Mathematical Logic

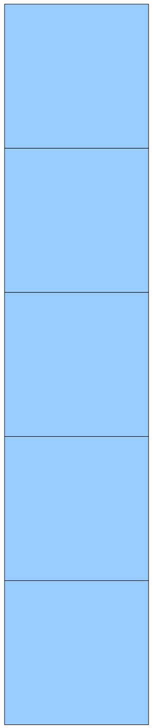
Part III

Announcements

- Problem Set 3 due Friday, October 21.
  - Stop by office hours!
  - Email us questions at **cs103@cs.stanford.edu**
- Clarification on late day policy:
  - You can use at most one late day per problem set.
  - Late days are 24-hour late days.
  - **Don't panic** if you got either of these details wrong; we will not penalize you this time around.
- Review session tonight, 370-370 at 7:00PM.

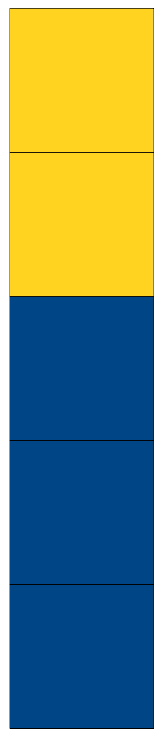
Recall: The Unstacking Game

Score: **0**



Recall: The Unstacking Game

Score: **0**



Recall: The Unstacking Game

Score: **0**

Recall: The Unstacking Game

Score: **0**

3 × 2 = 6

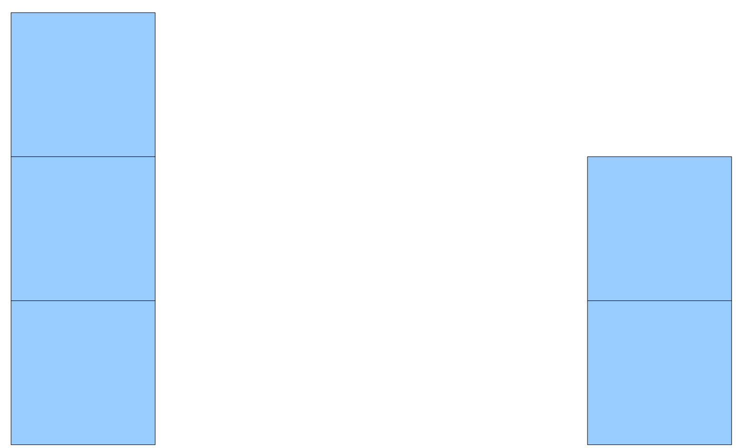
Recall: The Unstacking Game

Score: **6**

$3 \times 2 = 6$

Recall: The Unstacking Game

Score: **6**



Recall: The Unstacking Game

Score: **6**

Recall: The Unstacking Game

Score: **6**

Recall: The Unstacking Game

Score: **6**

$1 \times 1 = 1$

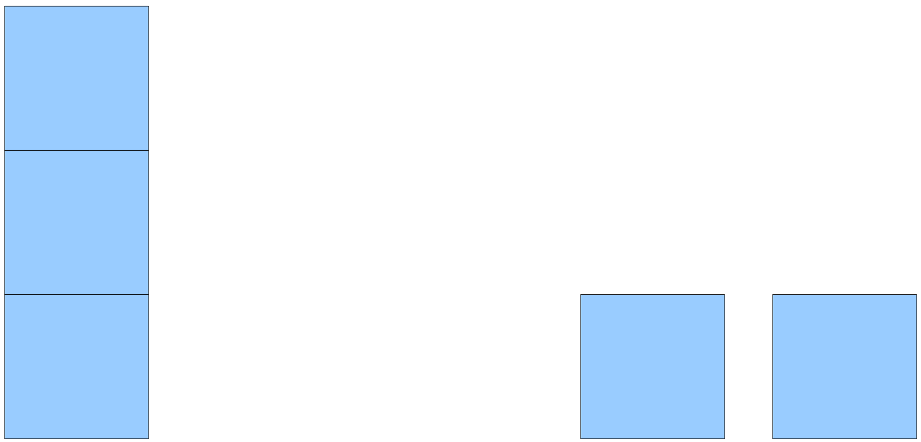
Recall: The Unstacking Game

Score: **7**

$1 \times 1 = 1$

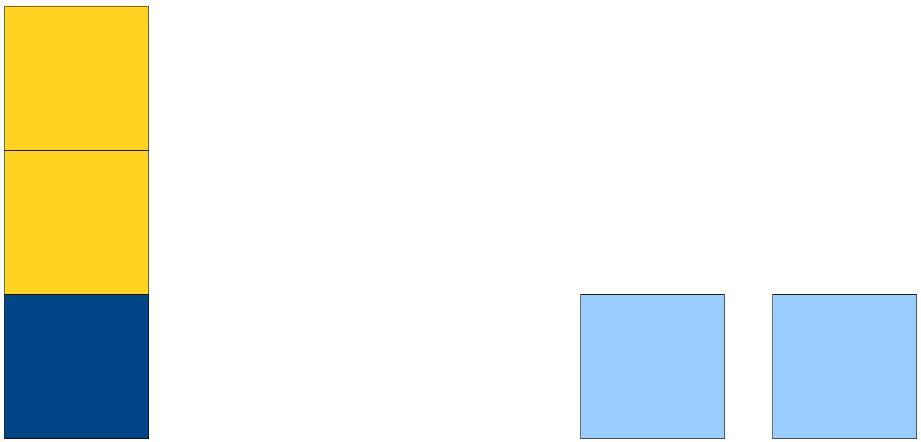
Recall: The Unstacking Game

Score: **7**



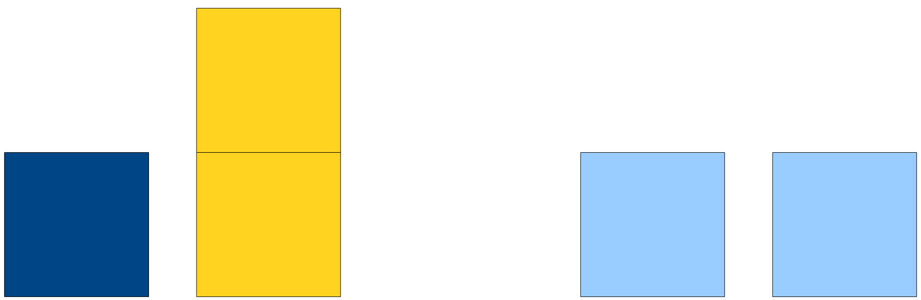
Recall: The Unstacking Game

Score: **7**



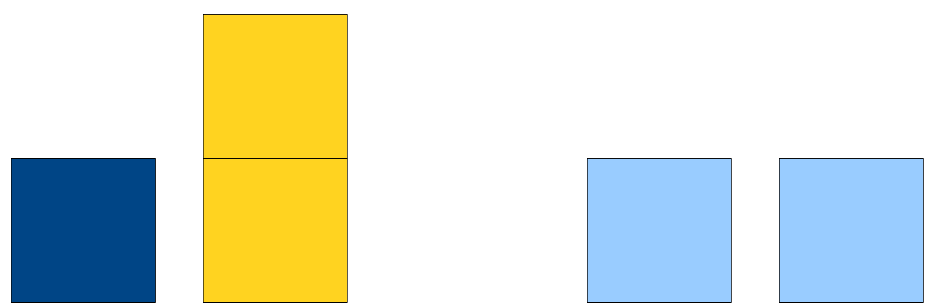
Recall: The Unstacking Game

Score: **7**



Recall: The Unstacking Game

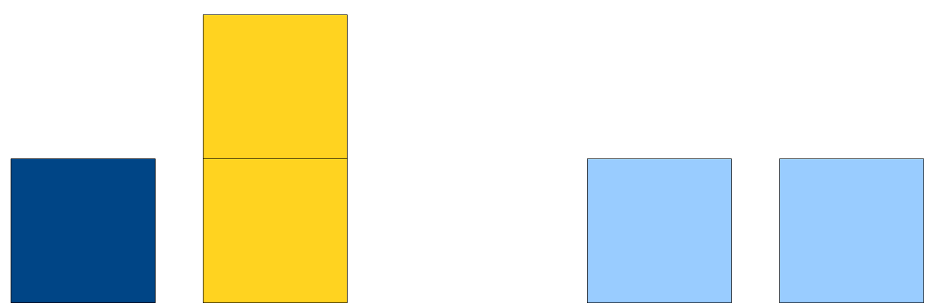
Score: **7**



$1 \times 2 = 2$

Recall: The Unstacking Game

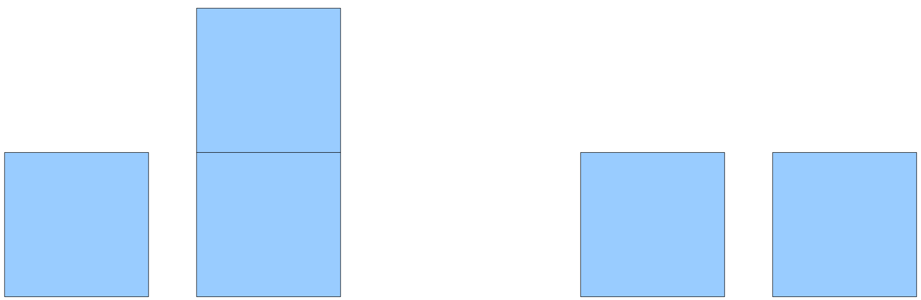
Score: **9**



$1 \times 2 = 2$

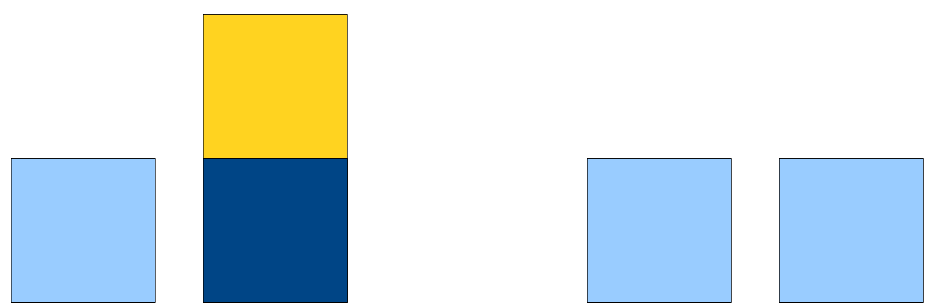
Recall: The Unstacking Game

Score: **9**



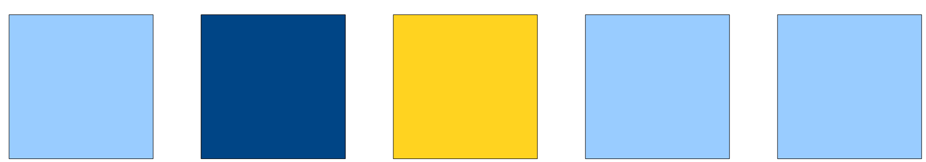
Recall: The Unstacking Game

Score: **9**



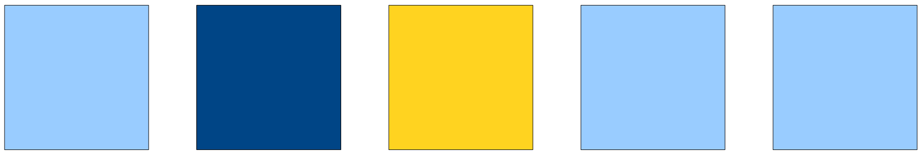
Recall: The Unstacking Game

Score: **9**



Recall: The Unstacking Game

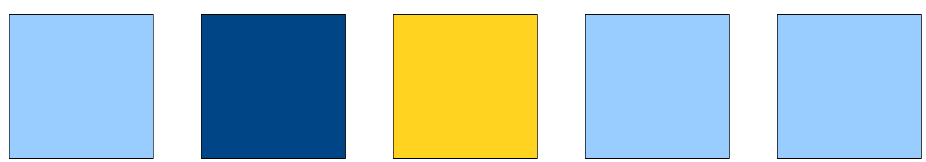
Score: **9**



$$1 \times 1 = 1$$

Recall: The Unstacking Game

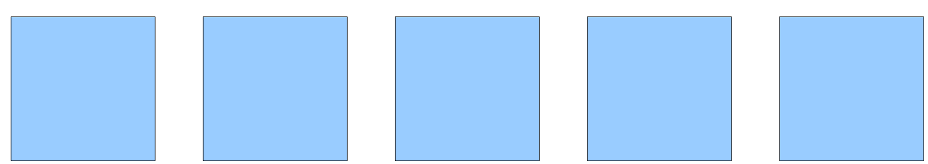
Score: **10**



$$1 \times 1 = 1$$

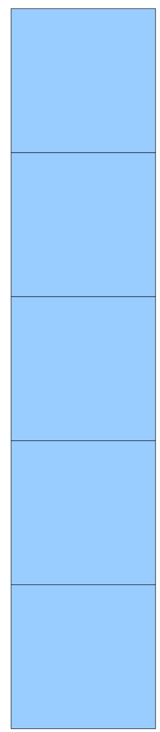
Recall: The Unstacking Game

Score: **10**



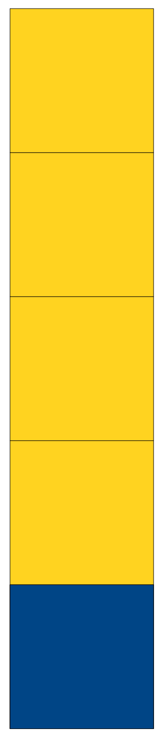
Another Example

Score: **0**



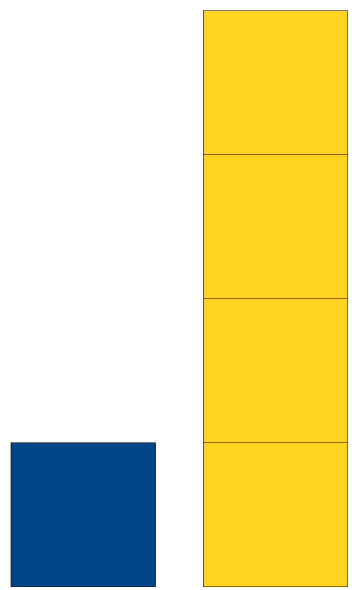
Another Example

Score: **0**



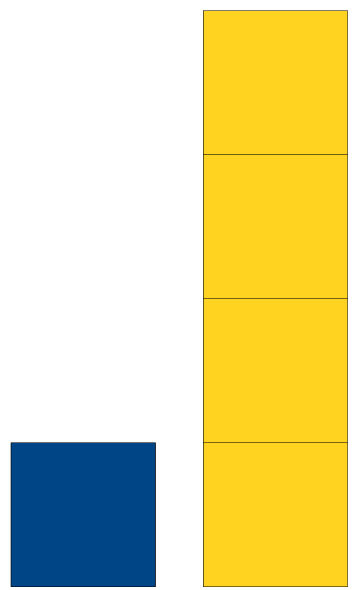
Another Example

Score: **0**



Another Example

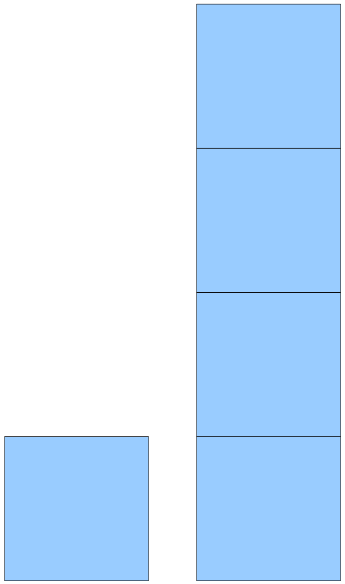
Score: **4**



$$1 \times 4 = 4$$

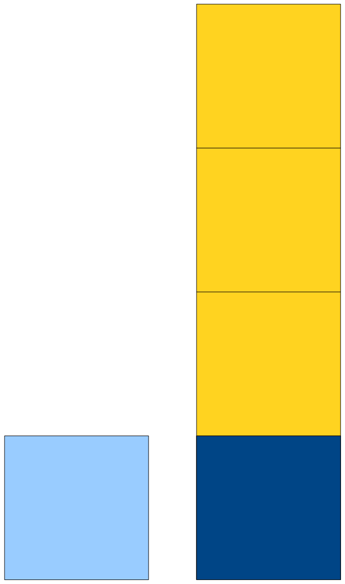
Another Example

Score: **4**



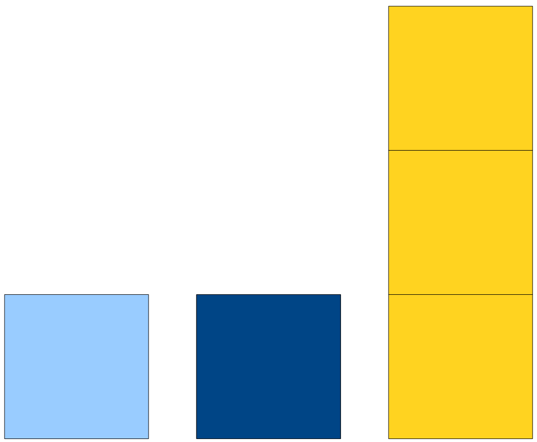
Another Example

Score: **4**



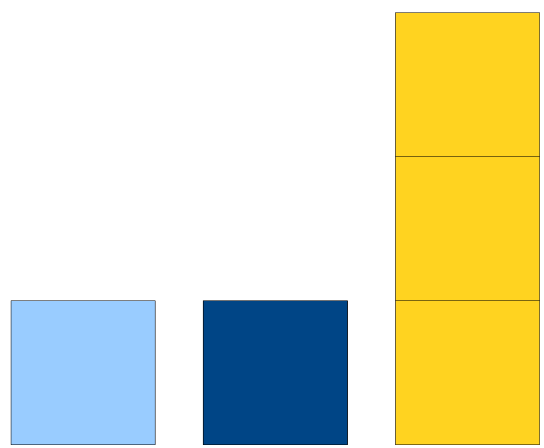
Another Example

Score: **4**



Another Example

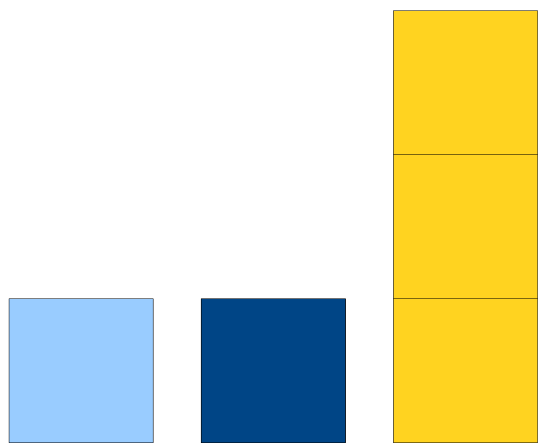
Score: **4**



$1 \times 3 = 3$

Another Example

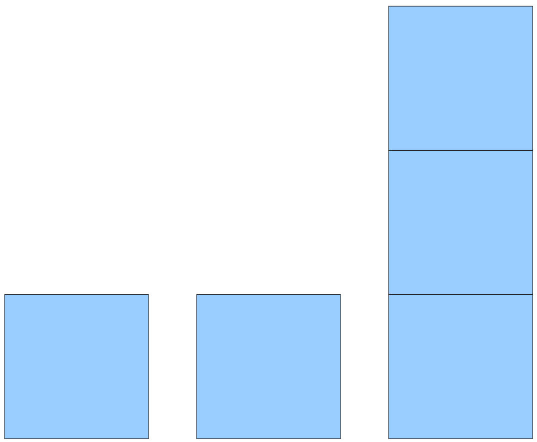
Score: **7**



$$1 \times 3 = 3$$

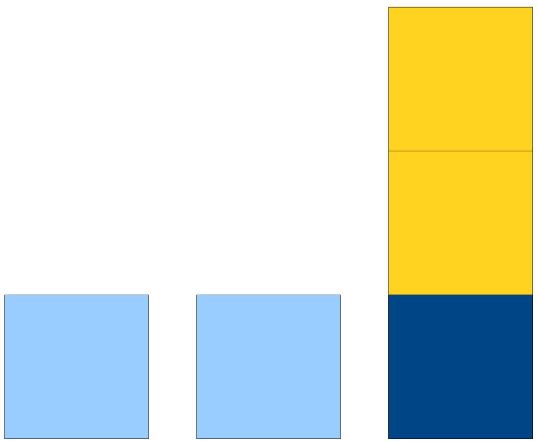
Another Example

Score: **7**



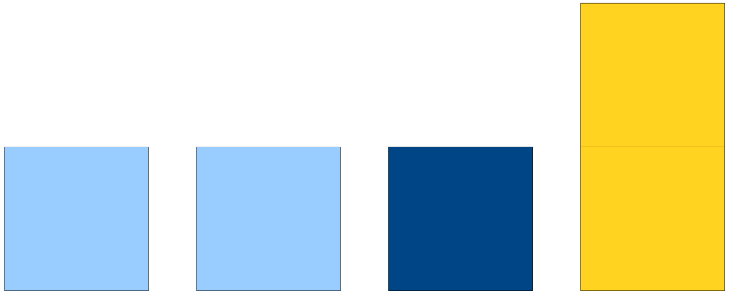
Another Example

Score: **7**



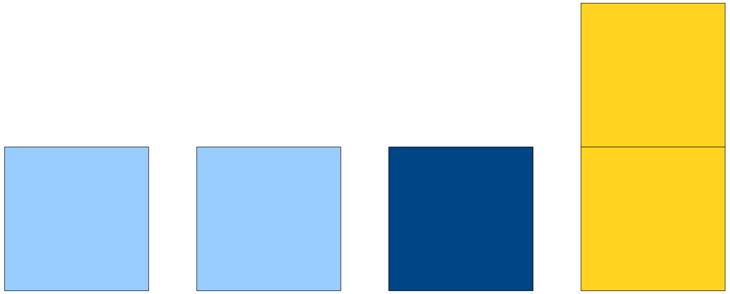
Another Example

Score: **7**



Another Example

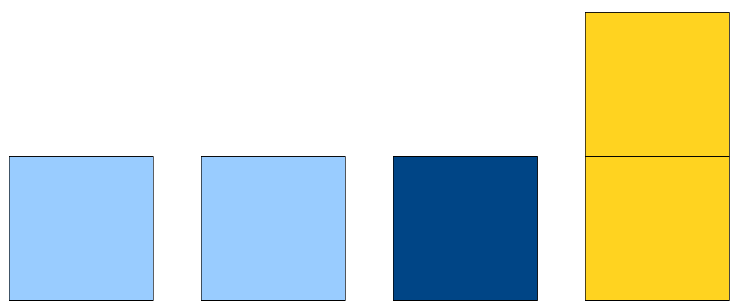
Score: **7**



$1 \times 2 = 2$

Another Example

Score: **9**



$1 \times 2 = 2$

Another Example

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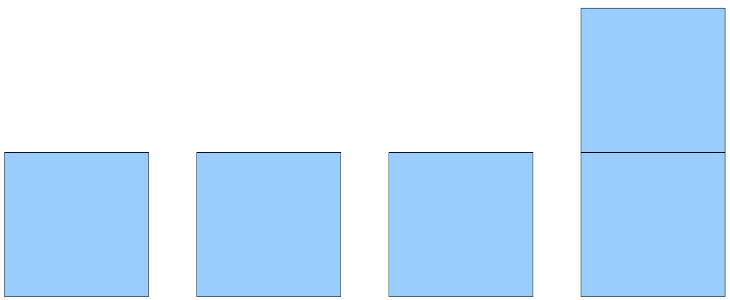

