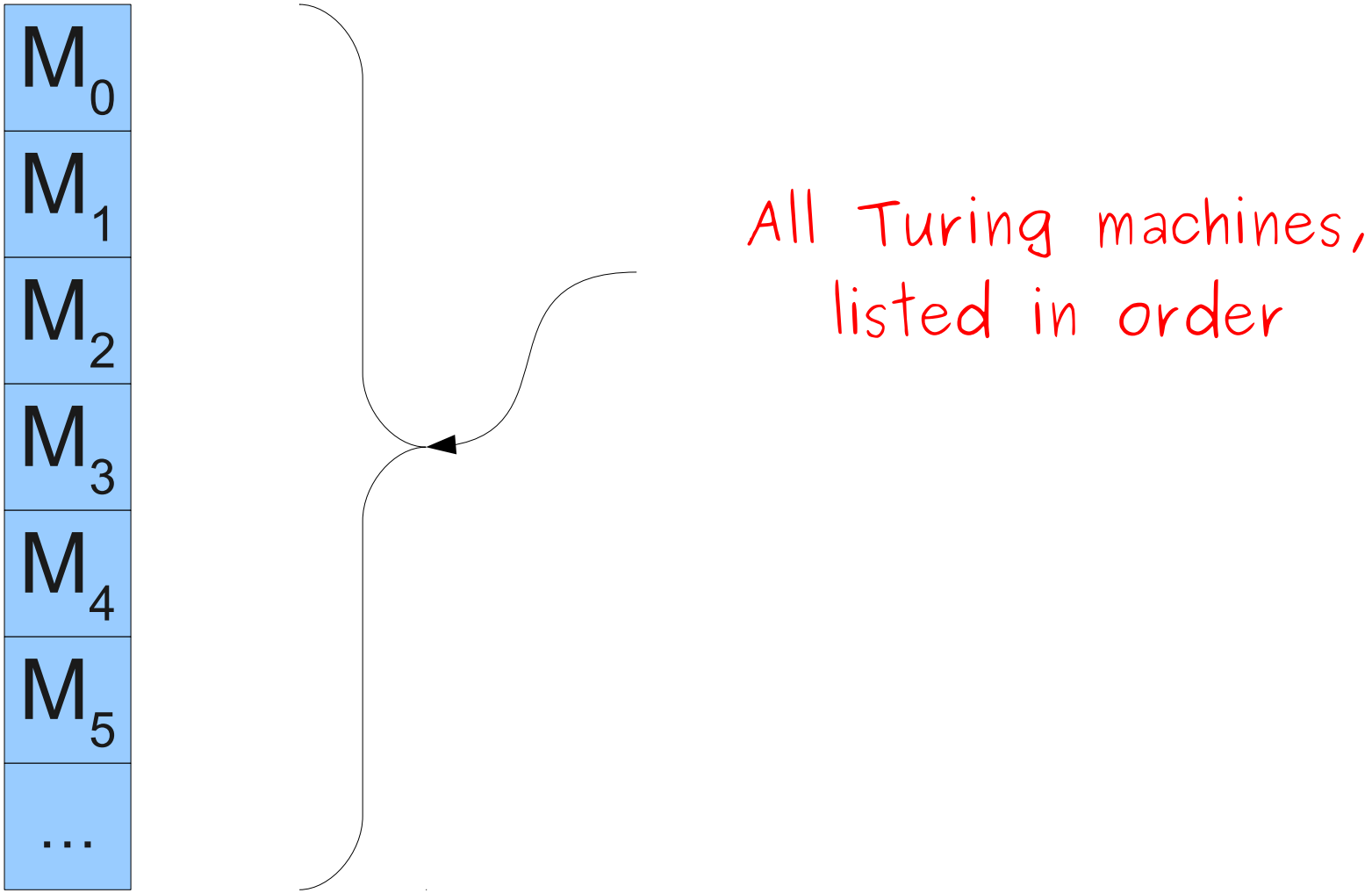


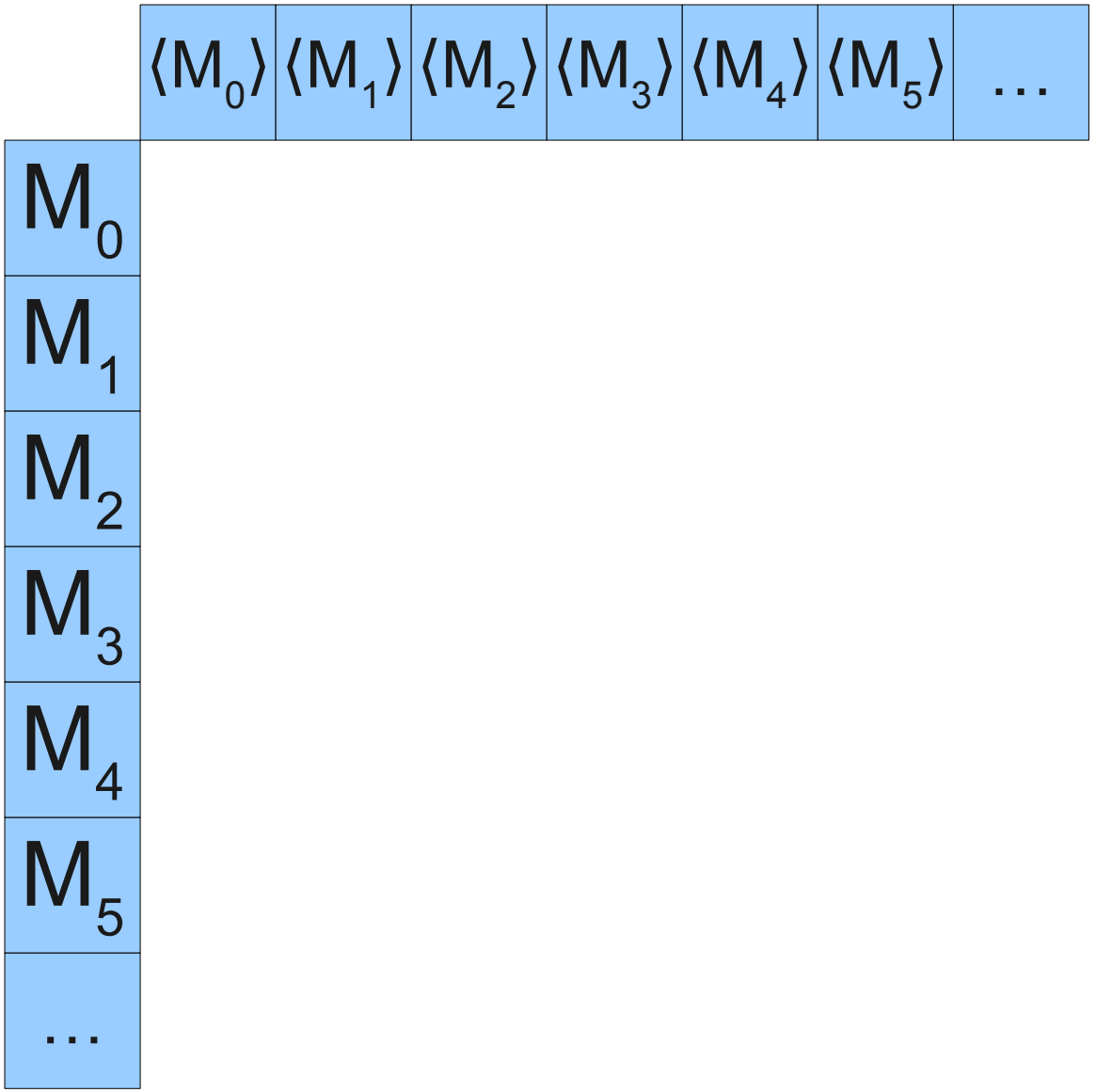
D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