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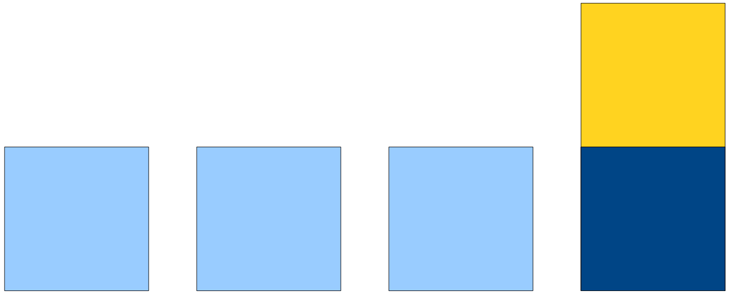

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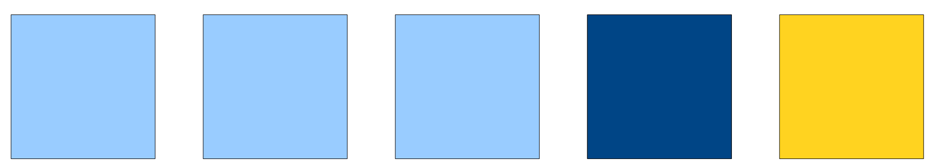
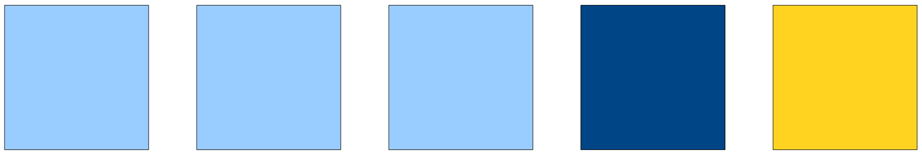
Score: **9**



Another Example

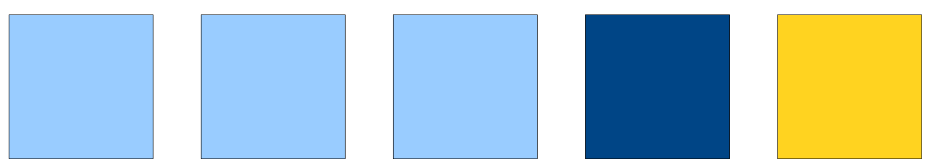
Score: **9**



$1 \times 1 = 1$

Another Example

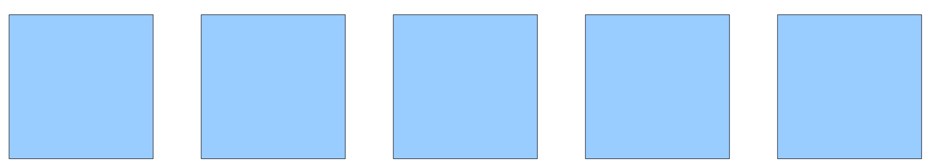
Score: **10**



$1 \times 1 = 1$

Another Example

Score: **10**



The Important Result

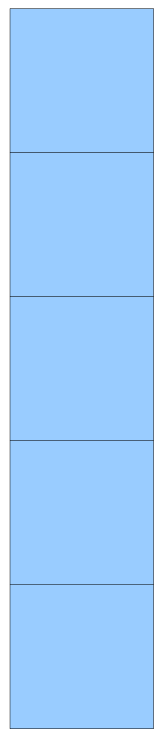
- The score for the unstacking game is **always** n(n – 1)/2, assuming that you start with n blocks in the original tower.
- For extra credit, I asked you to find an intuitive explanation for this phenomenon.
- We, the CS103 staff, proudly presents the best explanations we've seen.

What does the number $n(n – 1) / 2$ mean?

n(n – 1) / 2 is the number of unordered pairs of n distinct elements.

The First Intuition

Score: **0**



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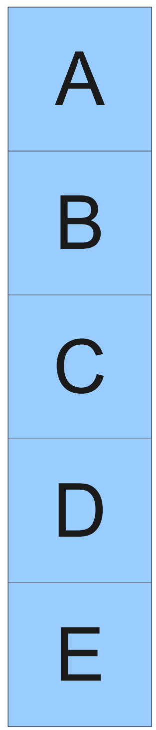
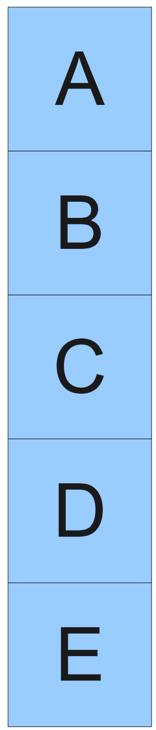
Score: **0**



The First Intuition

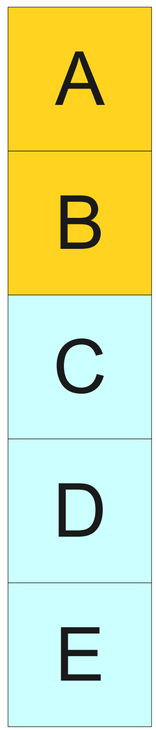
Score: **0**



AB, AC, AD,
AE, BC, BD,
BE, CD, CD,
DE

The First Intuition

Score: **0**



AB, AC, AD,
AE, BC, BD,
BE, CD, CD,
DE

The First Intuition

Score: **0**

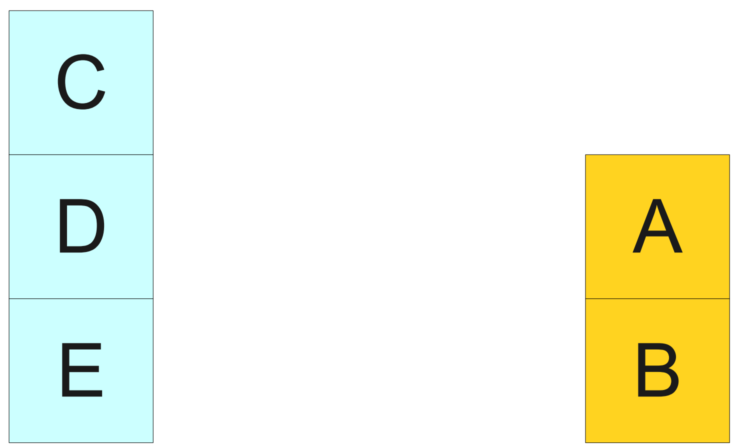
CD, CE, DE

AB

The First Intuition

Score: **0**

AC, AD, AE,
BC, BD, BE



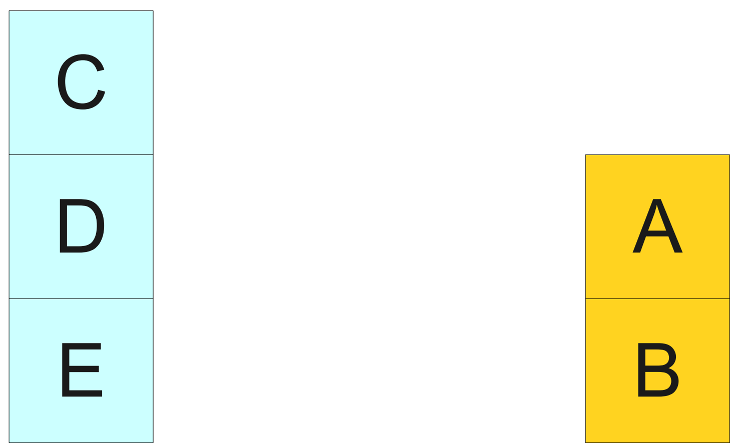
CD, CE, DE

AB

The First Intuition

Score: **6**

AC, AD, AE,
BC, BD, BE

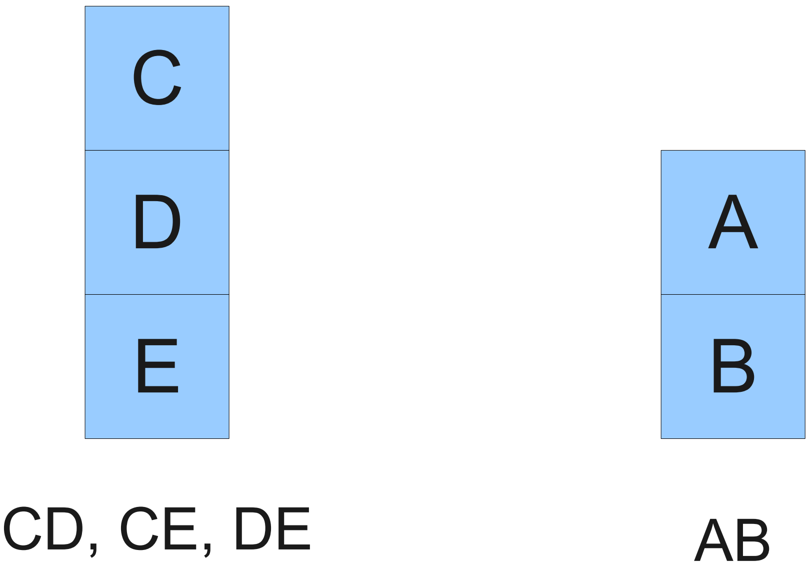


CD, CE, DE

AB

The First Intuition

Score: **6**



The First Intuition

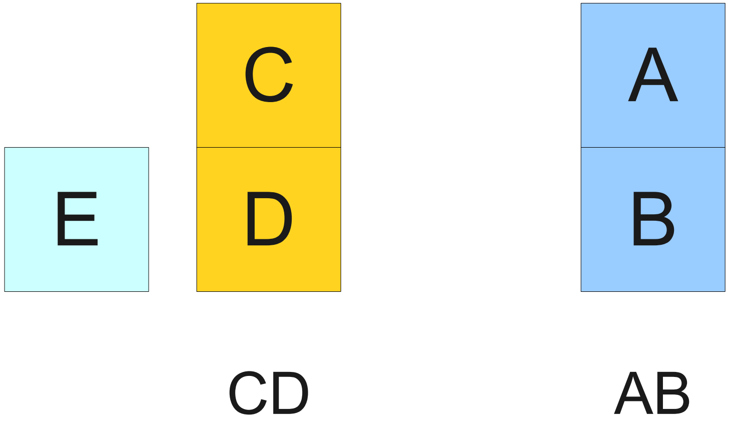
Score: **6**

CD, CE, DE

AB

The First Intuition

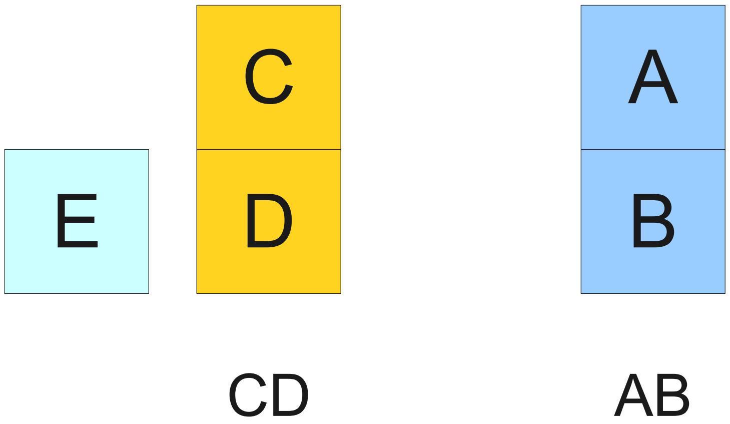
Score: **6**



The First Intuition

EC, CD

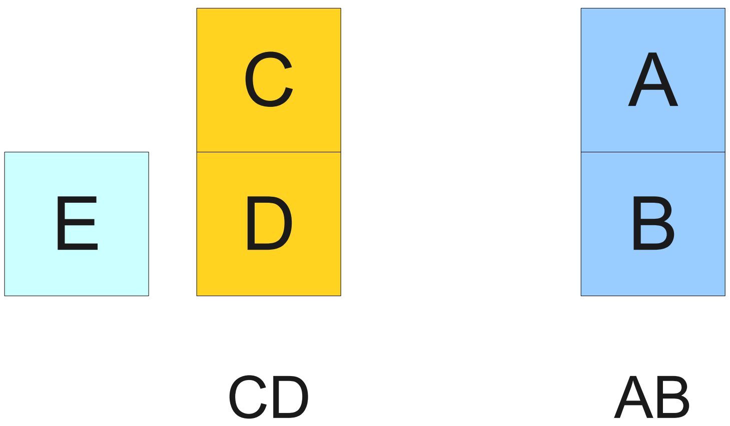
Score: **6**



The First Intuition

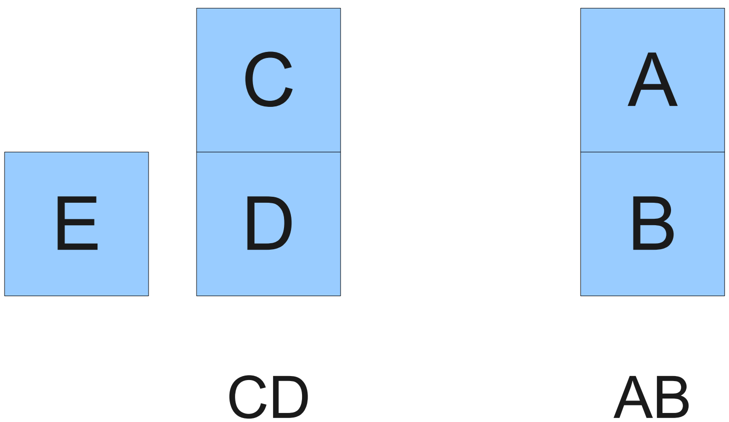
Score: **8**

EC, CD



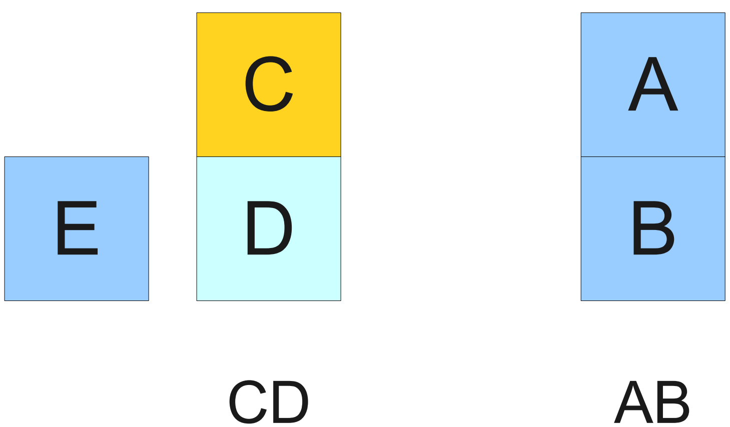
The First Intuition

Score: **8**



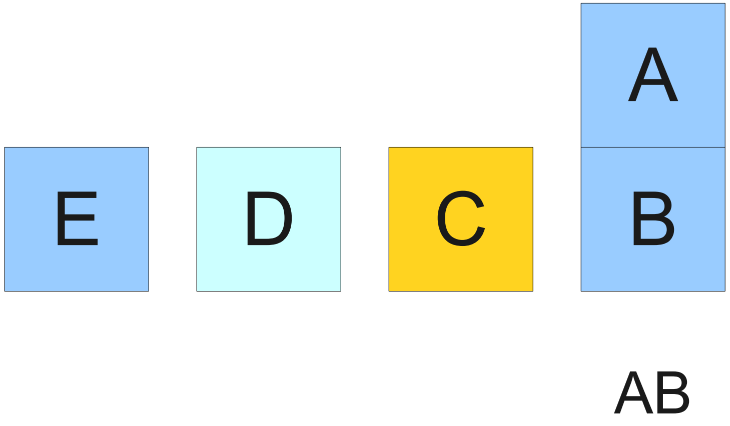
The First Intuition

Score: **8**



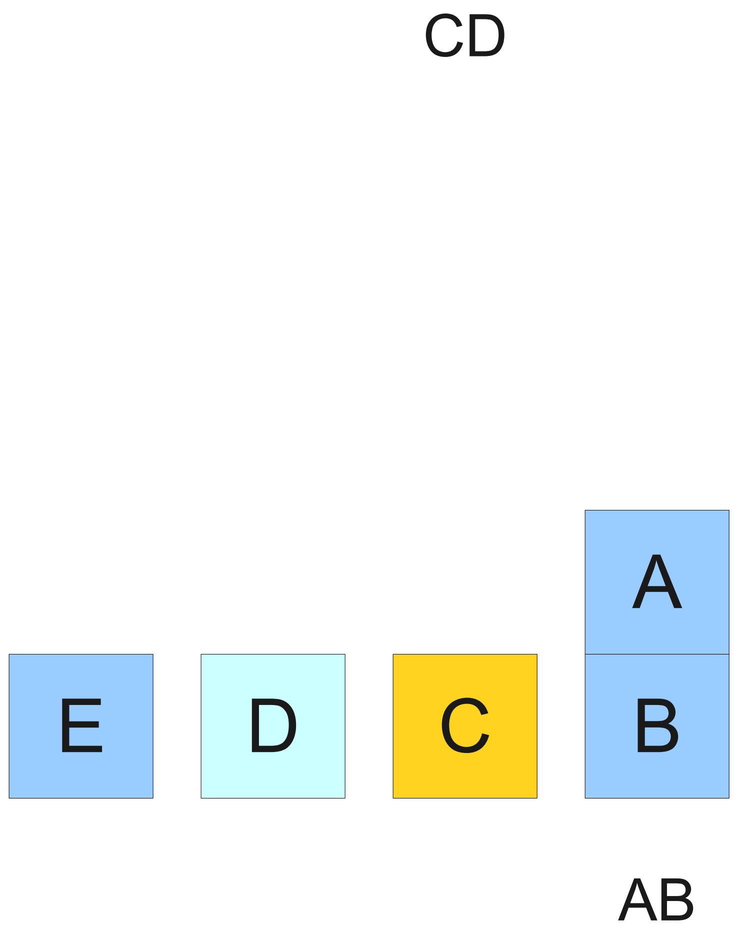
The First Intuition

Score: **8**



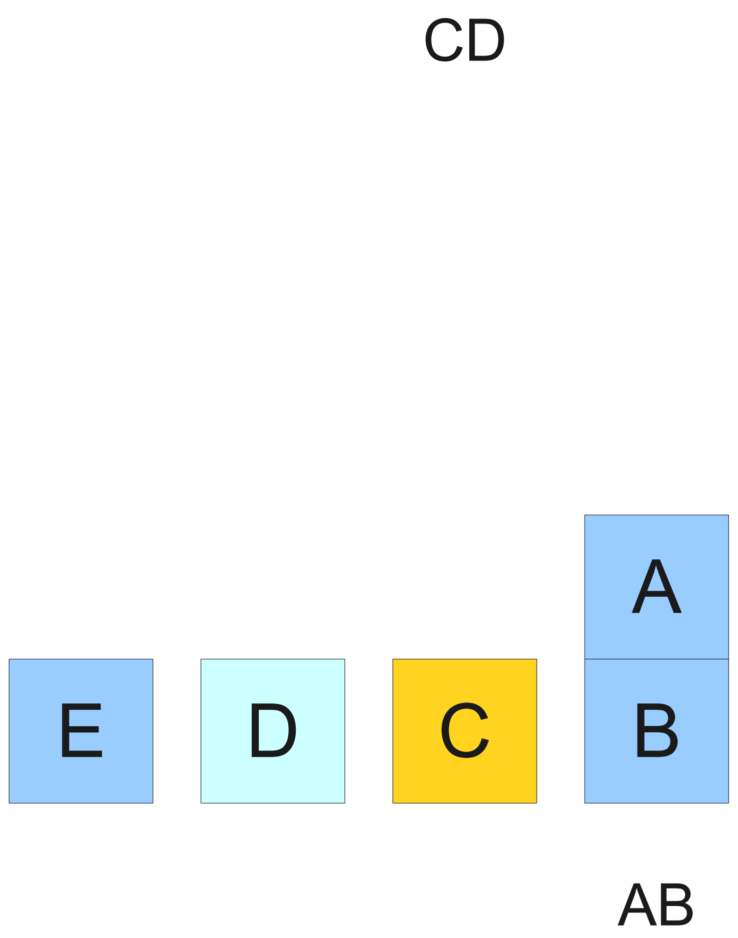
The First Intuition

Score: **8**



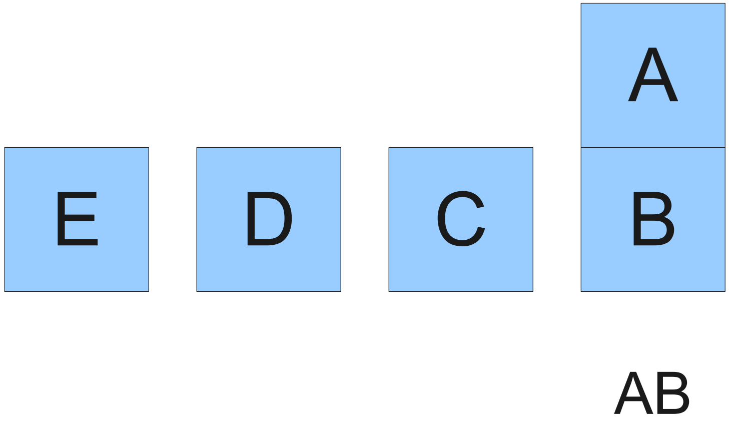
The First Intuition

Score: **9**



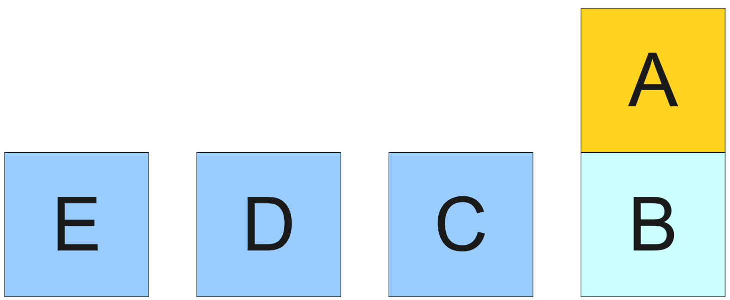
The First Intuition

Score: **9**



The First Intuition

Score: **9**



AB

The First Intuition

Score: **9**

E D C B A

The First Intuition

Score: **9**

AB

E D C B A

The First Intuition

Score: **10**

AB

E D C B A

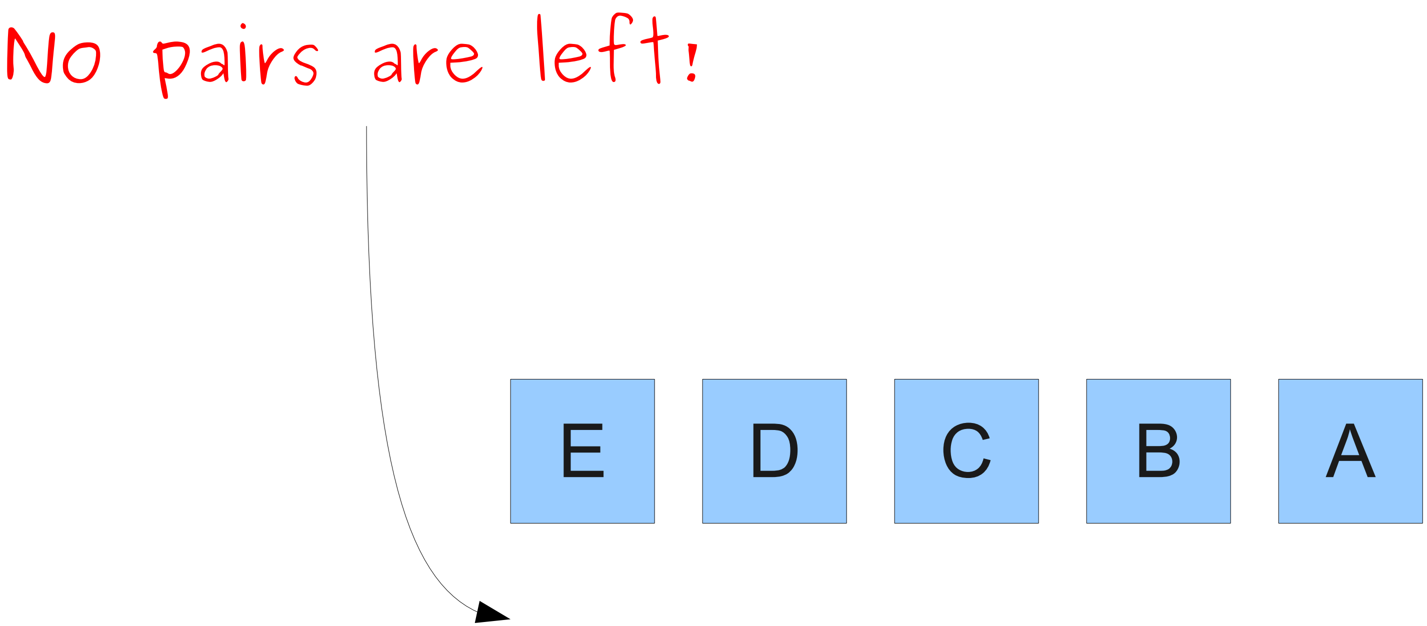
The First Intuition

Score: **10**

E D C B A

The First Intuition

Score: **10**

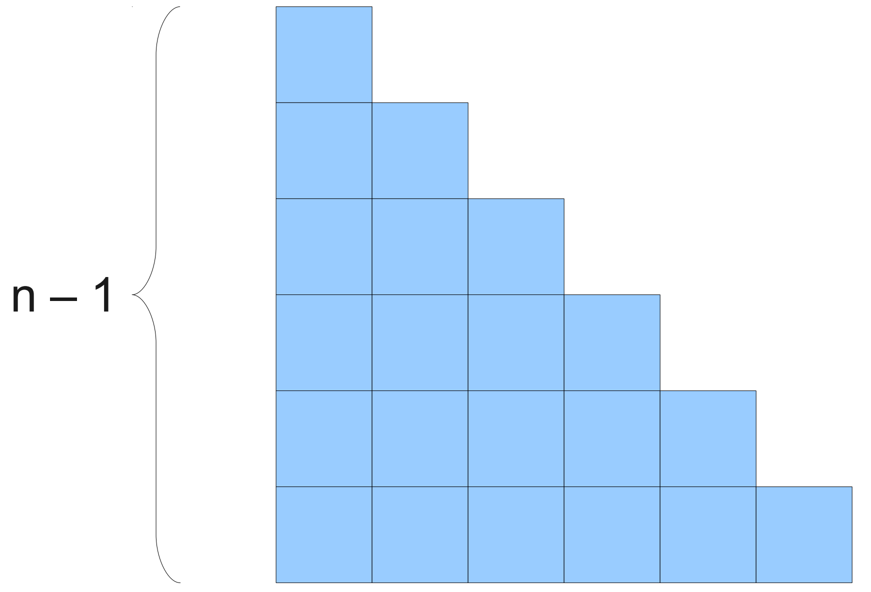


The First Intuition

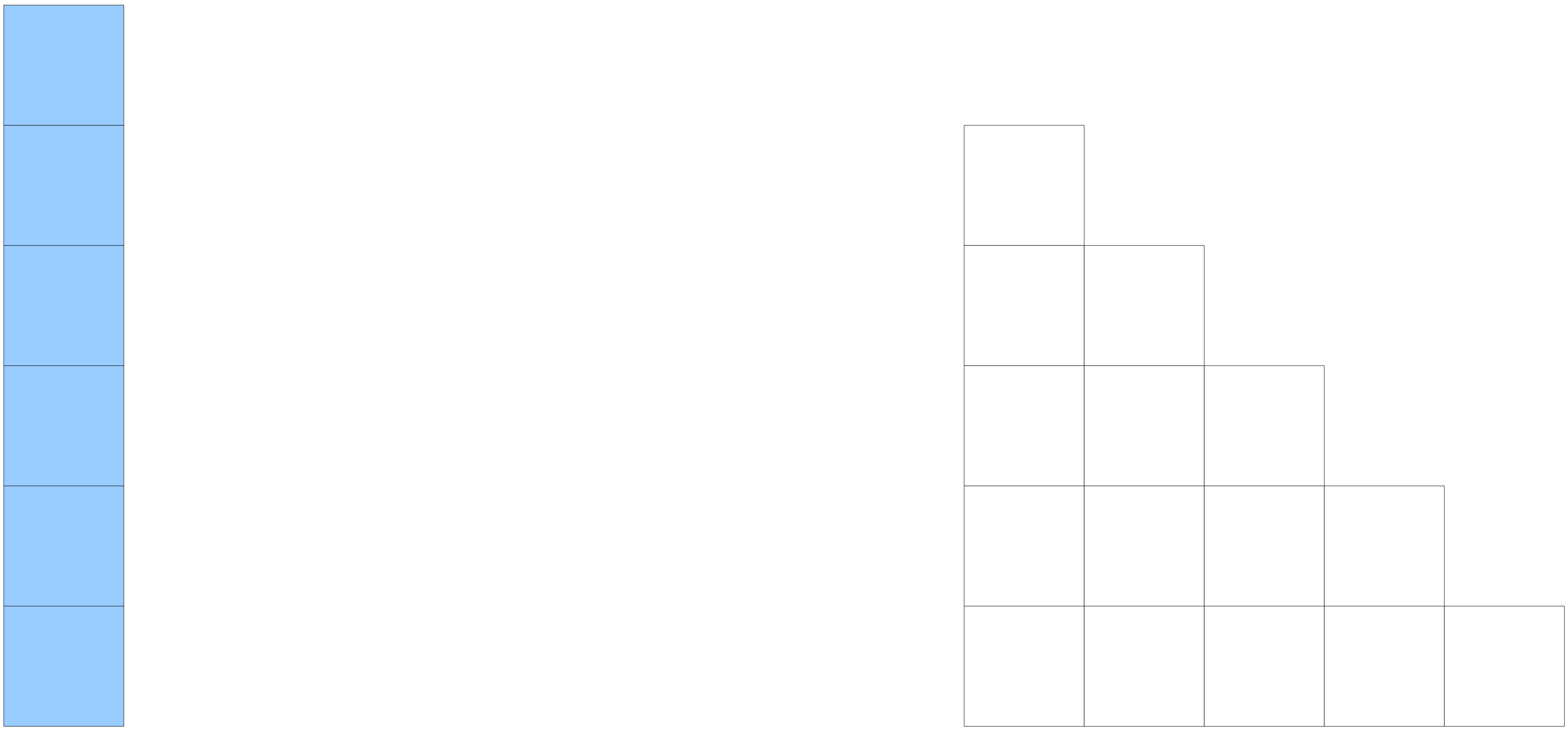
- At the start of the game, there are n(n – 1)/2 pairs of elements associated with the initial stack.
- Each move splits the pairs into three groups:
  - Pairs solely in the left stack.
  - Pairs solely in the right stack.
  - Pairs broken by the move.
- If the split is n – k and k, there are k(n – k) pairs broken by the move.
- The score for the move is k(n – k), which is the number of broken pairs.
- At the end of the game, each stack has height one.
- All pairs are eventually broken, so the total score is equal to the total number of pairs: n(n – 1) / 2.

The Second Intuition

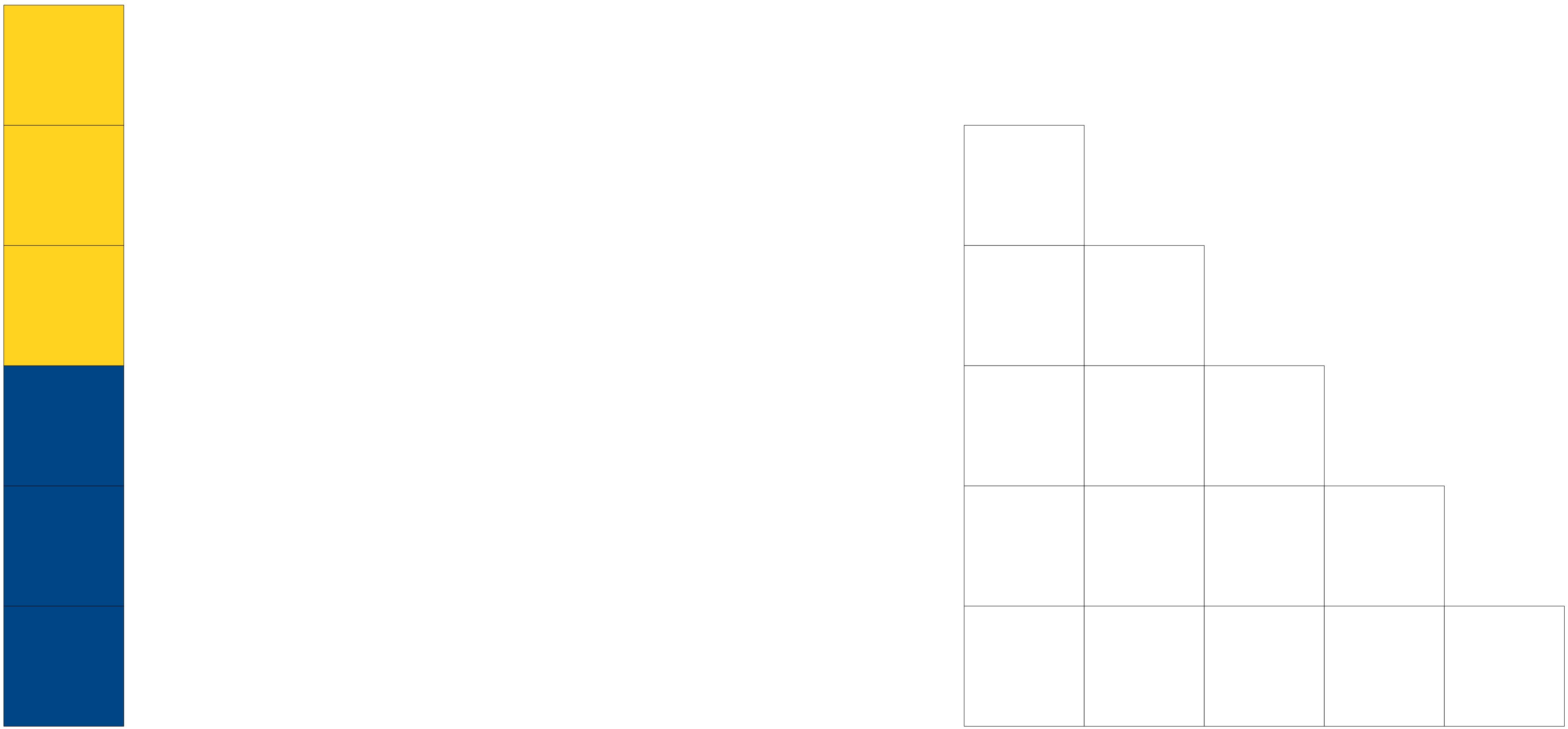
n(n – 1) / 2 is the number of squares in this triangle:



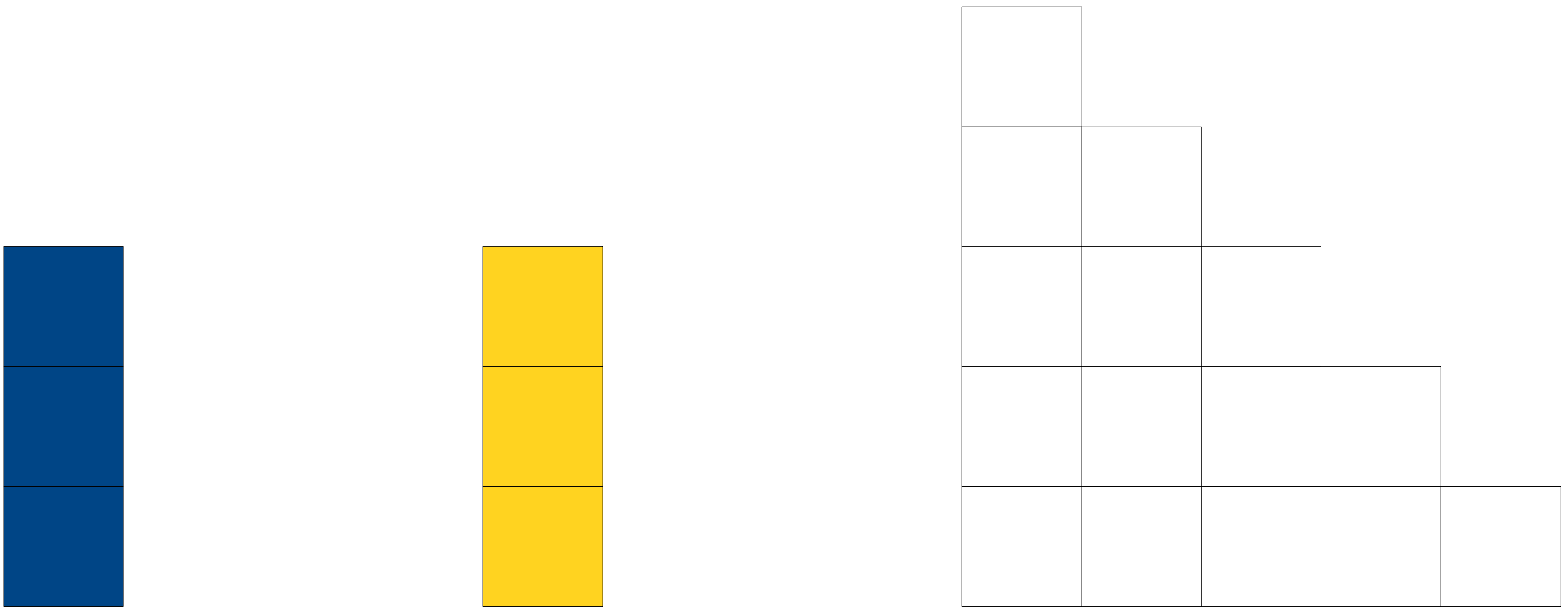
The Second Intuition



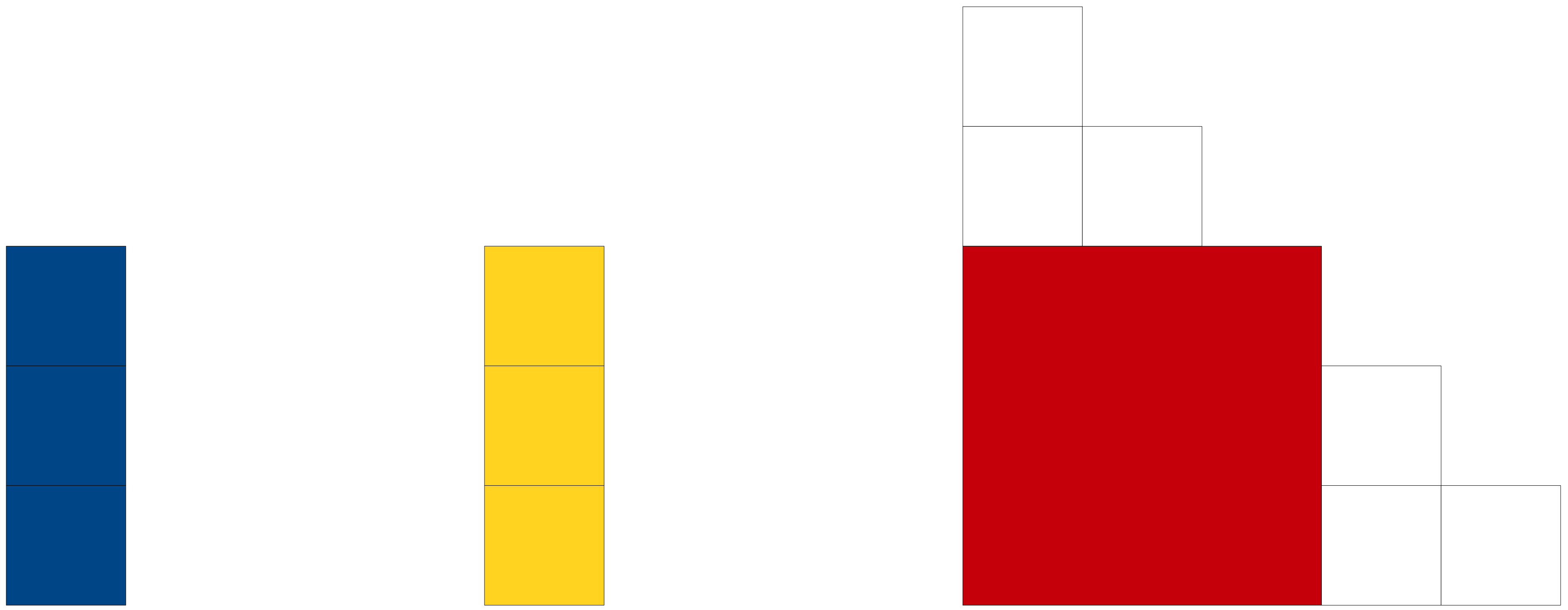
The Second Intuition



The Second Intuition



The Second Intuition



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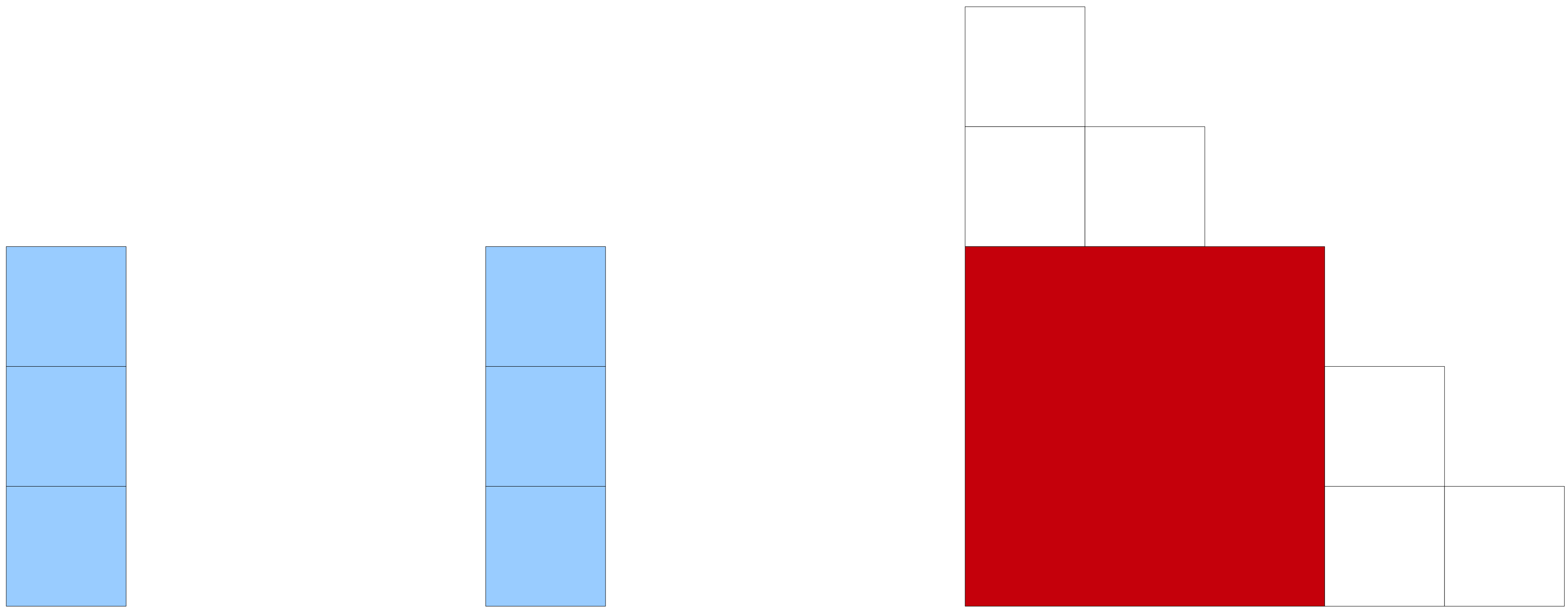