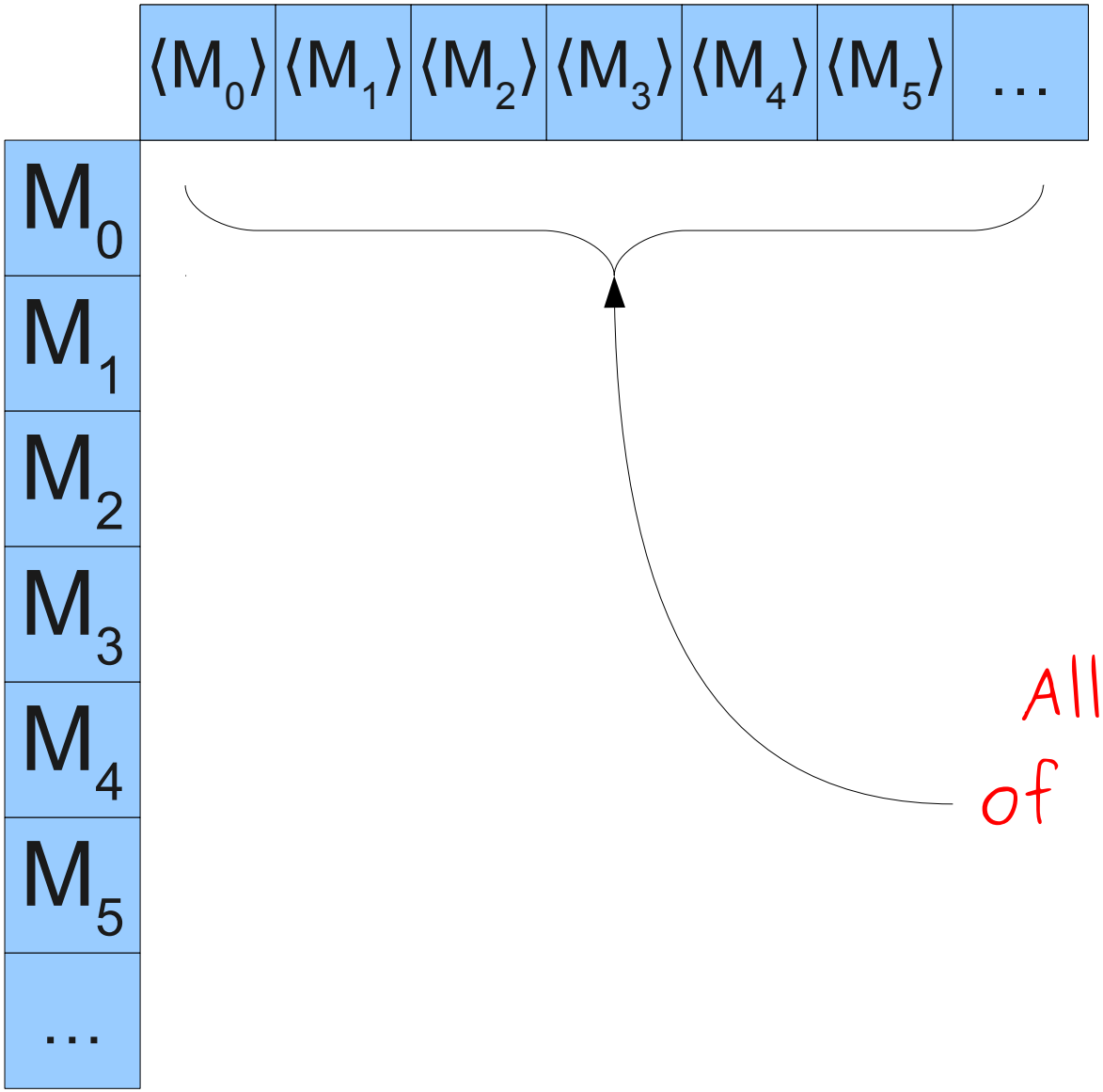
If *HALT* is decidable, we can build
a TM that does the following on input ⟨M⟩:

Accept if M loops infinitely on ⟨M⟩
Loop infinitely if M halts on ⟨M⟩


M_0
M_1
M_2
M_3
M_4
M_5
...


M_0
M_1
M_2
M_3
M_4
M_5
...
All Turing machines,
listed in order


⟨M0⟩
⟨M1⟩
⟨M2⟩
⟨M3⟩
⟨M4⟩
⟨M5⟩
…
M0
M1
M2
M3
M4
M5
…



All descriptions of TMs, listed in order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1							
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

H	H	H	L	H	L	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…
	H	H	H	L	H	L	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

This is the language L(D)

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

What TM has this behavior?

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…
	L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…
	L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…
	L	L	L	H	L	H	…

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

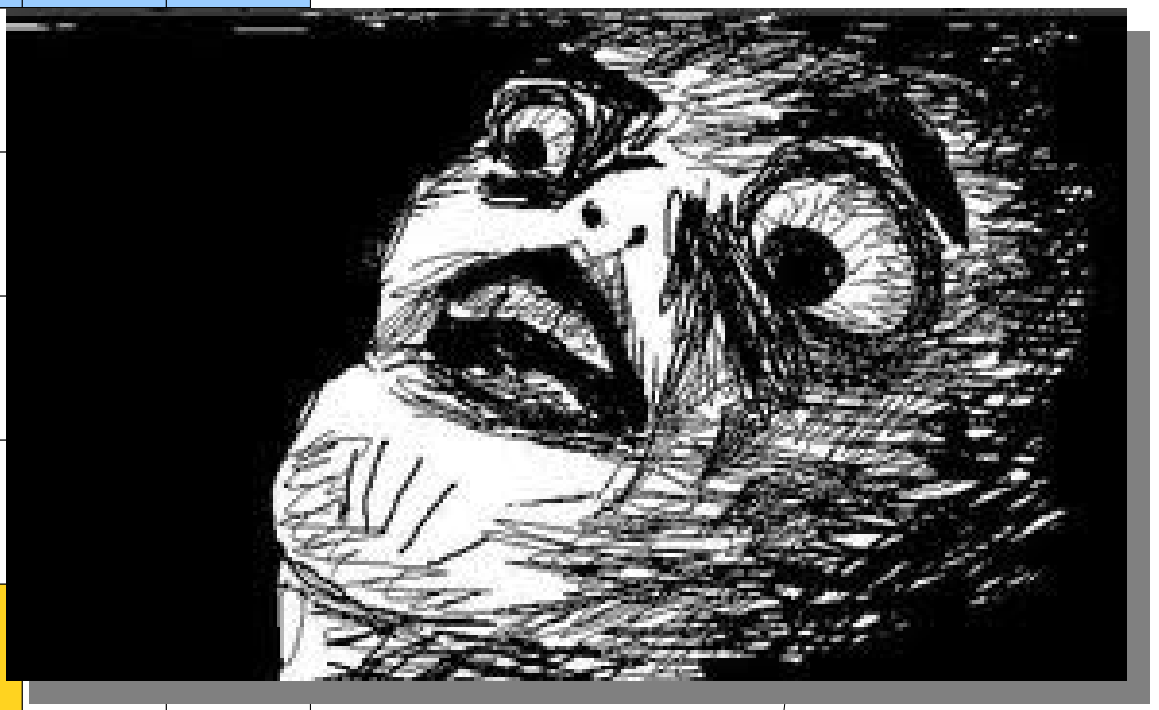
	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H	L	…
M_1	H	H	H	H	H	H	…
M_2	H	H	H	H	H	H	…
M_3	L	H	H	L	H	H	…
M_4	H	L	H	L	H	L	…
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…

L	L	L	H	L	H	…

No TM has this behavior!