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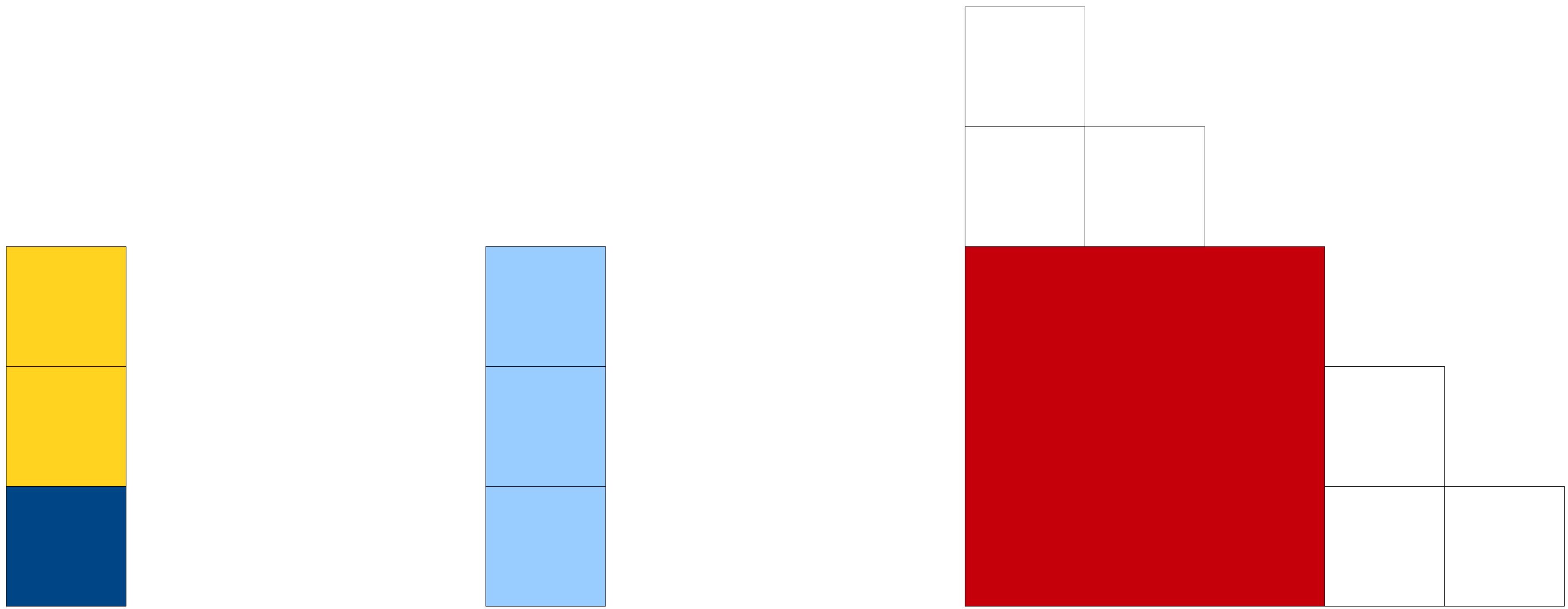

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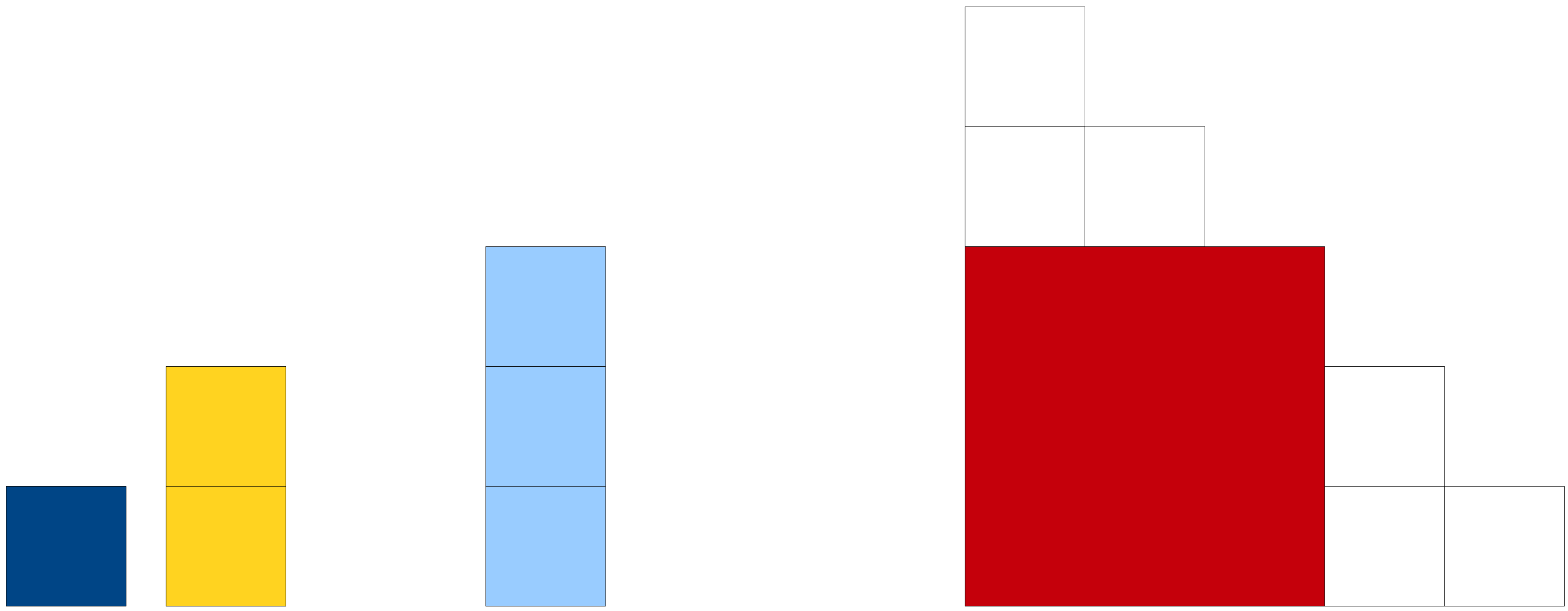

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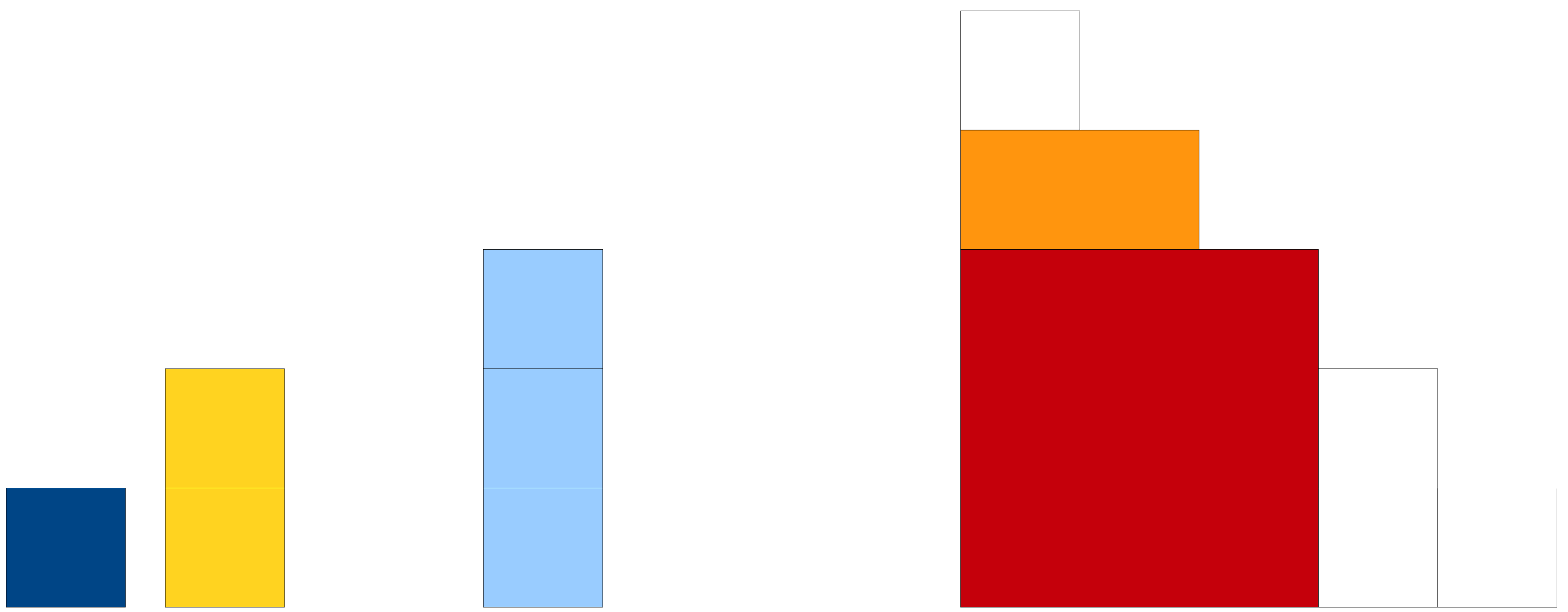
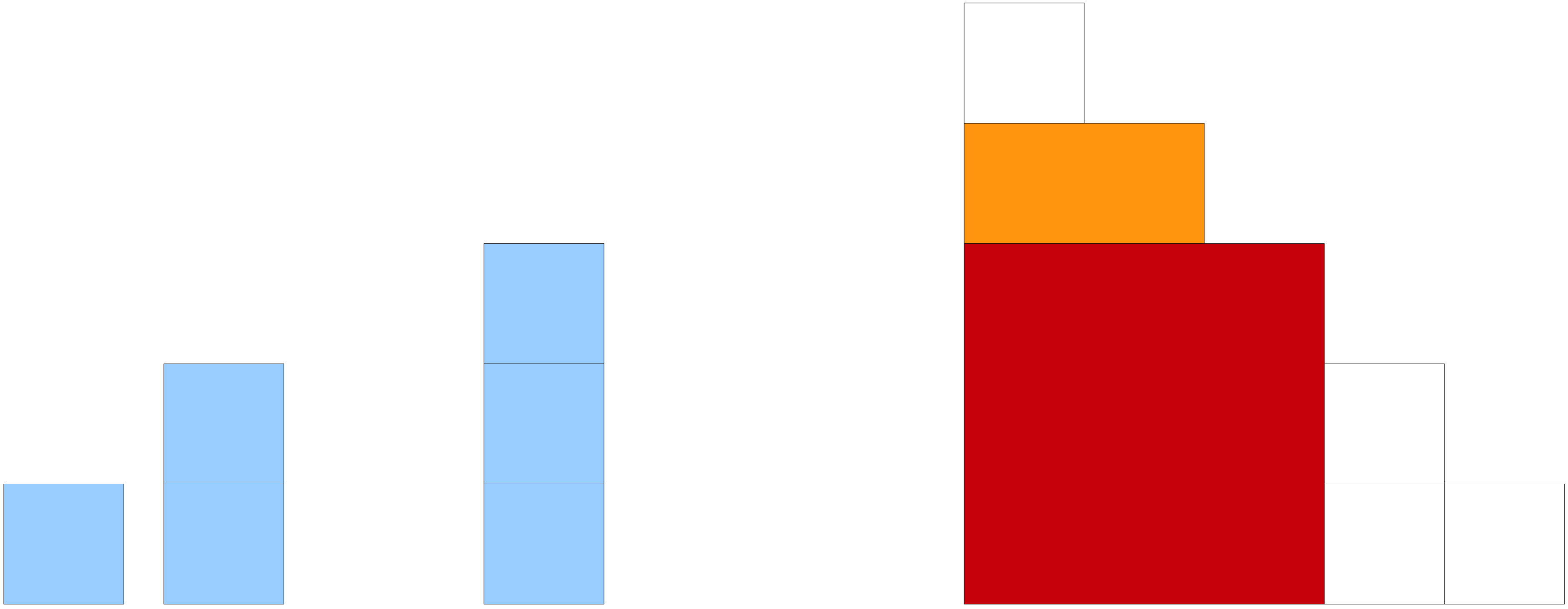


The Second Intuition



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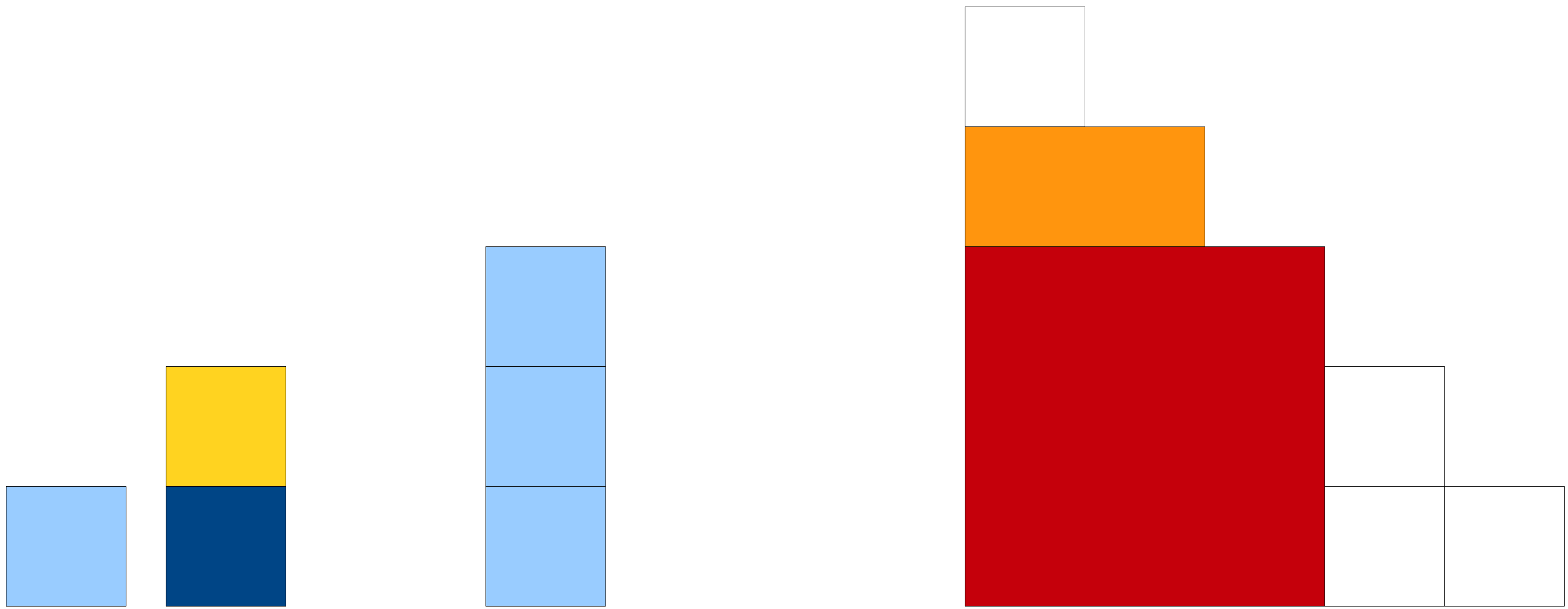

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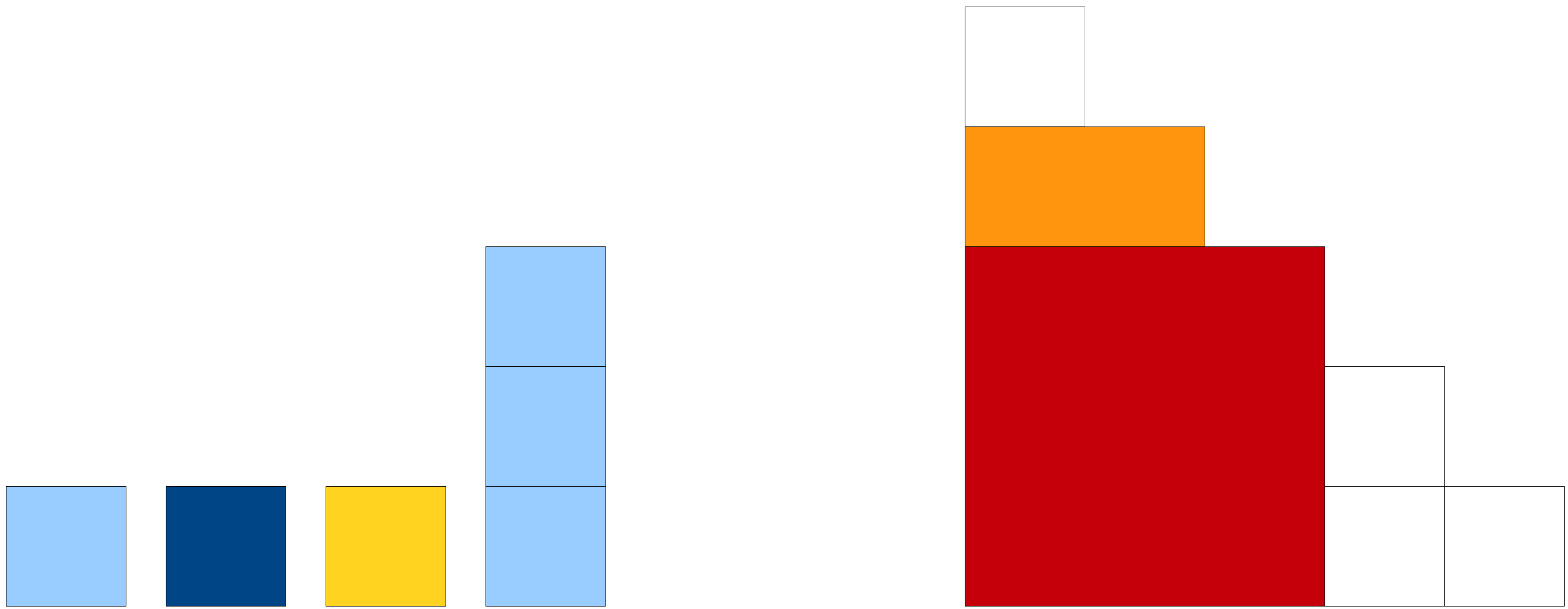

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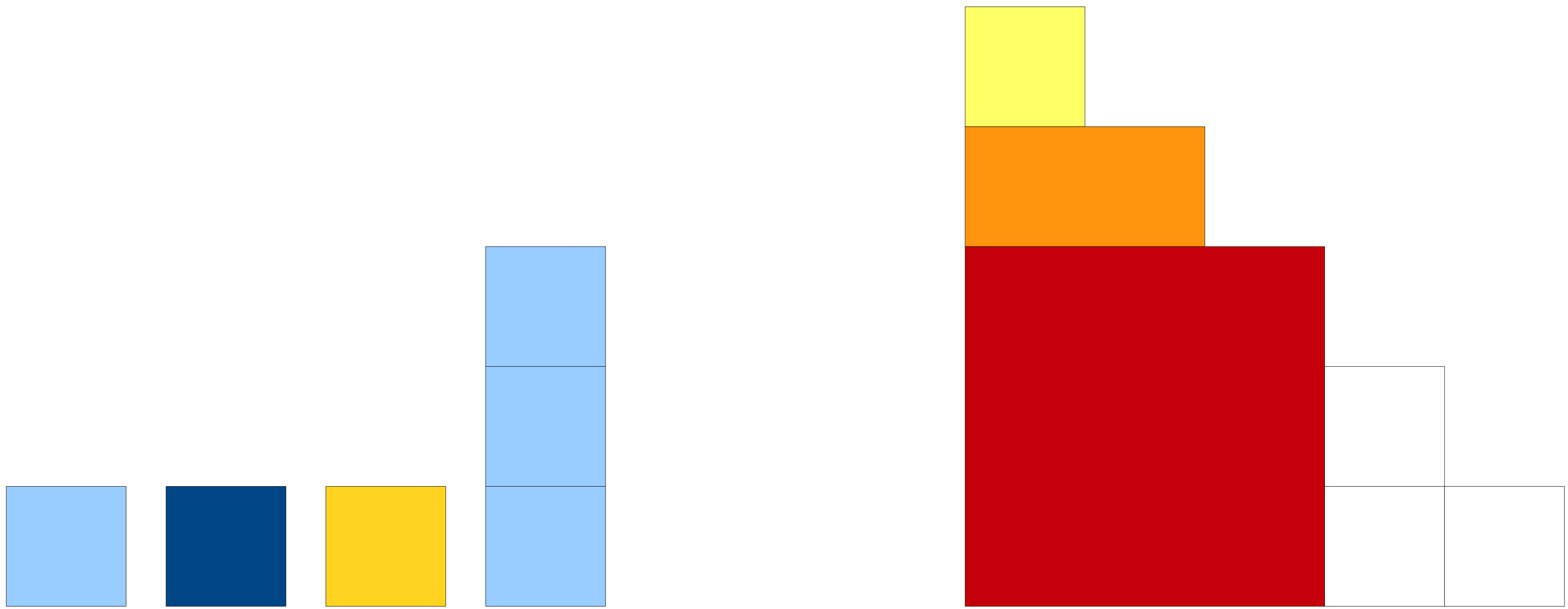

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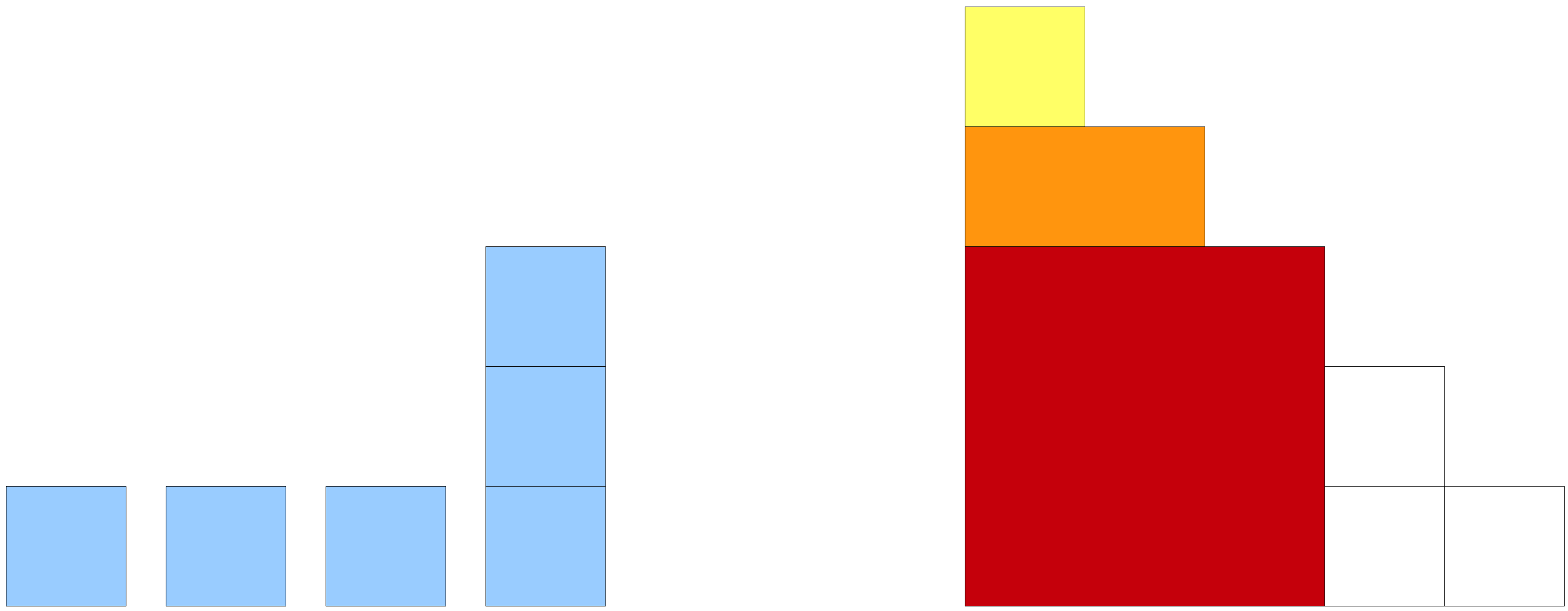

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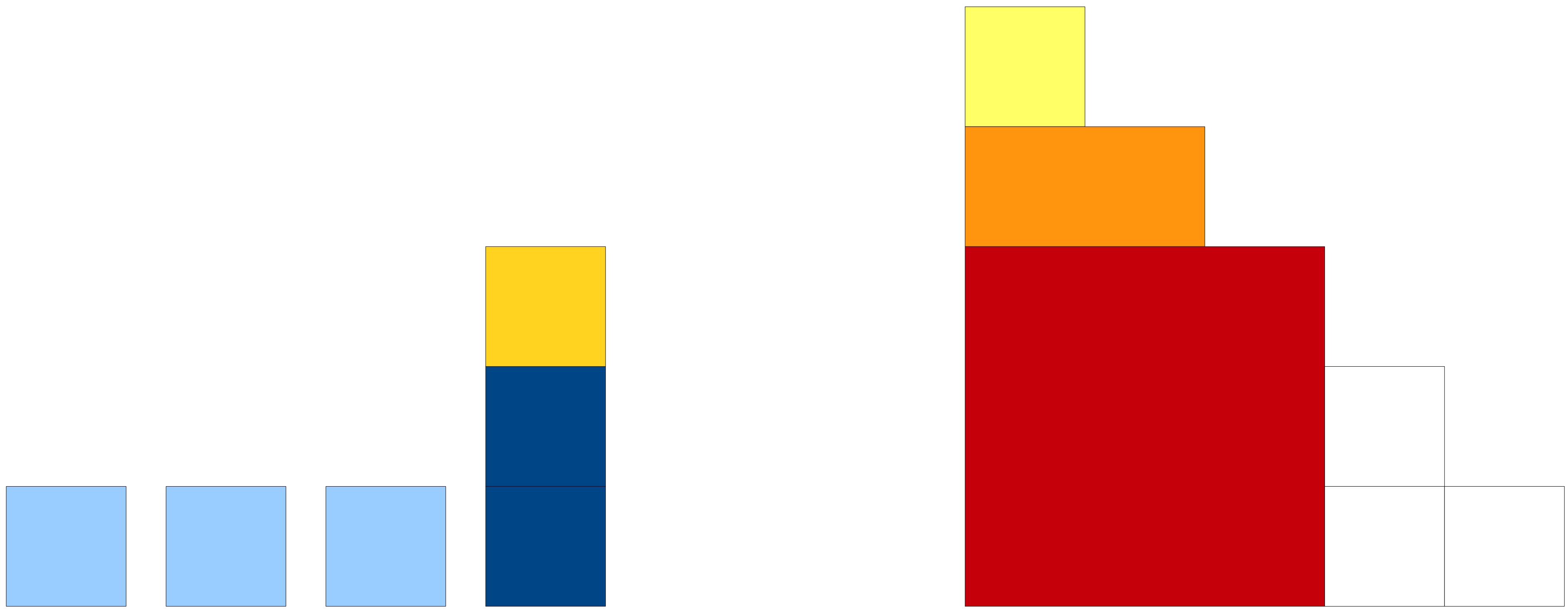

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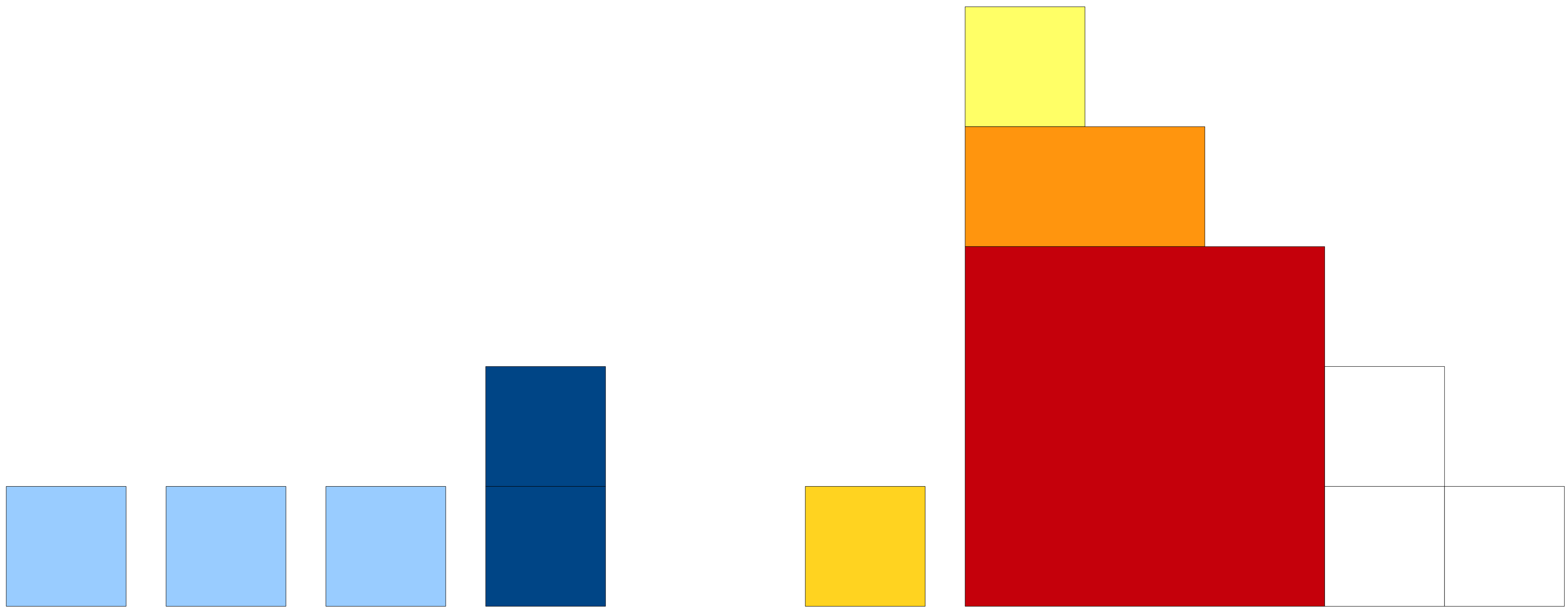

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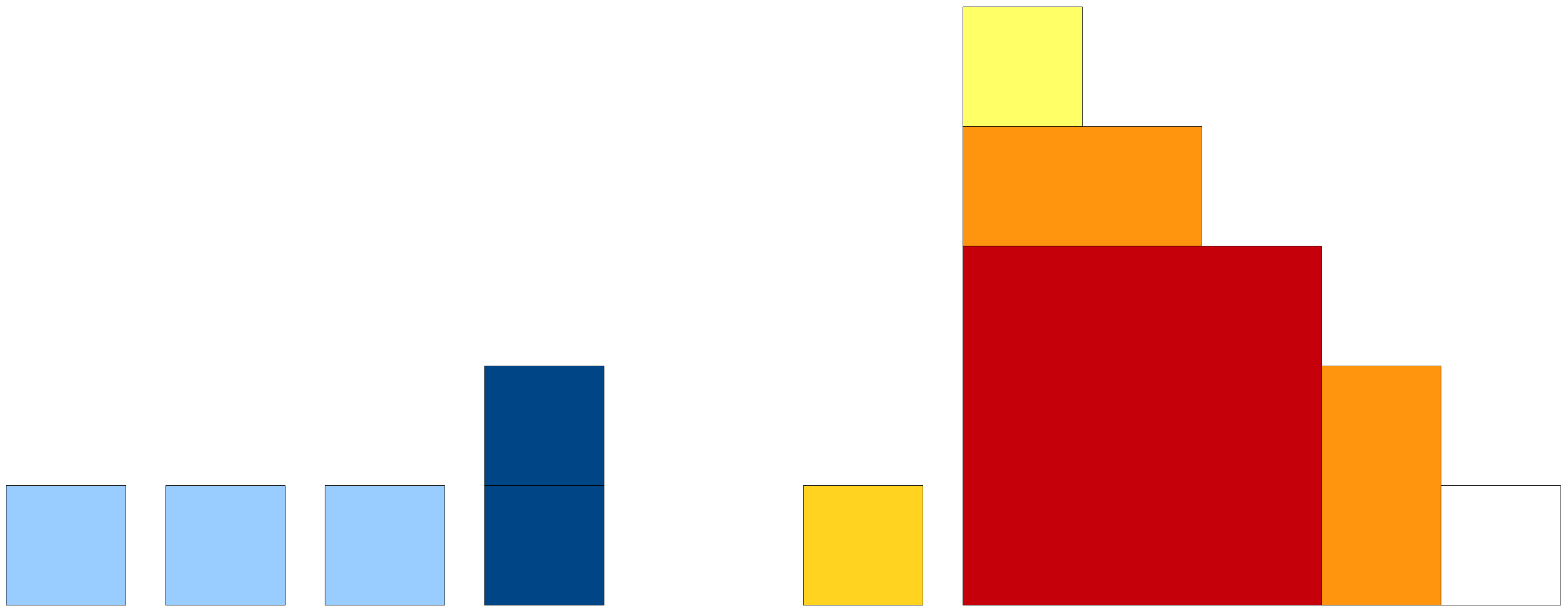

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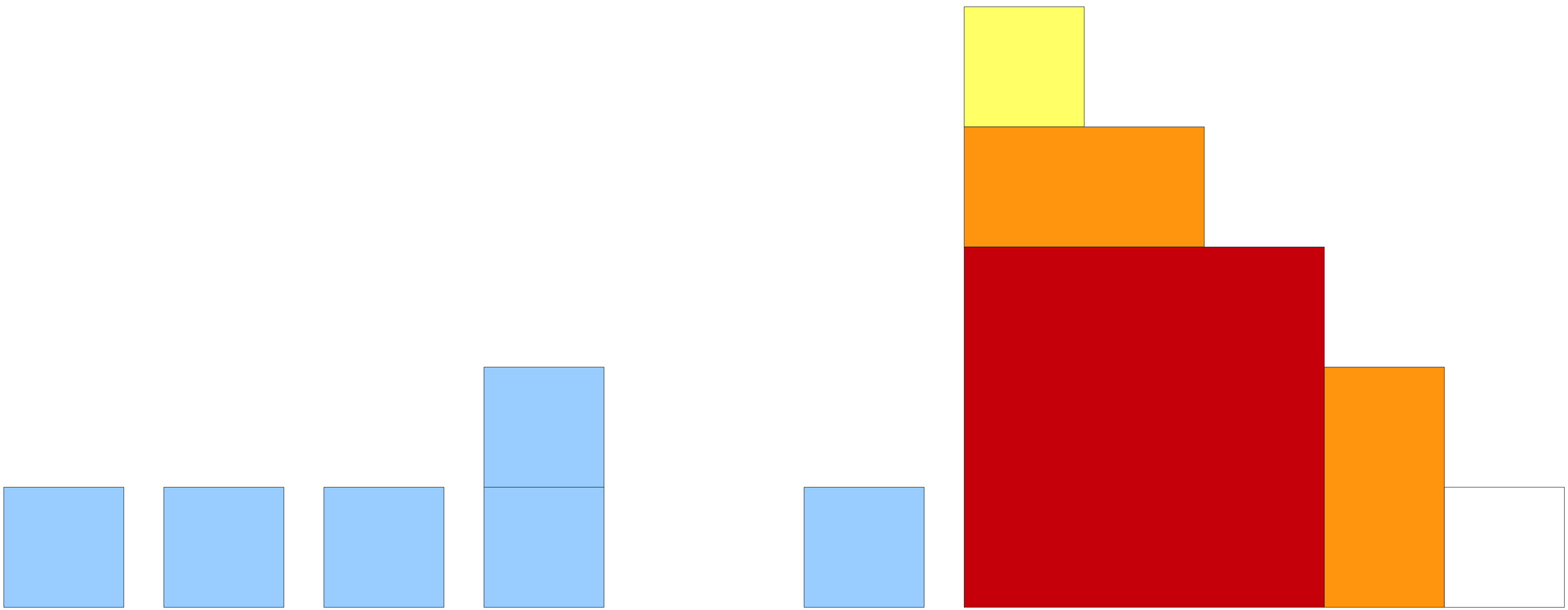

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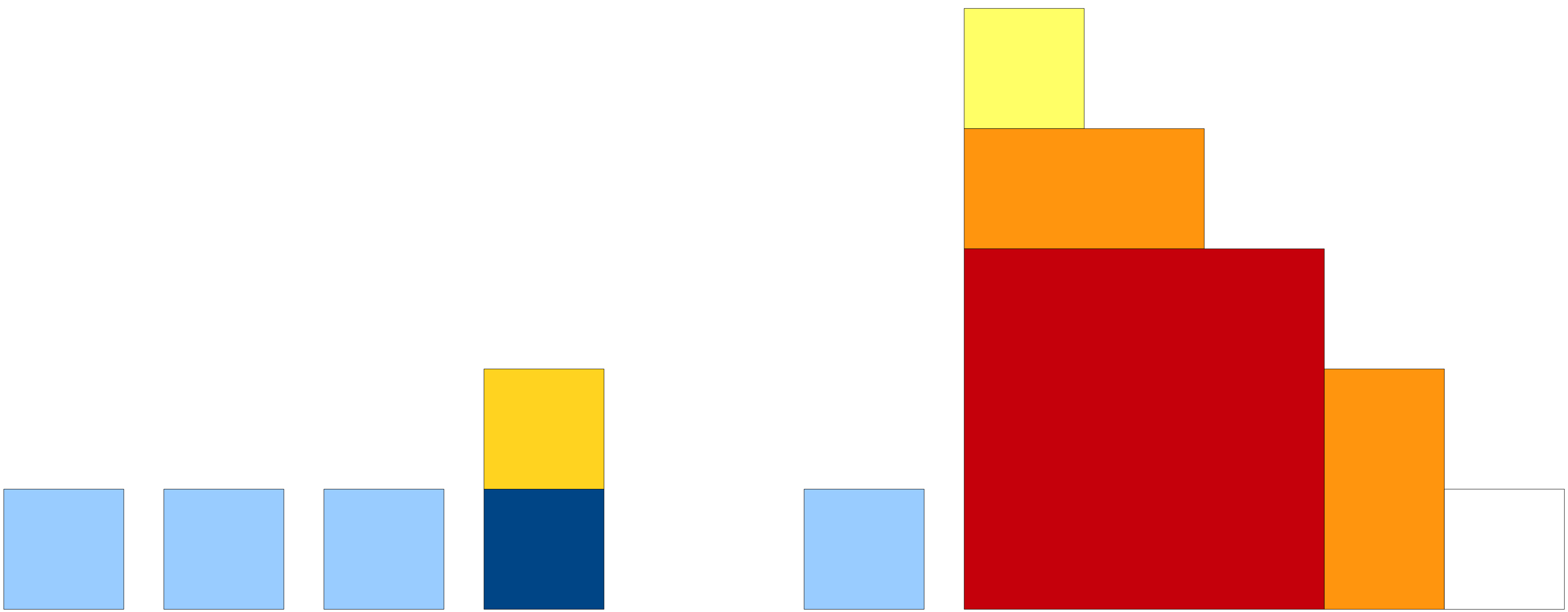

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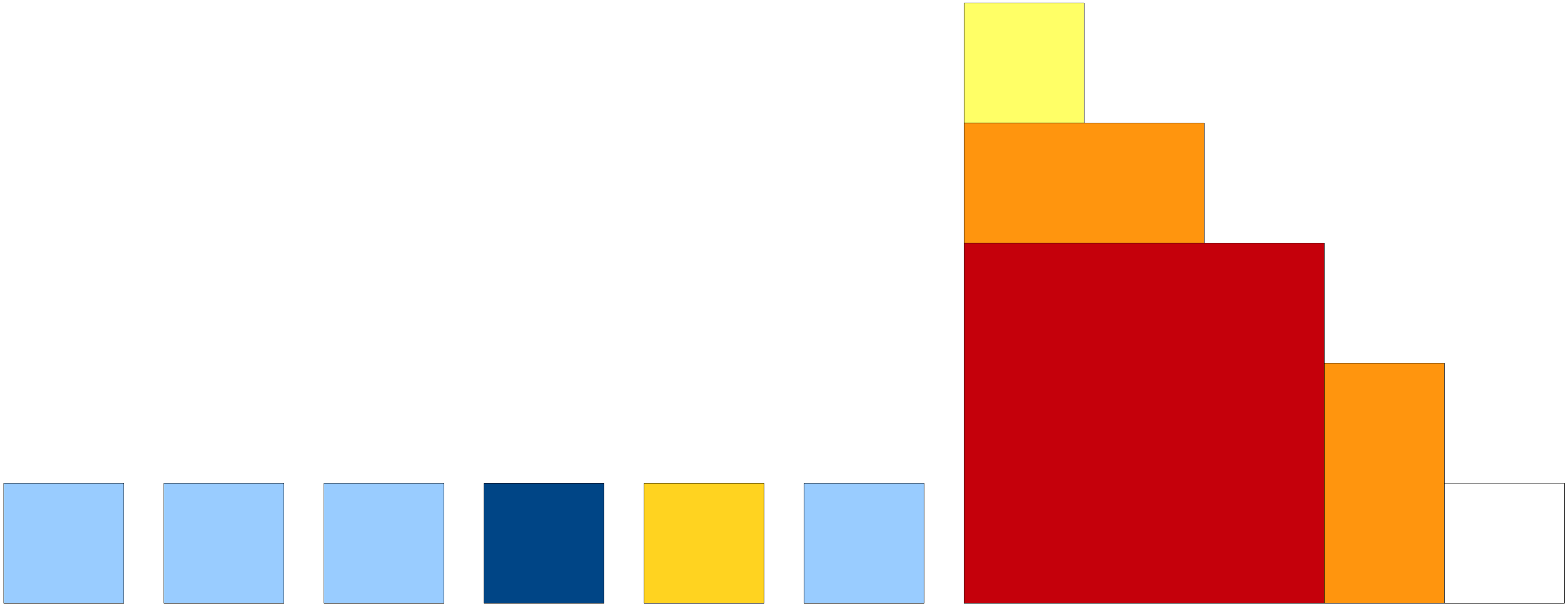