D = “On input ⟨M⟩,
If M halts on ⟨M⟩, loop infinitely.
If M loops on ⟨M⟩, accept.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	H	L	L	H	H		
M_1	H	H	H	H	H		
M_2	H	H	H	H	H		
M_3	L	H	H	L	H		
M_4	H	L	H	L	H		
M_5	L	L	H	H	L	L	…
…	…	…	…	…	…	…	…



L	L	L	H	L	H	…

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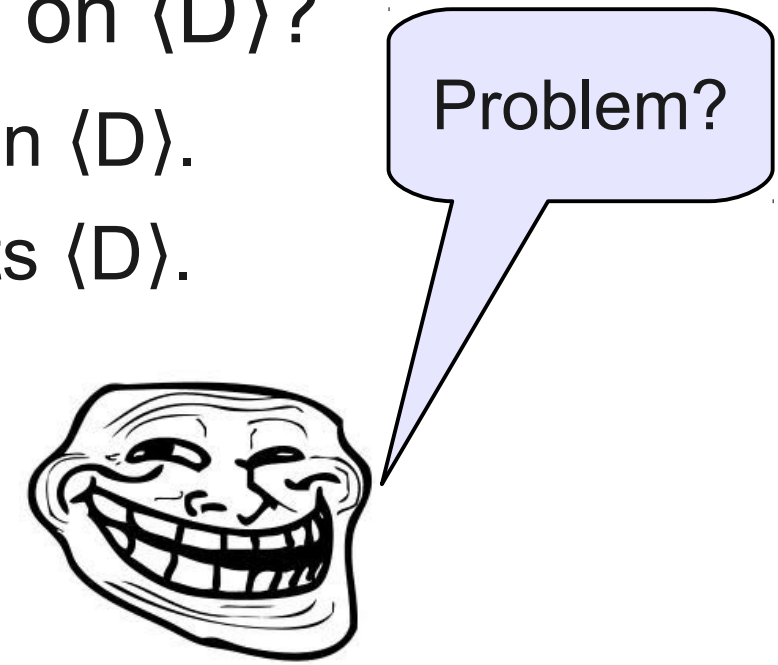

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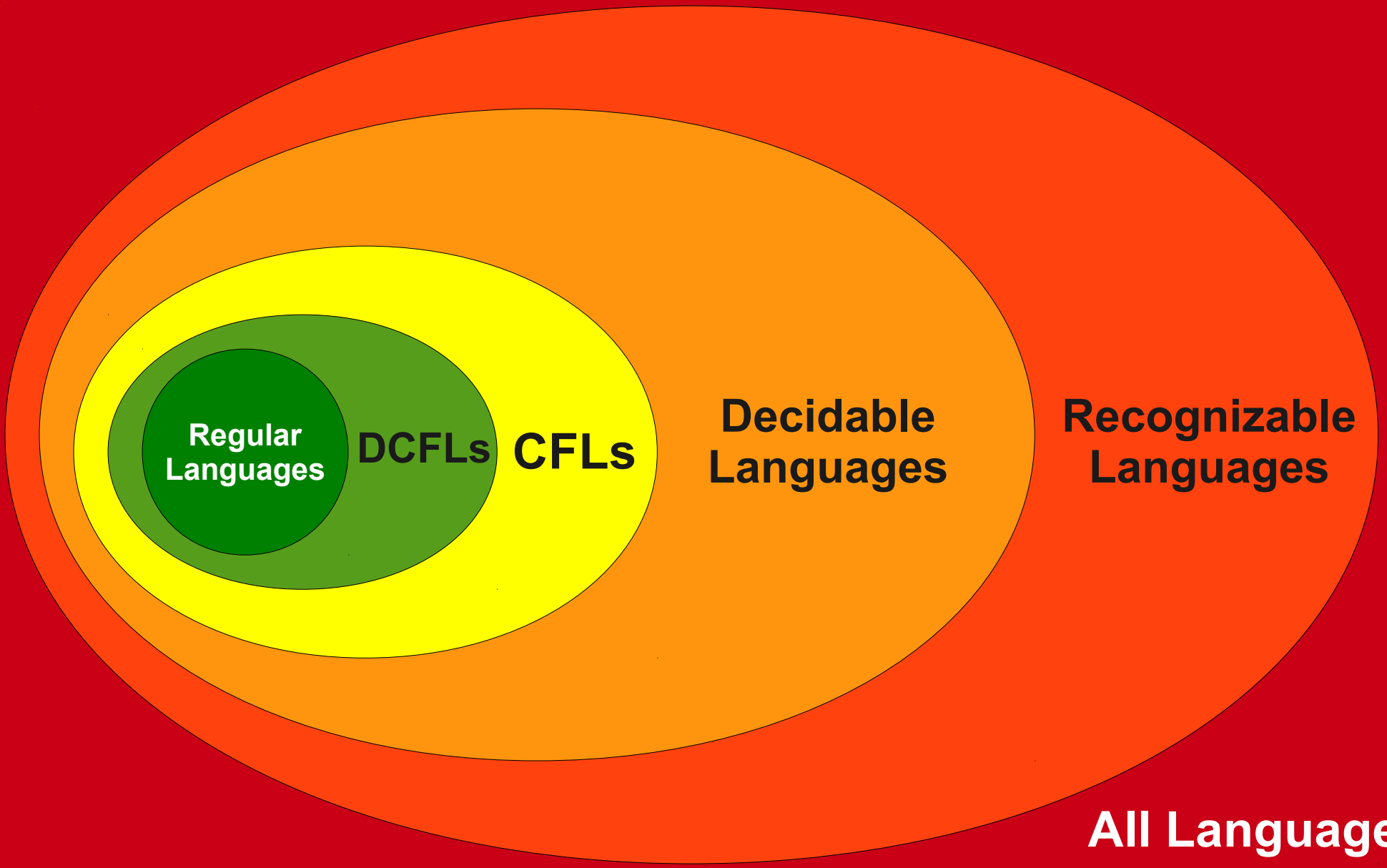
Indirect Self-Reference

- The proof that *HALT* is undecidable, while more complex than the proof that L_D is unrecognizable, uses the same basic trick.
- Assume, for the sake of contradiction, that you can build a TM that operates over descriptions of TMs.
- Feed this TM its own description.
- Derive a contradiction where the TM must have the opposite of its own behavior.

Where is *HALT*?