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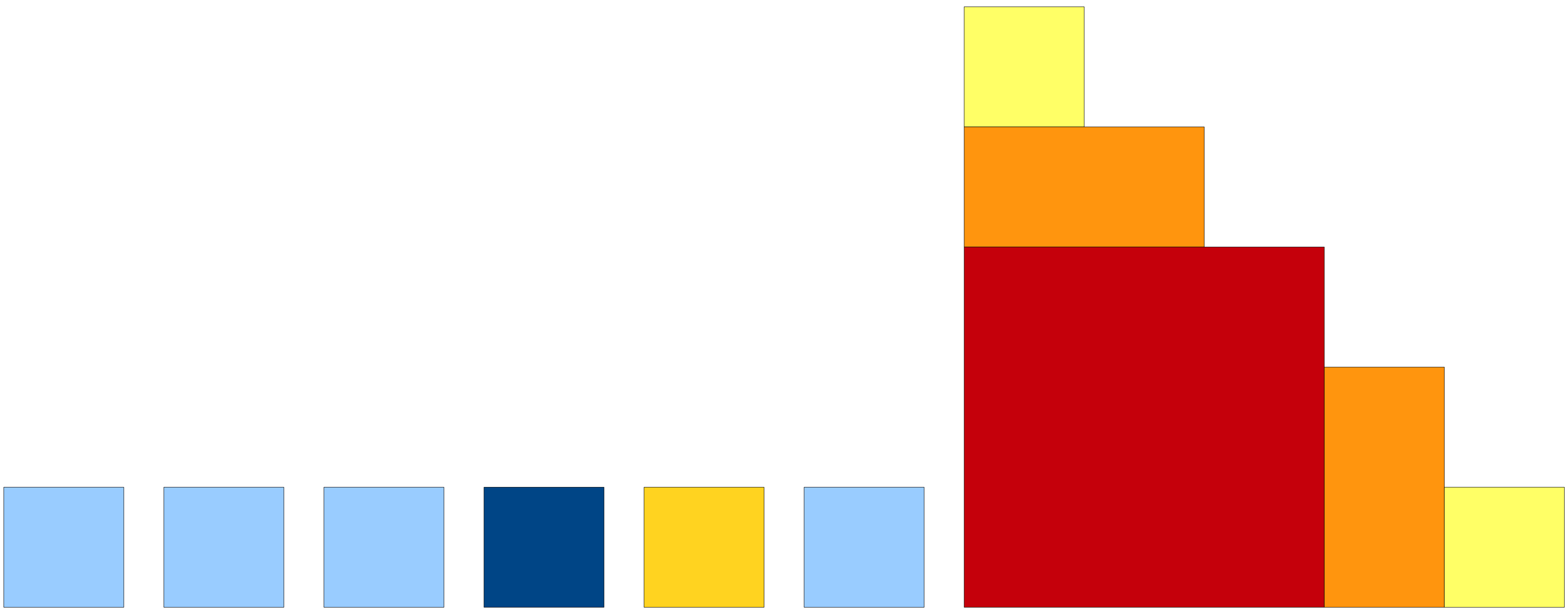

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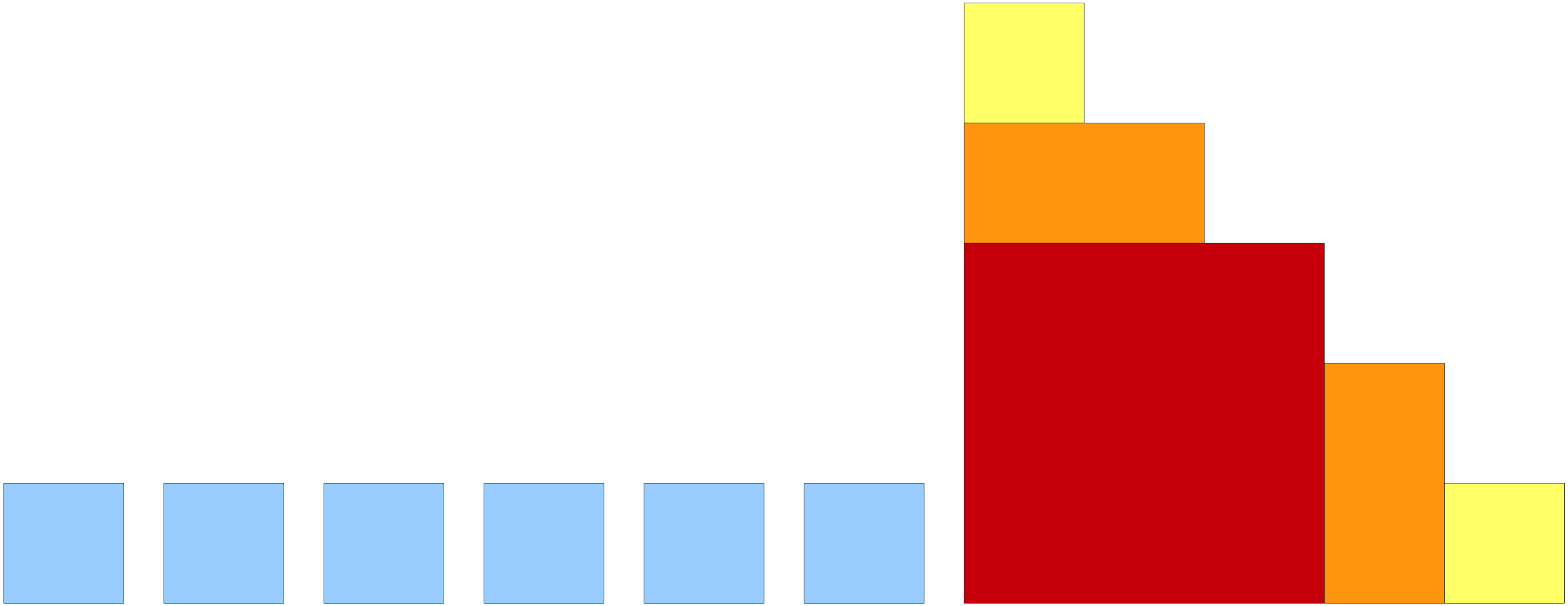
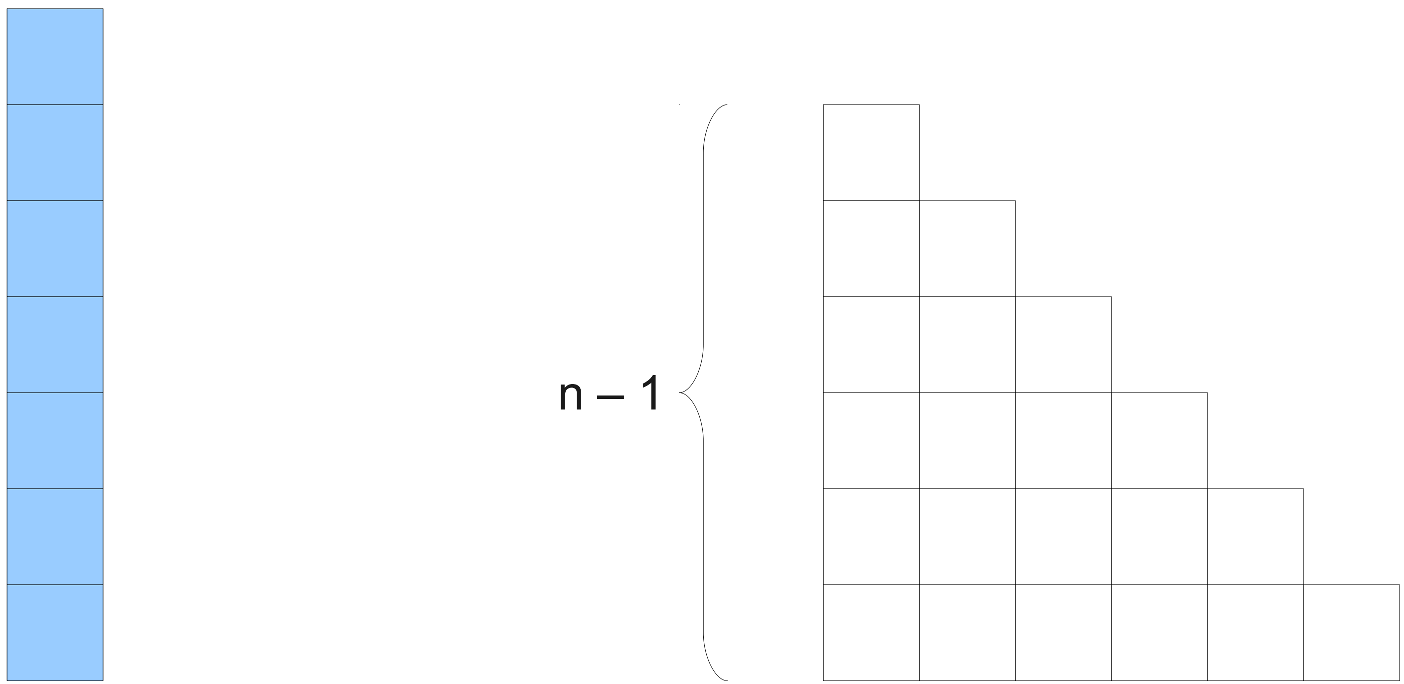


A Cute Induction Proof



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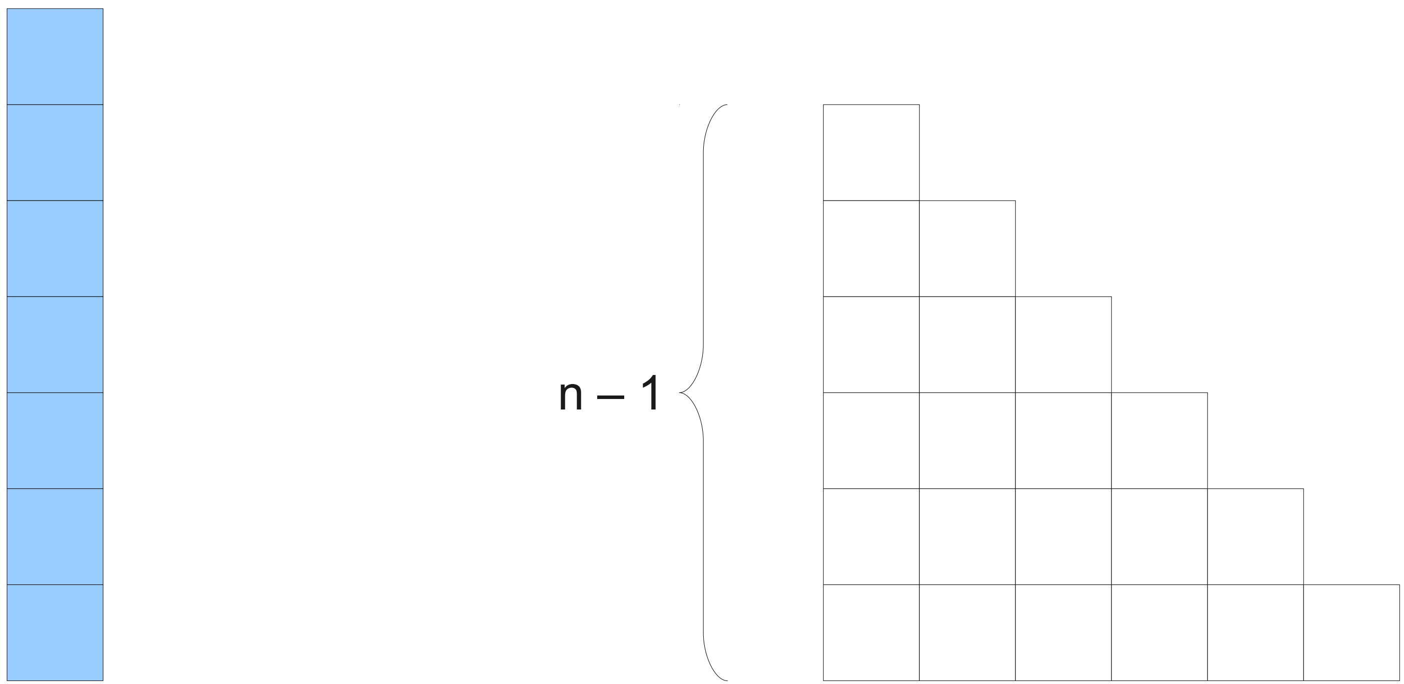

Show that this stack...

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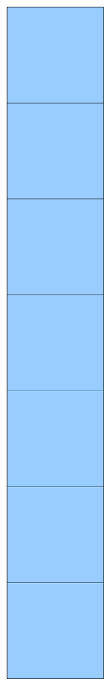

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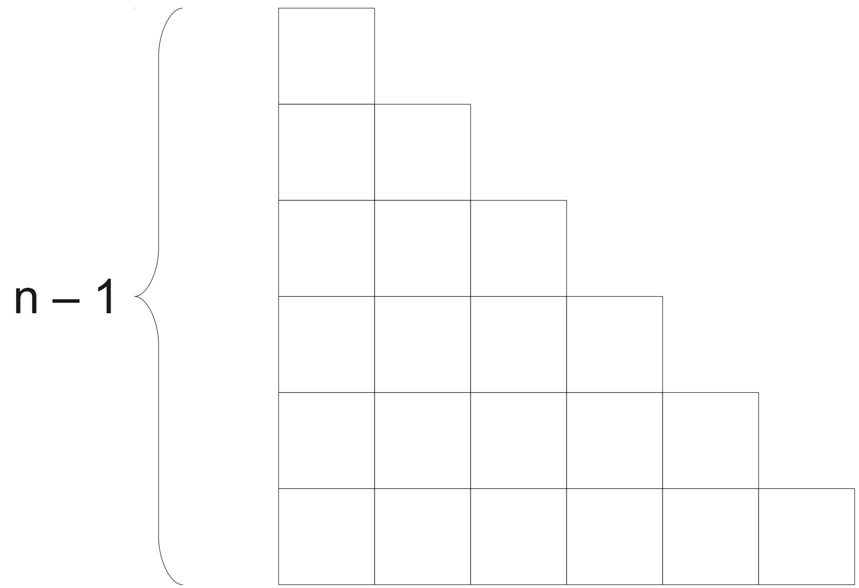
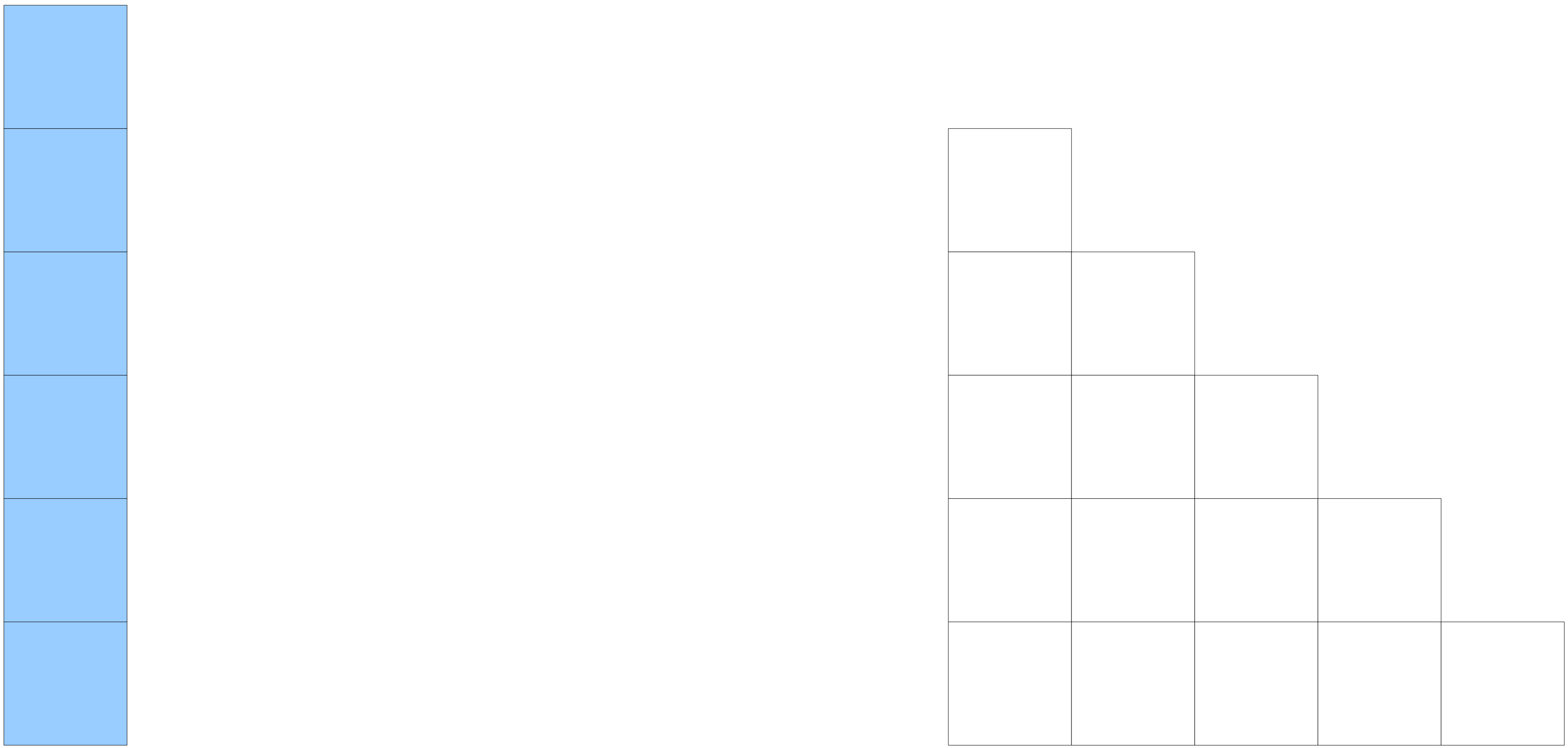


has enough points to fill this triangle.

The Second Intuition



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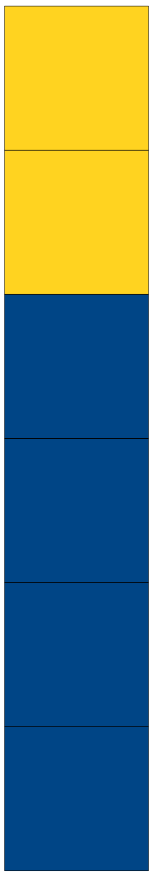

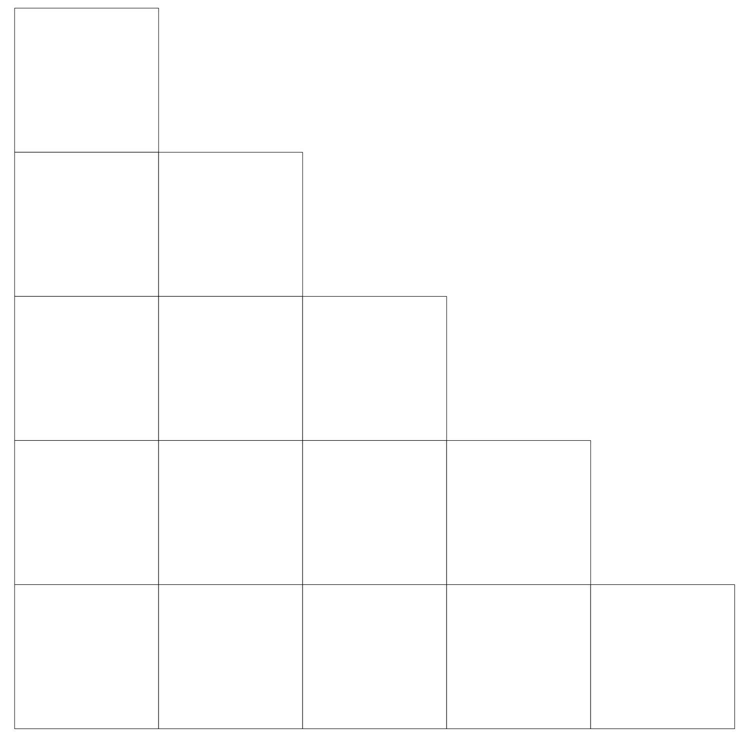

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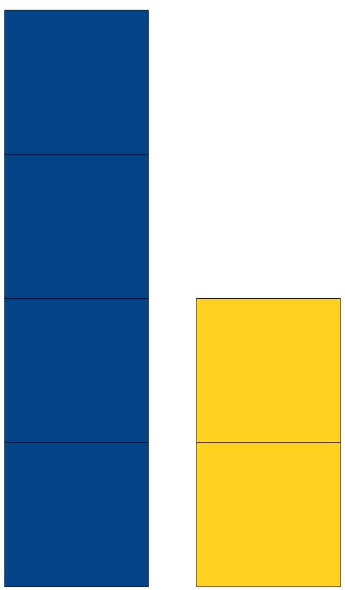

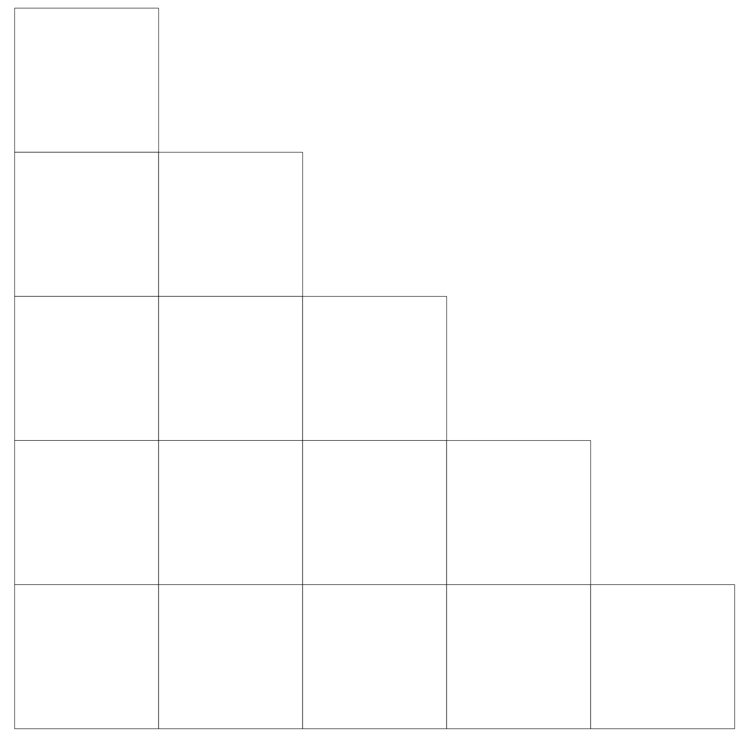

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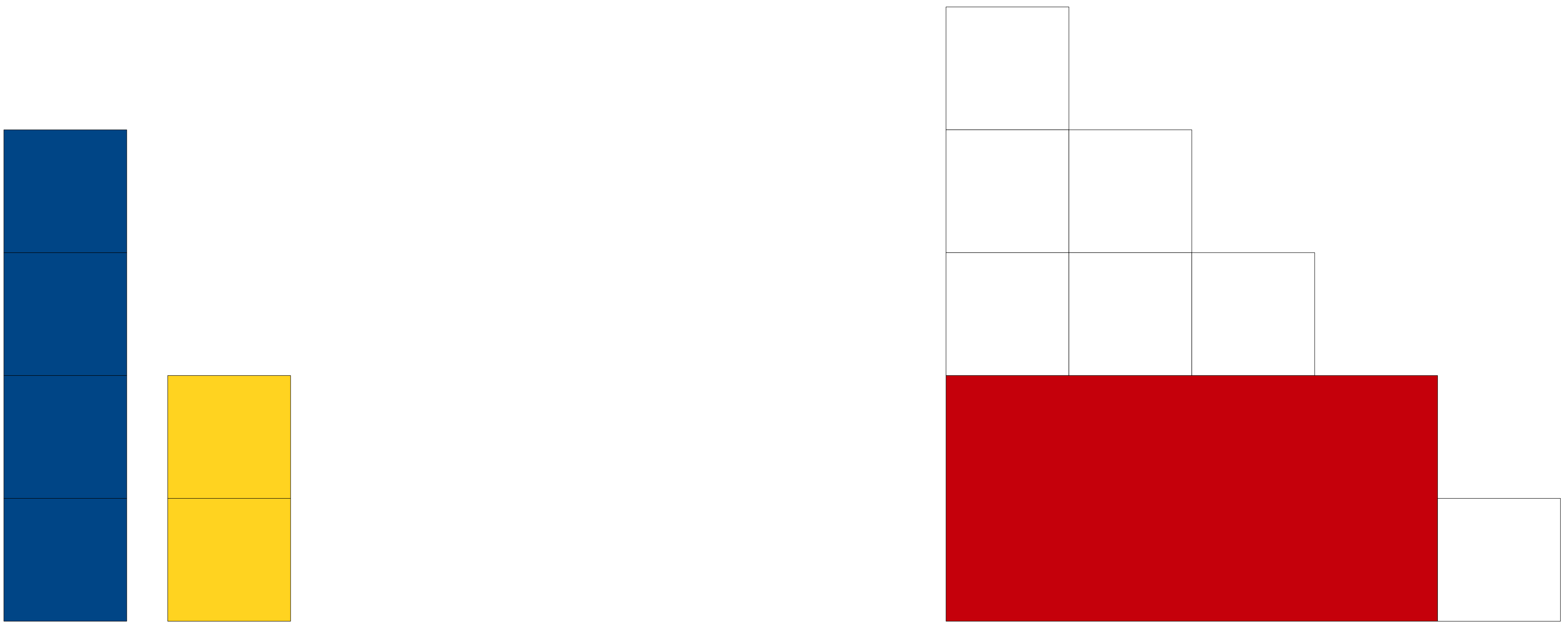

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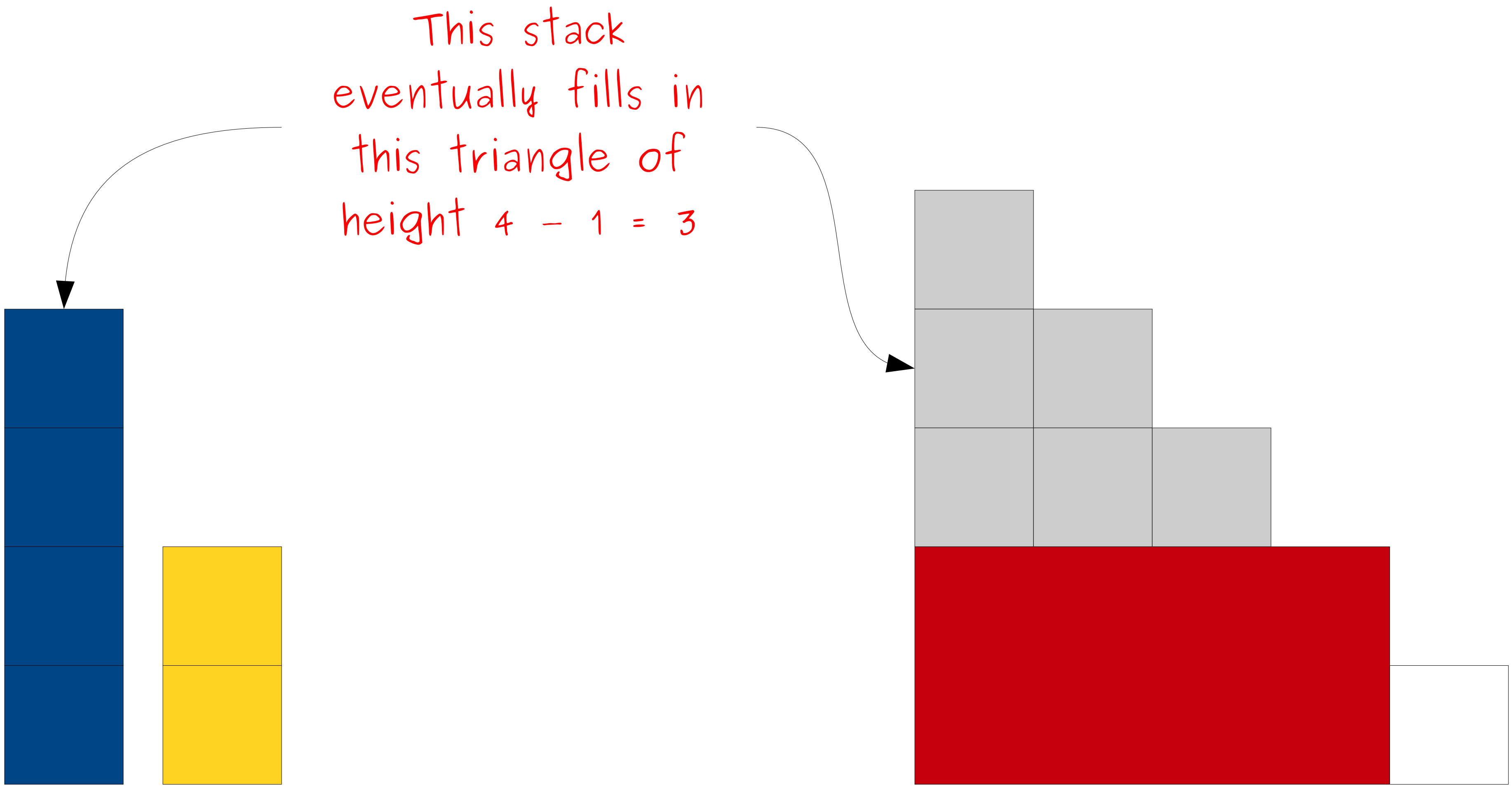

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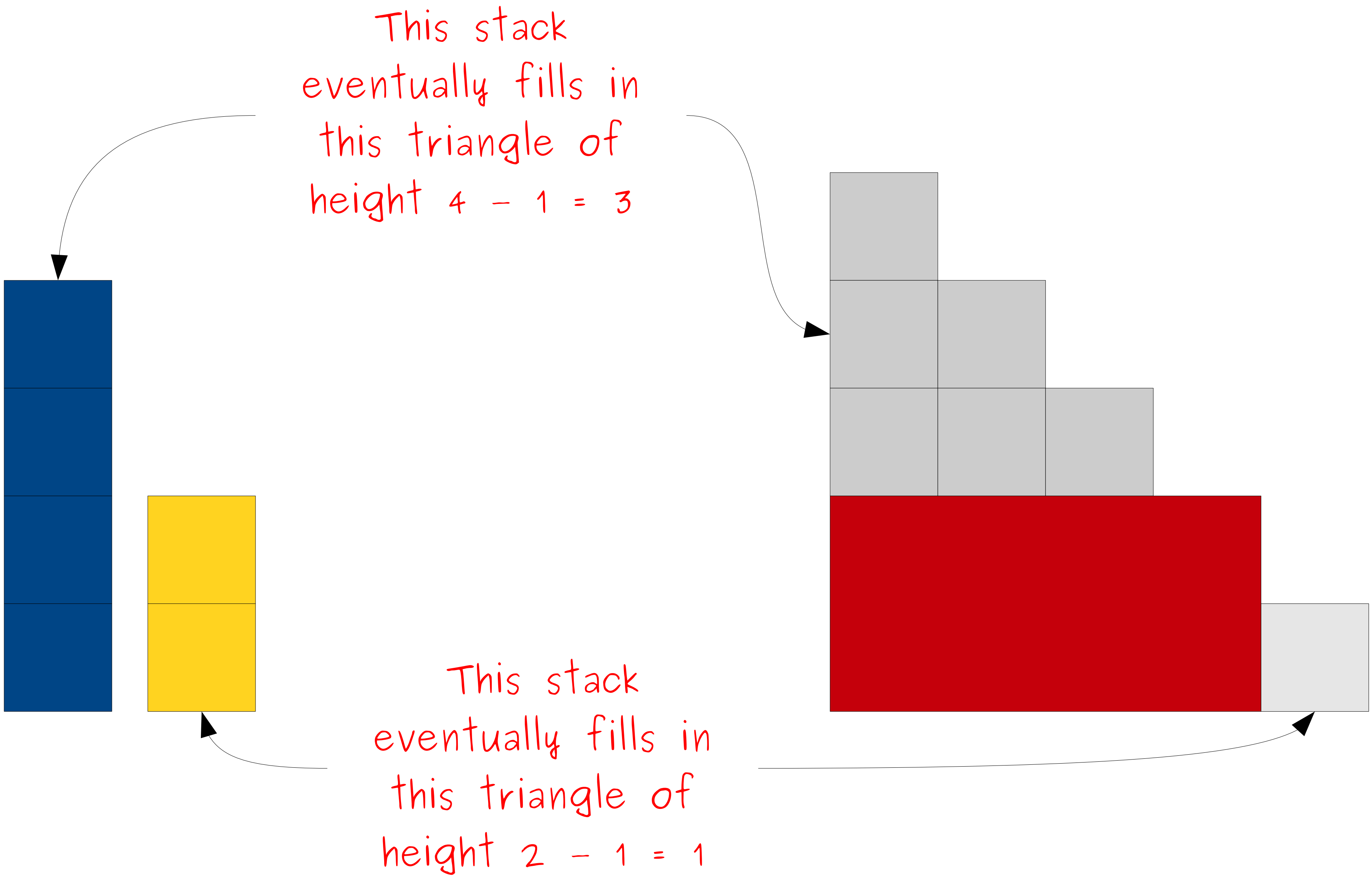

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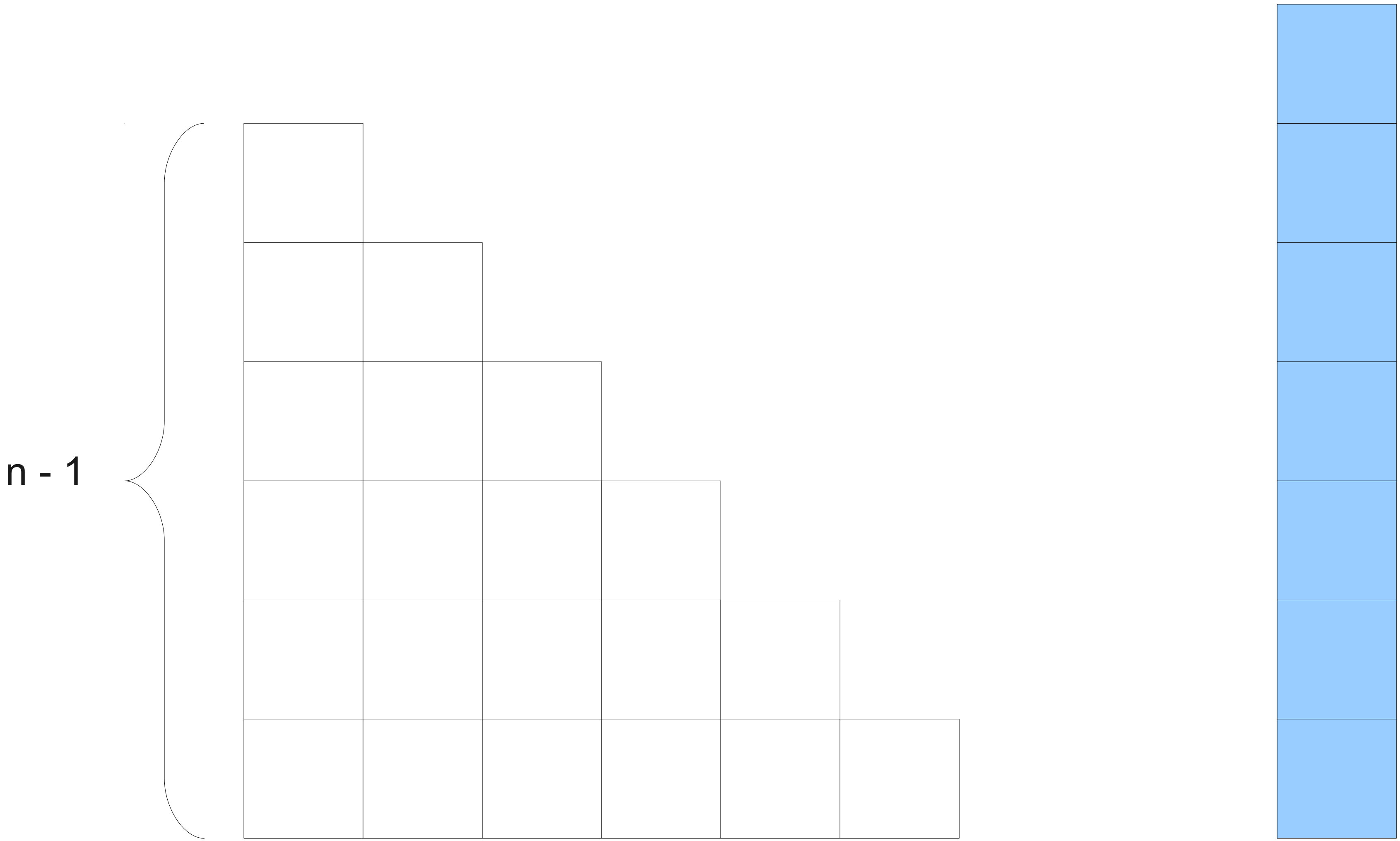

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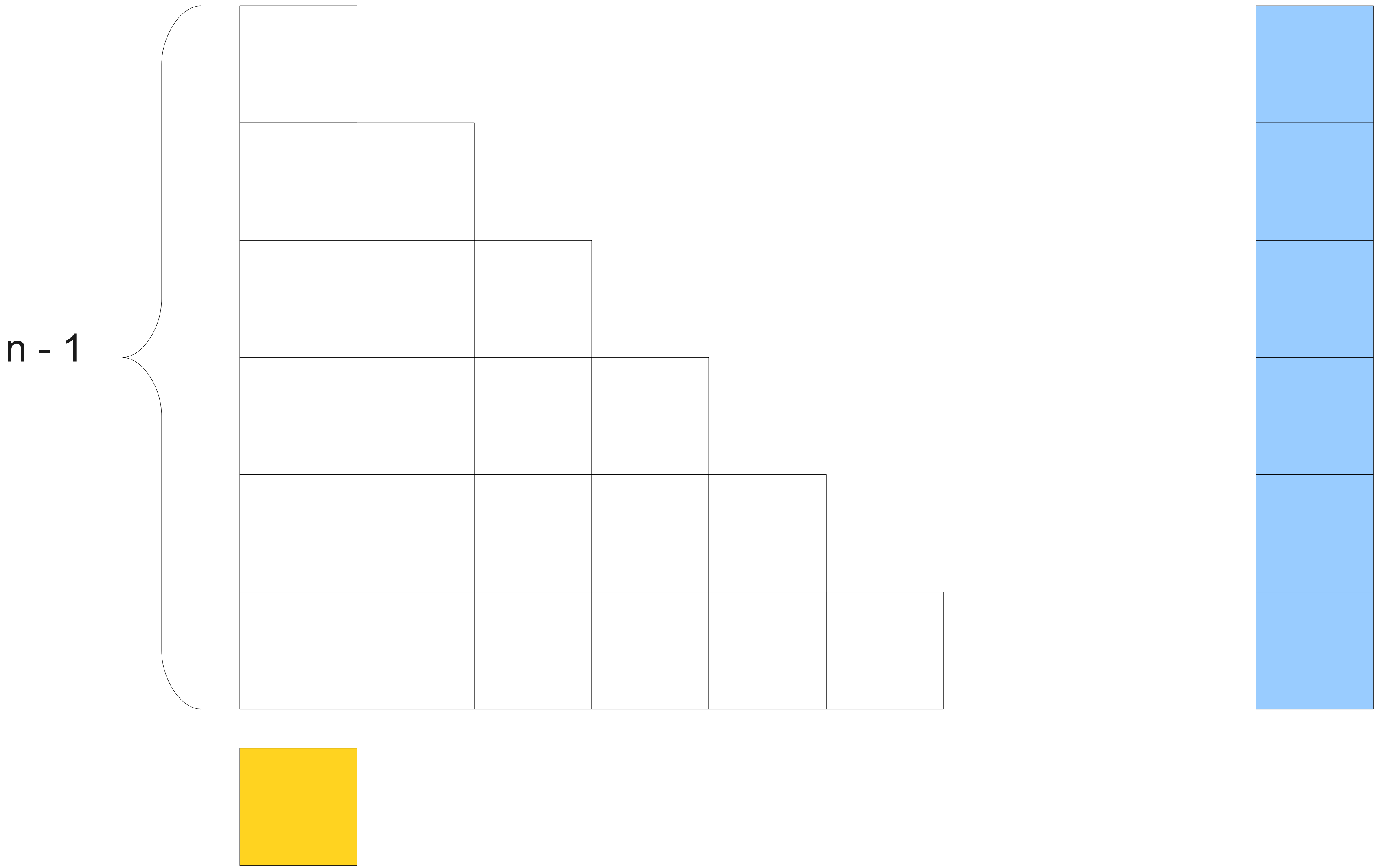

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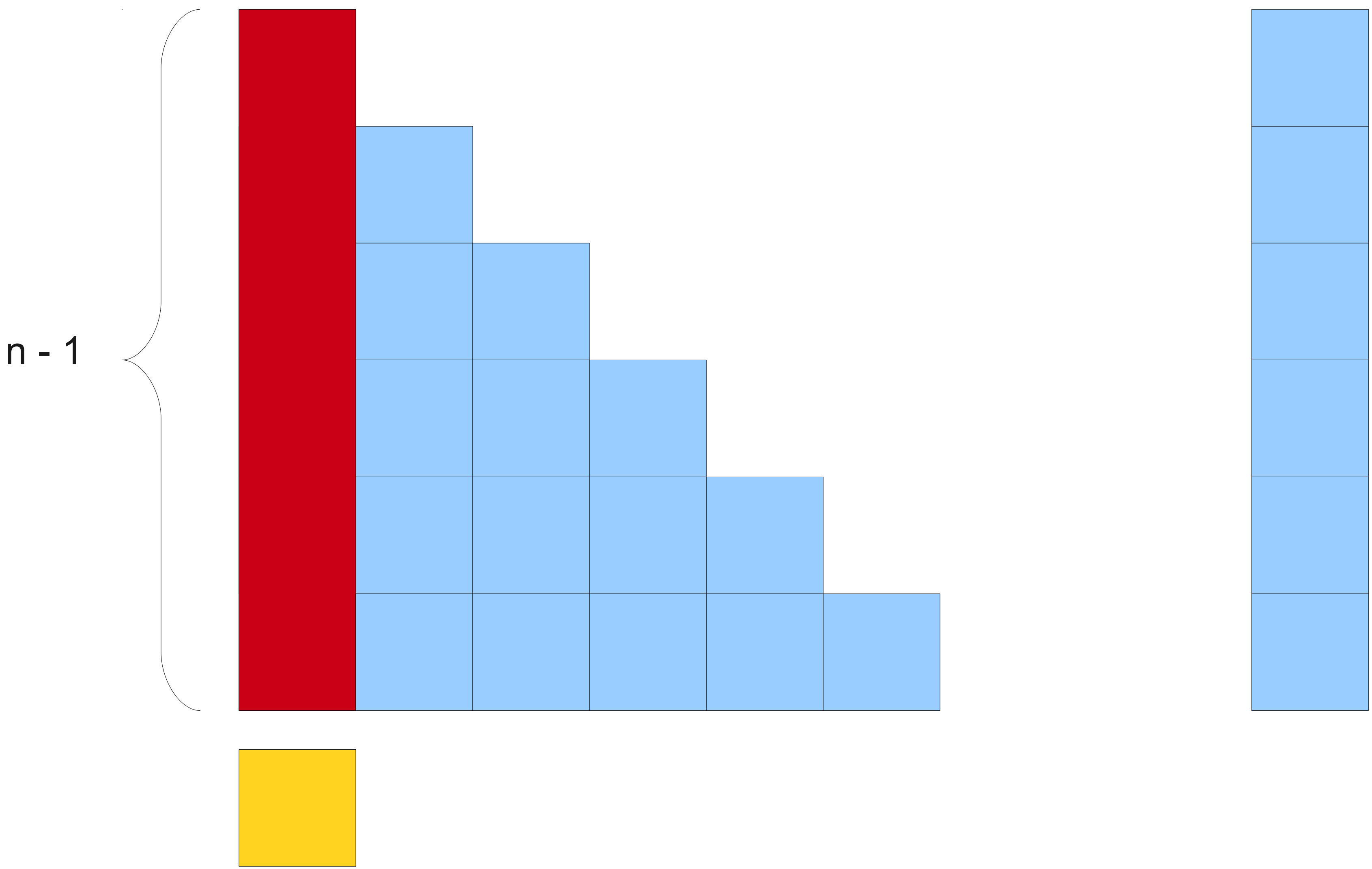