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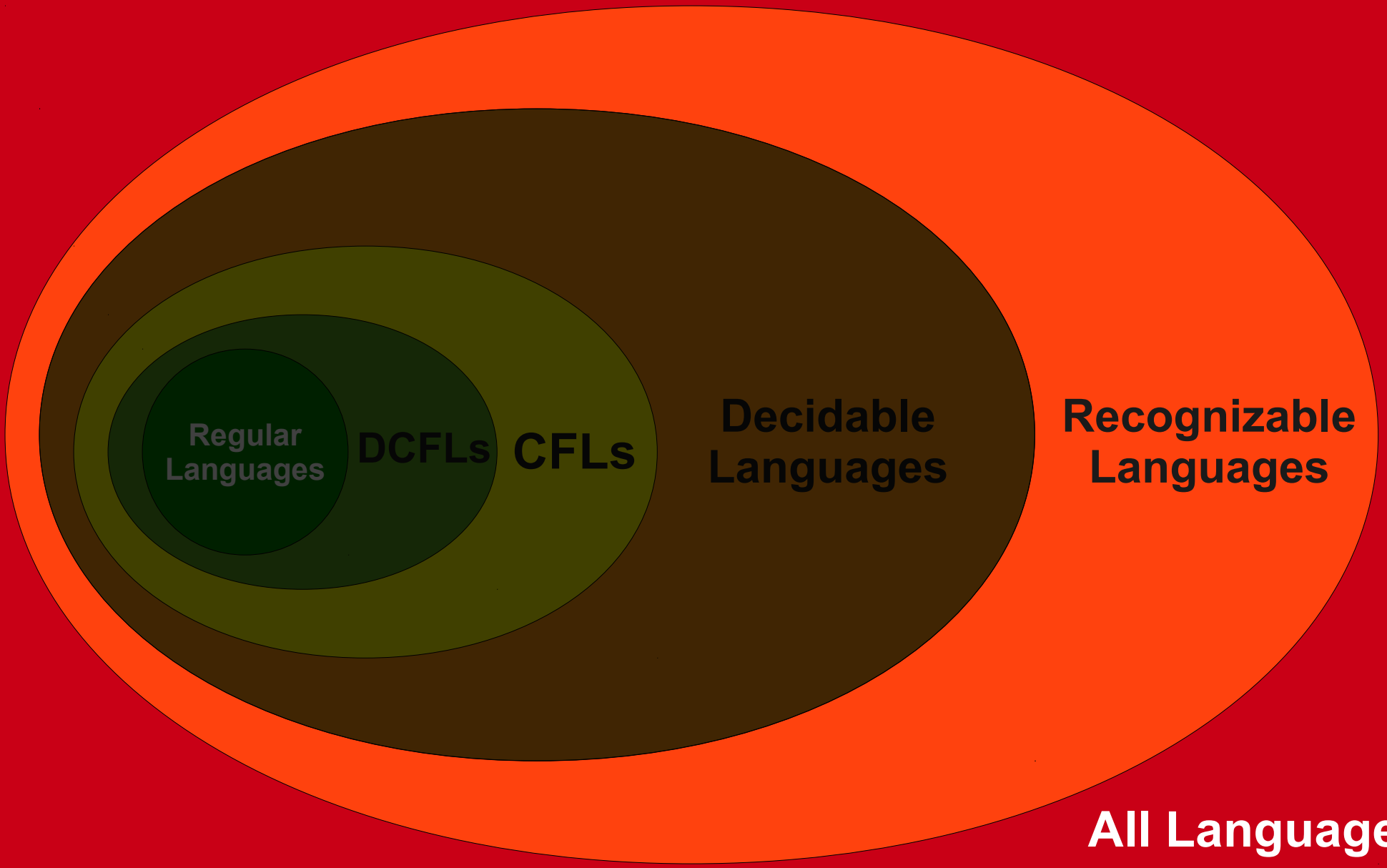

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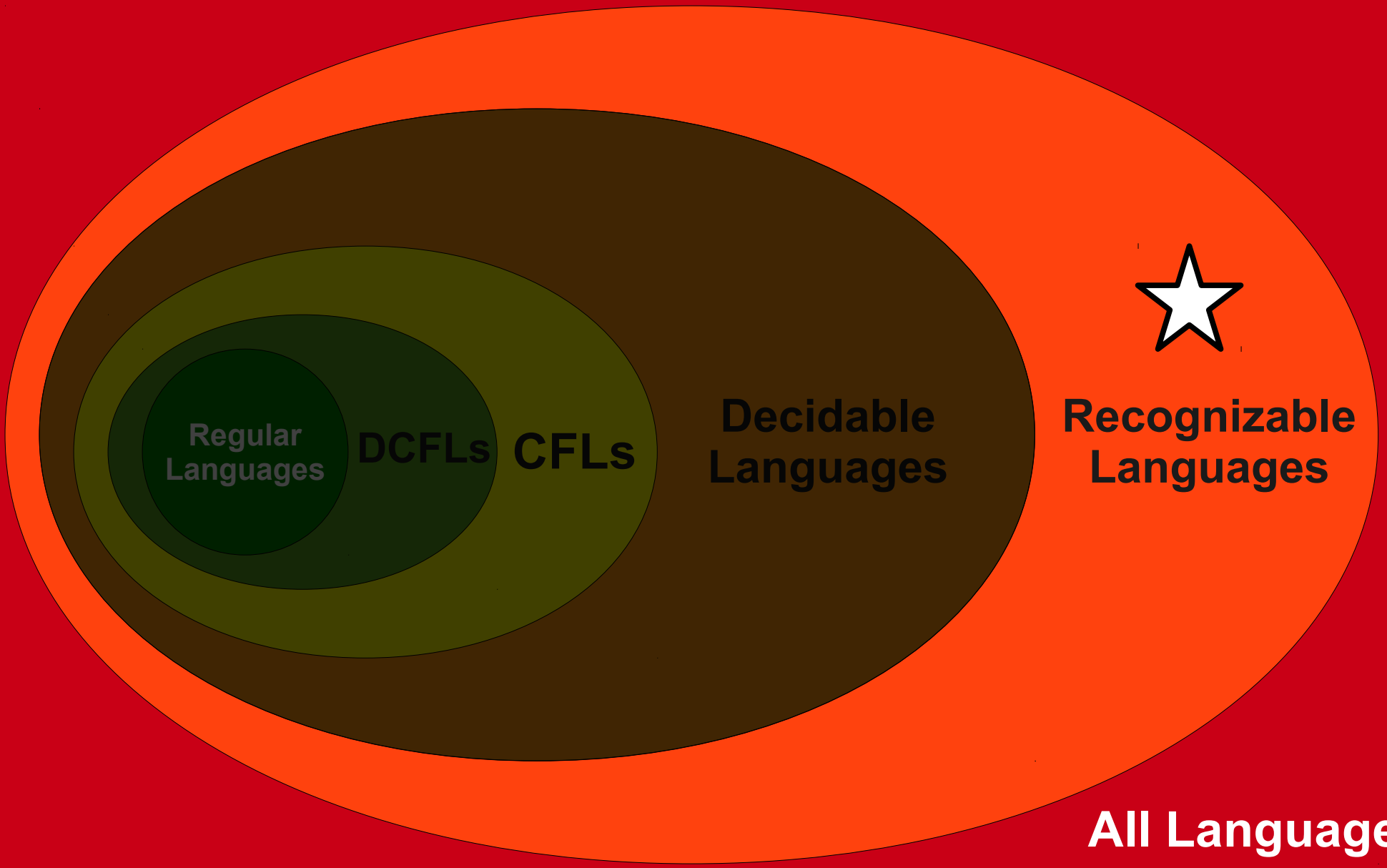

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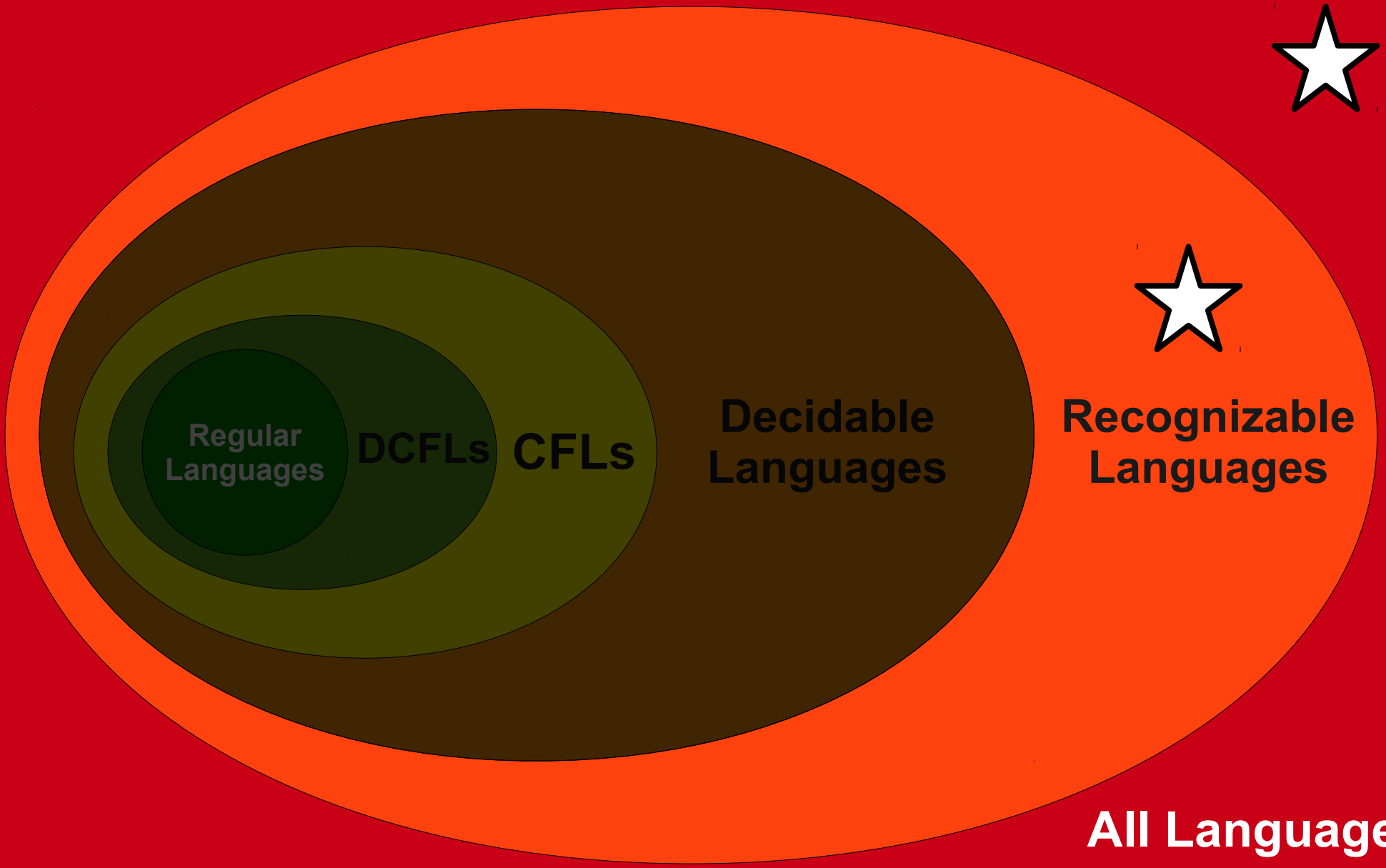
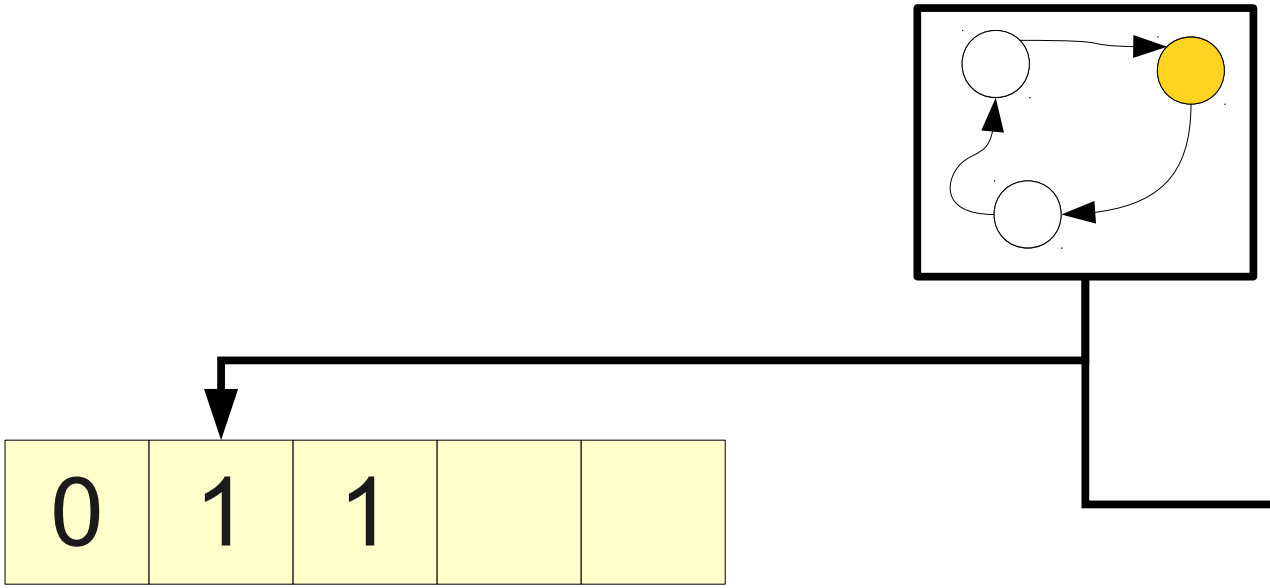
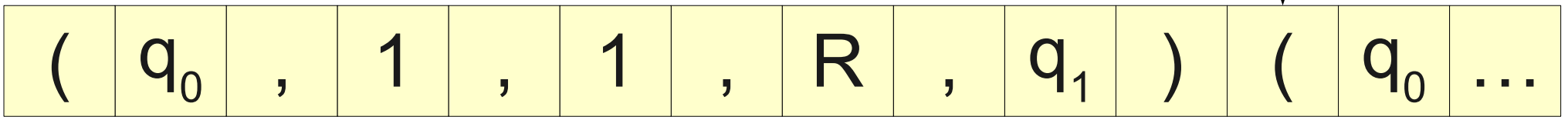


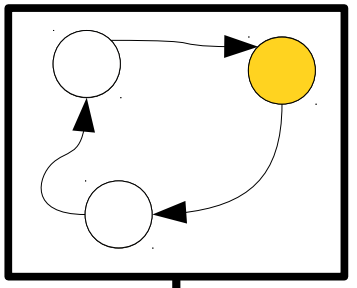
The Universal Turing Machine

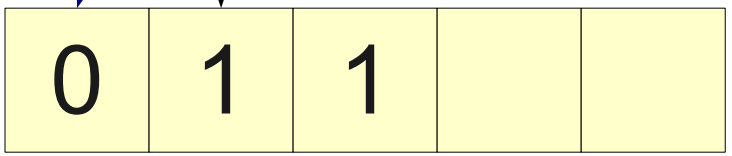


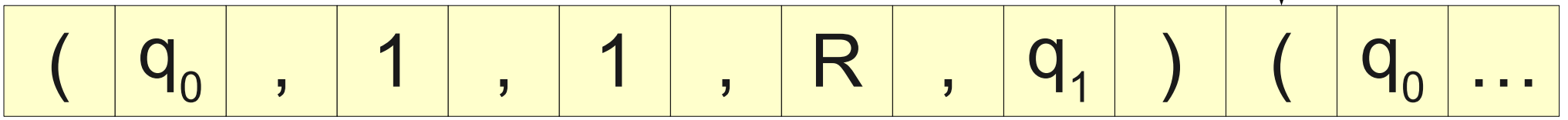


The Universal Turing Machine

The work tape holds the infinite tape from the TM being simulated.







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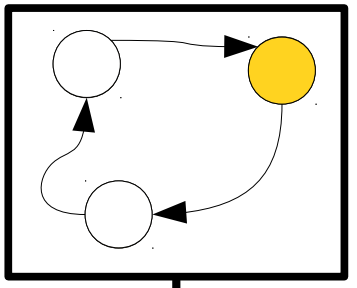

0	1	1		

(	q_0	,	1	,	1	,	R	,	q_1	)	(	q_0	...

The program tape holds the description of the TM to simulate.

The Universal Turing Machine

- The **universal Turing machine** U is a Turing machine that accepts ⟨M, *w*⟩ (an encoding of a machine M and string w) and simulates the behavior of M running on *w*.
- U accepts iff M accepts *w*.
- U rejects iff M rejects *w*.
- U loops iff M loops on *w*.
- The language of U is called A_{TM}:

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ accepts } w \}$$

HALT is Recognizable

- We can **recognize** *HALT* by using the universal Turing machine.
- Construct a TM U' that works as follows:
  - U' = “On input ⟨M, w⟩:
    - Using U, run M on w.
    - If M accepts w, accept.
    - If M rejects w, accept.”
- Note that U' is not a decider, because M might loop forever on w.

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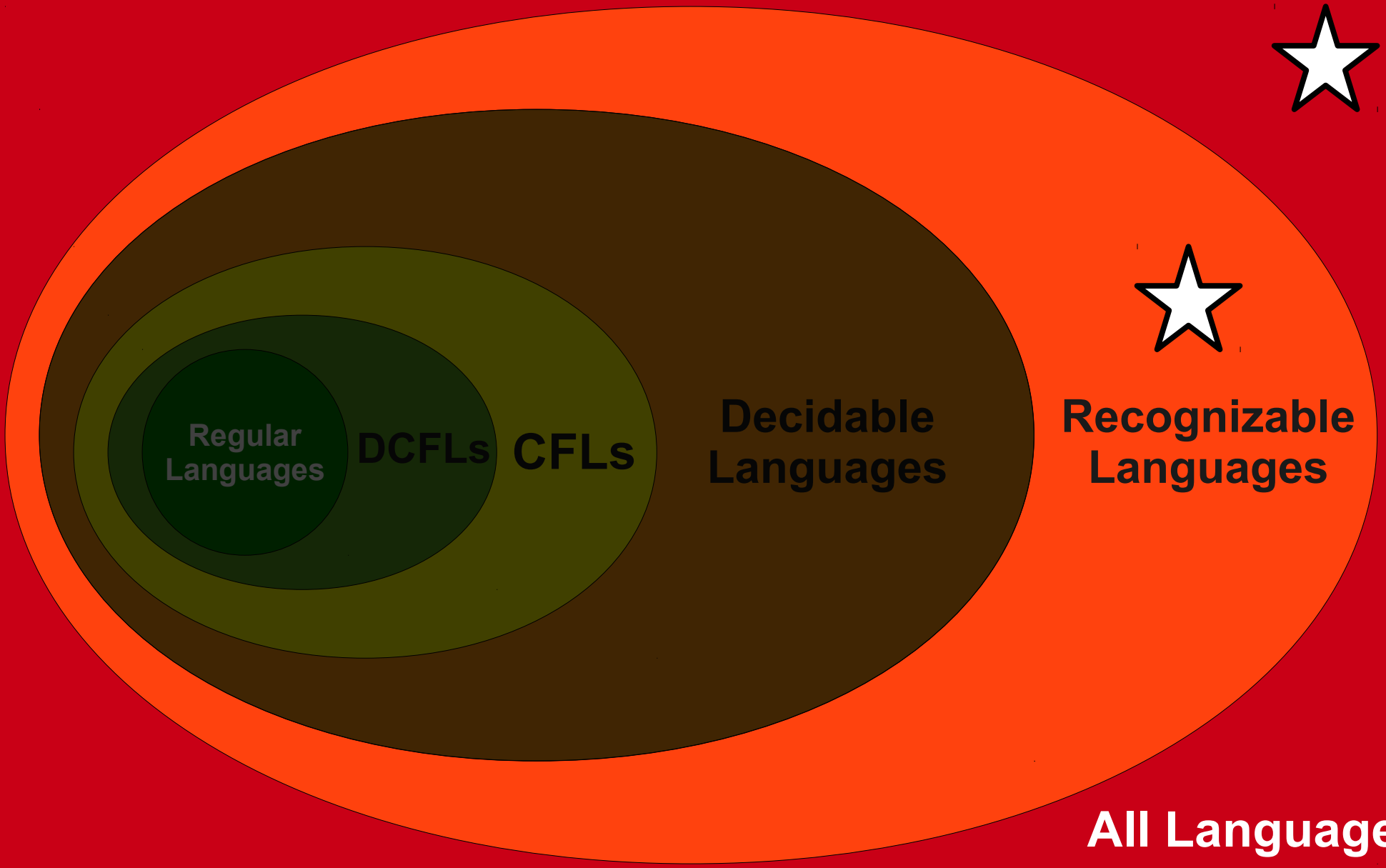

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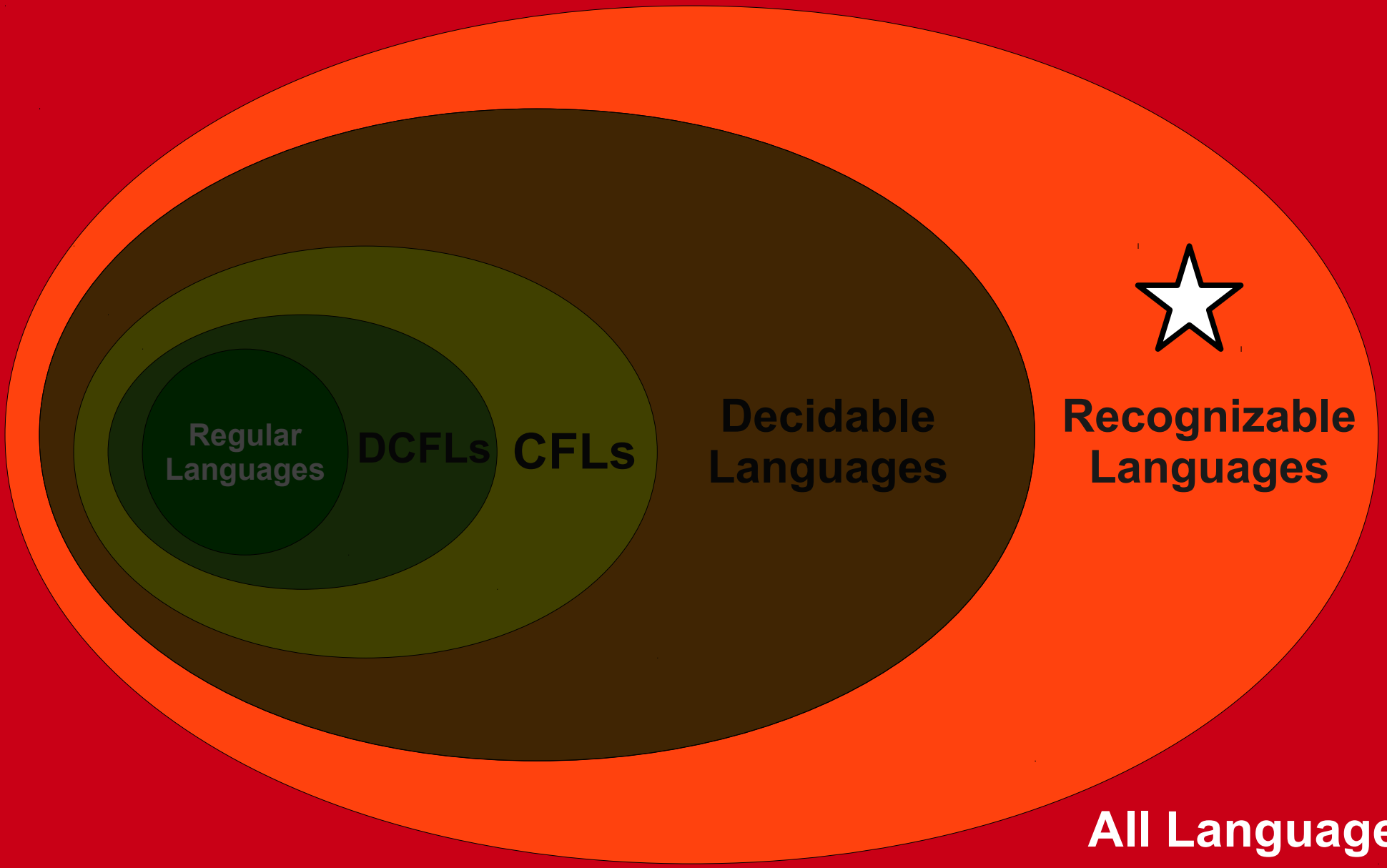

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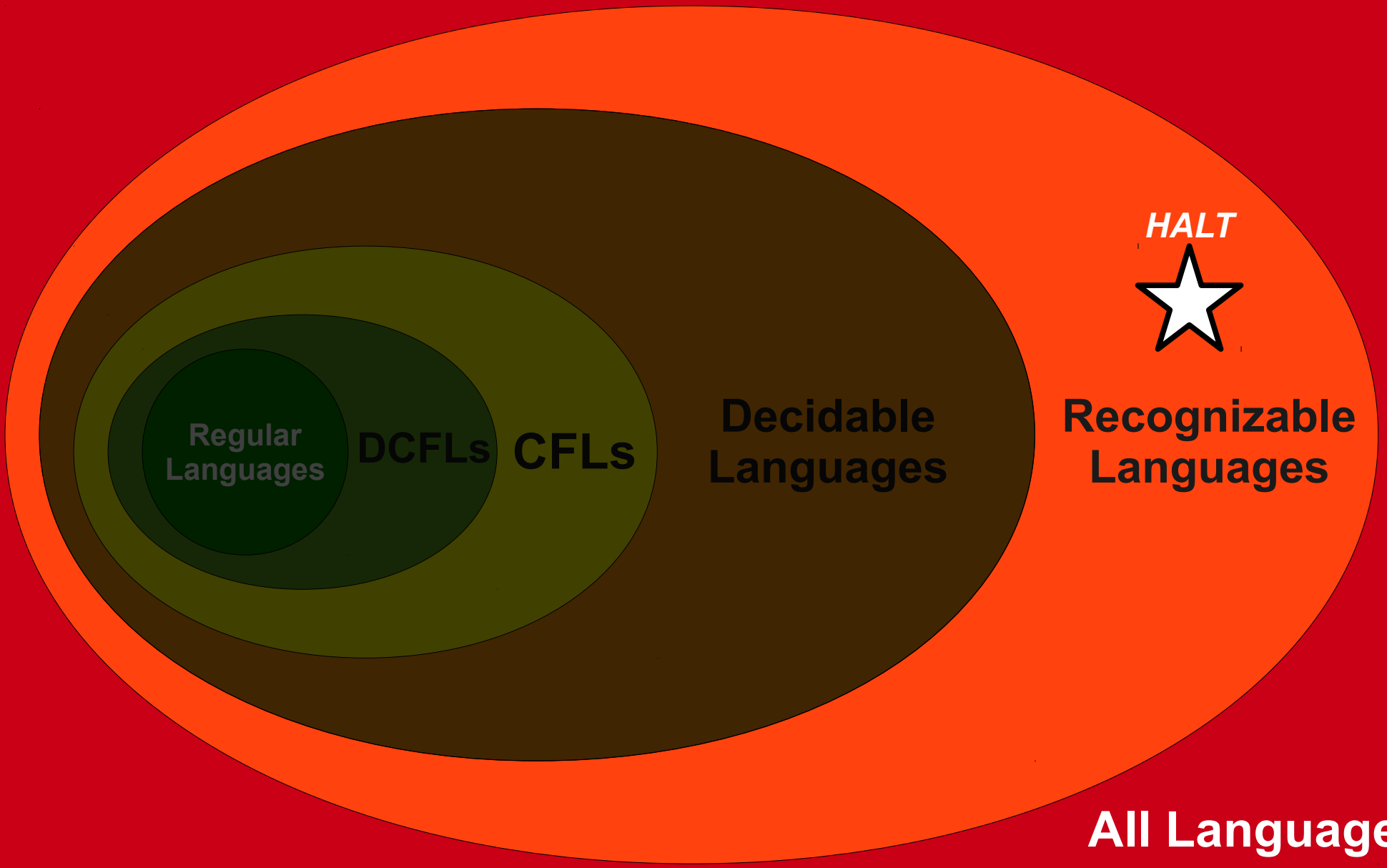

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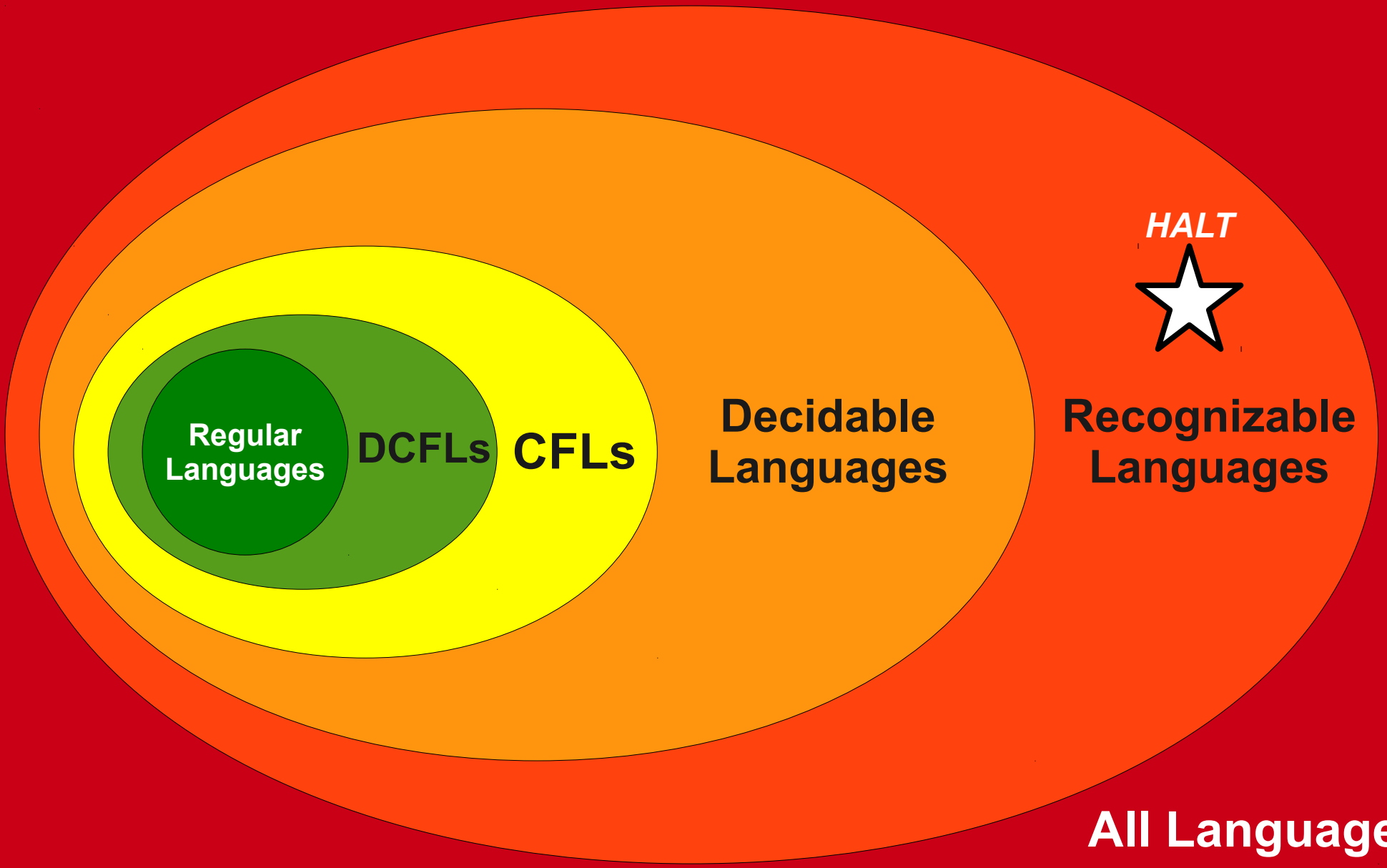

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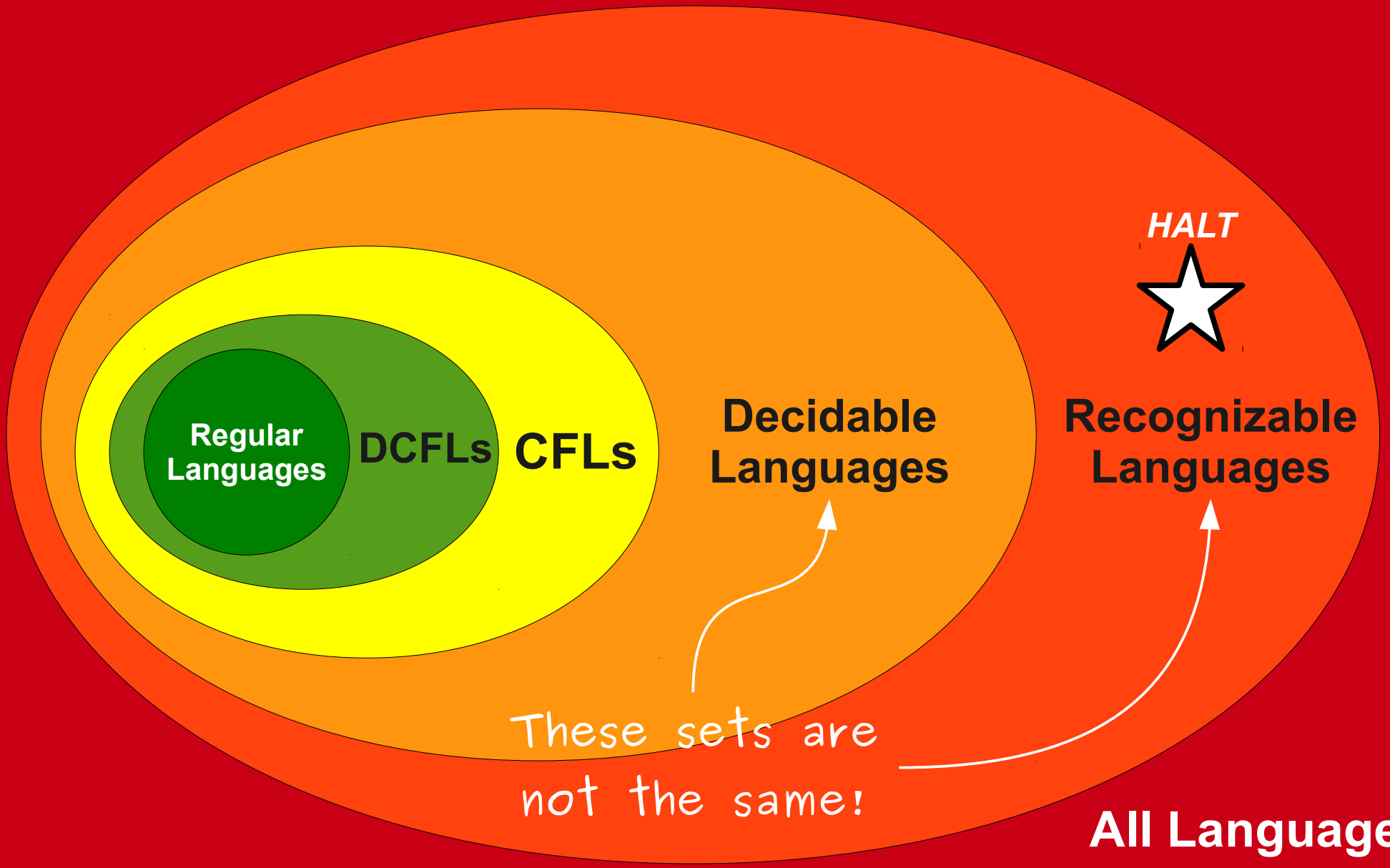

The Halting Problem is Undecidable

- The fact that *HALT* is undecidable immediately gives some important results.
  - There are problems that are **recognizable** but not **decidable**.
  - There is no general procedure for converting a **recognizer** into a **decider**.
  - As we will see, there is **no general algorithm** for reasoning about what a given program will or will not do.
- This is one of the most famous and important proofs in computer science.

Summary

- The **diagonalization language** L_D is not Turing-recognizable.
  - No feasible computing device is powerful enough to even *verify* that a string is in L_D!
- The **halting problem** *HALT* is Turing-recognizable, but not Turing-decidable.
  - We can confirm that a TM halts on some input by simulating it to see what happens.
  - However, we can never **decide** whether a TM will halt on some input.

Next Time

- **Finding more unsolvable problems**.
  - Closure properties of recognizable and decidable languages.
  - Mapping reductions.