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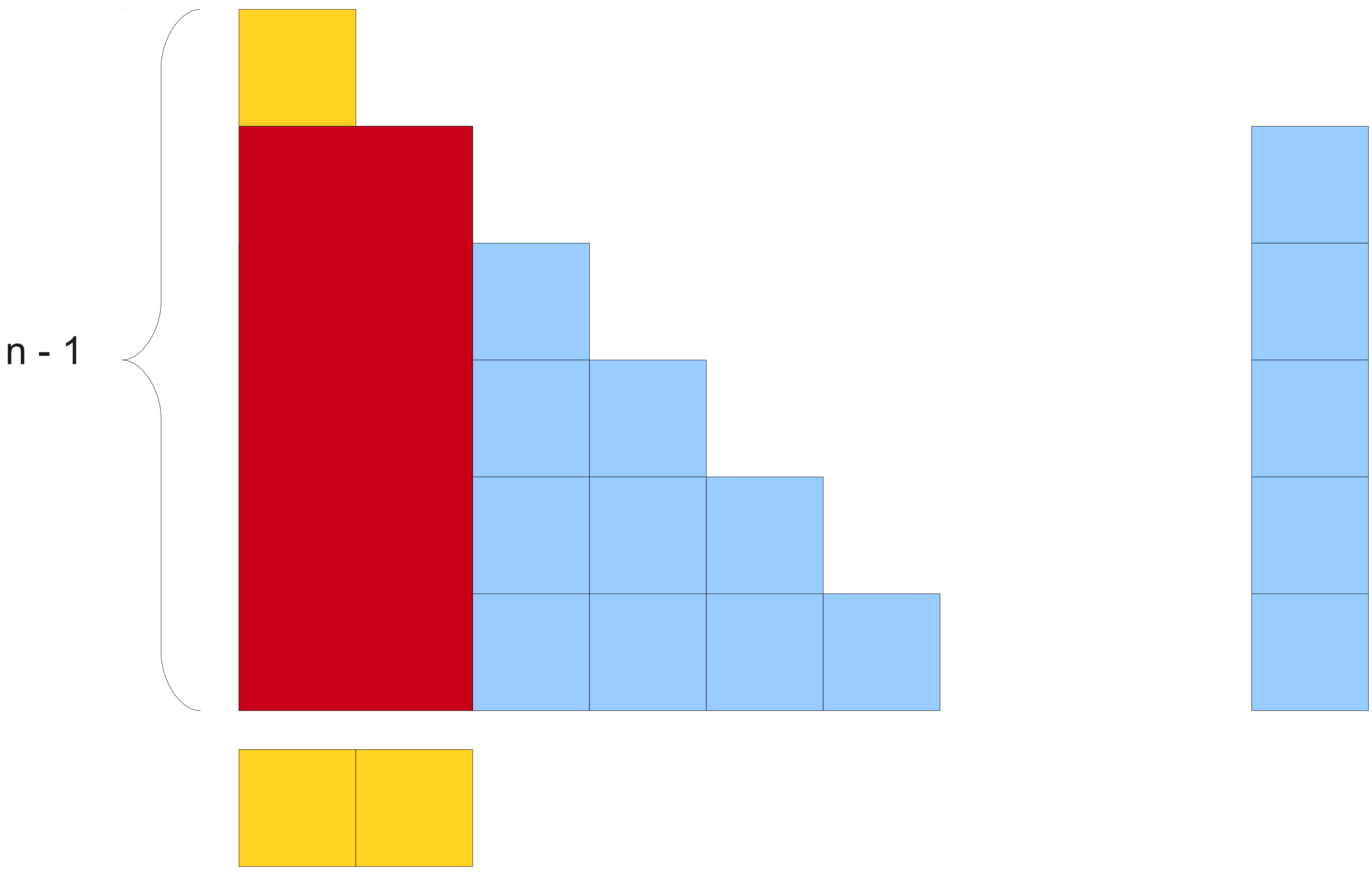

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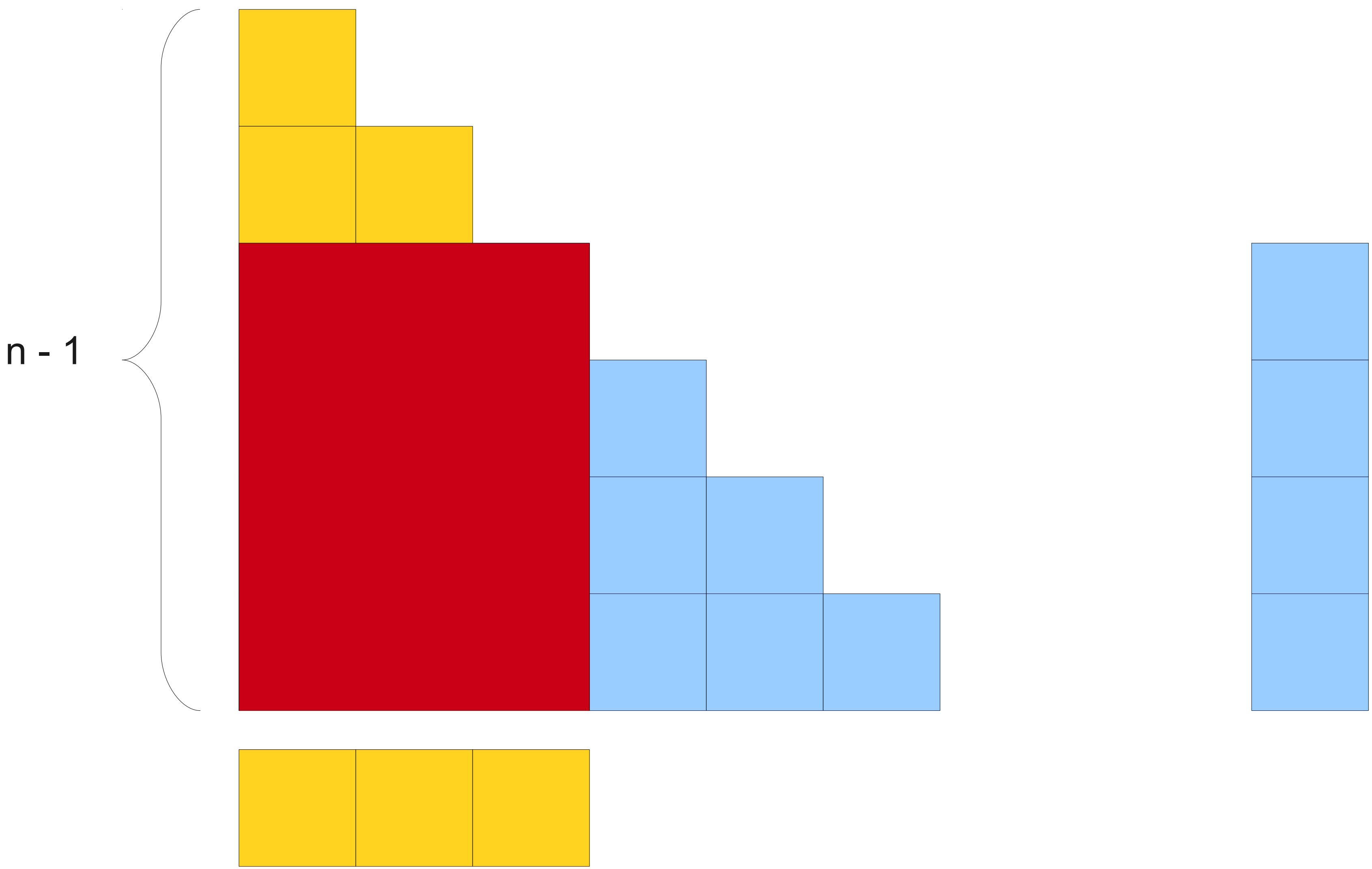

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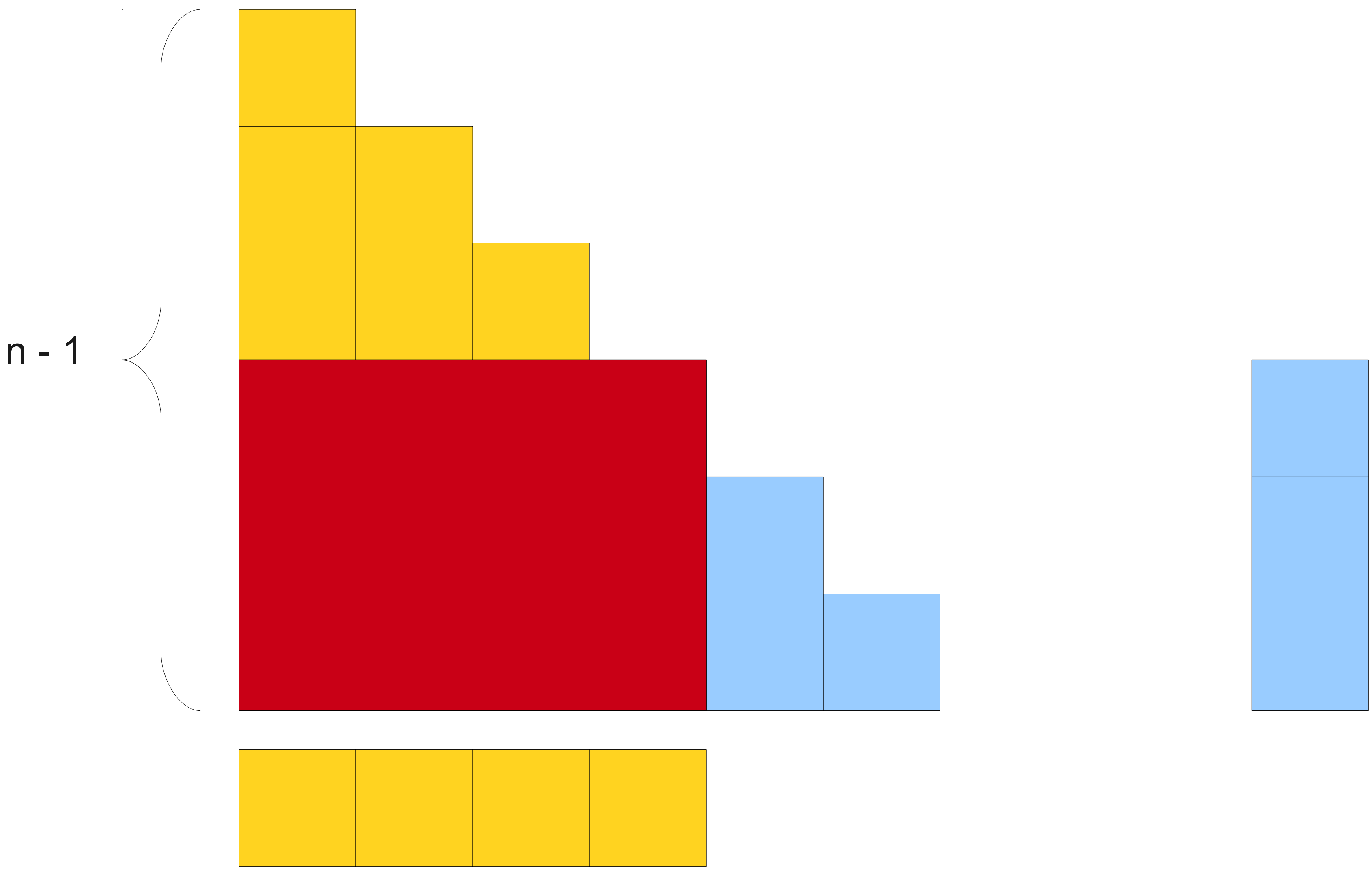

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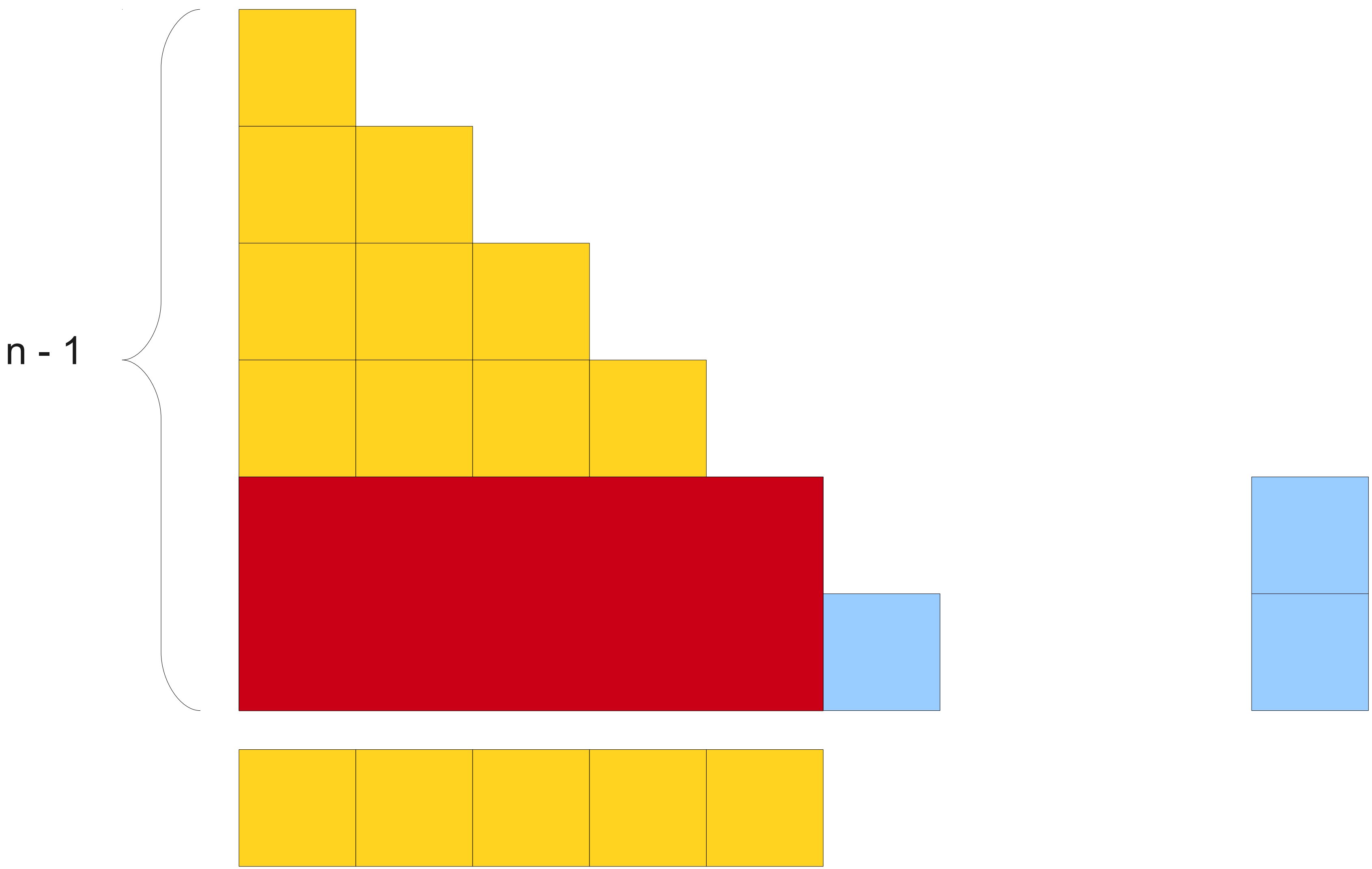

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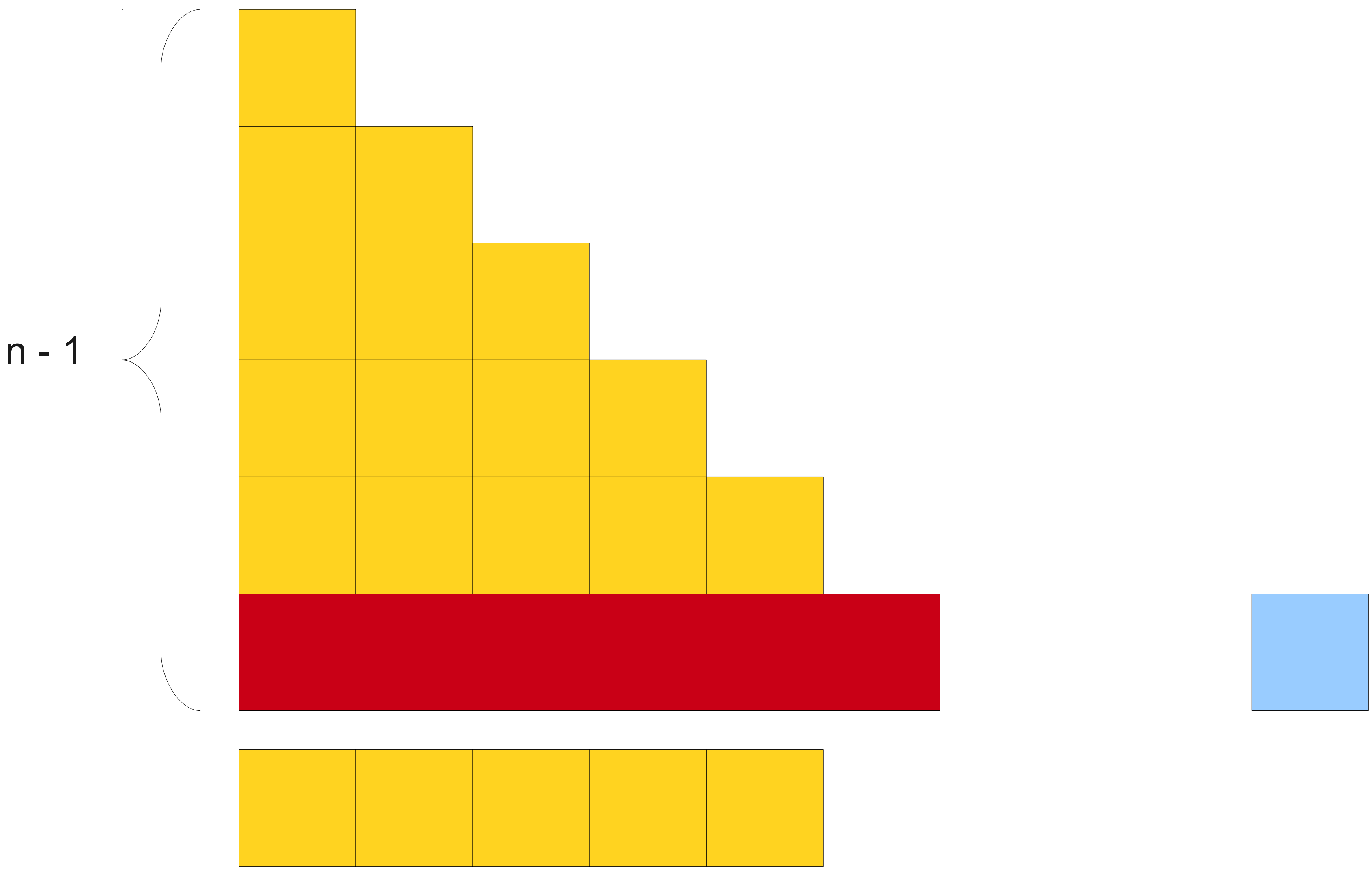

Why This Matters

- **Intuition is about telling stories**.
  - Looking at n(n – 1) / 2 as a number of pairs lets you account for the result by talking about how pairs are broken at each move.
  - Looking at n(n – 1) / 2 as the shape of the pyramid lets you account for the result by showing that each move gives a different way of building the same pyramid.
- **Proofs are about making the logic rigorous**.
- Math is the combination of these two factors.

First-Order Logic, Continued

Recap from Last Time

- First-order logic uses **constants** to refer to objects in the domain.
- **Predicates** take objects and evaluate to either true or false.
- **Functions** map objects to other objects.
- **Quantifiers** allow us to talk about multiple objects at the same time:
  - The **universal quantifier** ∀ states that something is true for **all objects**.
  - The **existential quantifier** ∃ states that something is true for **at least one object**.

A Bad Translation

Everyone who can outrun velociraptors won't get eaten.

$$\forall x.\ (FasterThanVelociraptors(x) \land \neg WillBeEaten(x))$$

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“Any time P(x), then Q(x)”

translates as

$$\forall x.\ (P(x) \rightarrow Q(x))$$

Another Bad Translation

If some velociraptor can open windows,
then it can eat me.

$$\exists x. (\text{Velociraptor}(x) \land \text{OpensWindows}(x) \rightarrow \text{EatsMe}(x))$$

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If some velociraptor can open windows,
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$\exists x. (Velociraptor(x) \land OpensWindows(x) \rightarrow EatsMe(x))$

I'm not a velociraptor!

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$$\forall x.\ (\text{Velociraptor}(x) \land \text{OpensWindows}(x) \rightarrow \text{EatsMe}(x))$$

Note that this is a universal quantifier even though we're using the word "some" in here!

“If some x satisfies P(x), then Q(x)”

translates as

$$\forall x.\ (P(x) \rightarrow Q(x))$$

Even More Bad Translations

There is a velociraptor that can open windows and eat me.

$\exists x. (Velociraptor(x) \land OpensWindows(x) \to EatsMe(x))$

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“There is some P(x) where Q(x)”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

The Takeaway Point

- **Natural language is often imprecise**.
- First-order logic gives us an unambiguous language for encoding general statements.
- However, you have to get the translation right!

Quantifying over Sets

- The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, P(x) holds."

- This is not technically a part of first-order logic; it is a shorthand for

$$\forall x.\ (x \in S \rightarrow P(x))$$

- How might we encode this concept?

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- How might we encode this concept?

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Answer: $\exists x.\ (x \in S \land P(x))$.

Note the use of $\land$ instead of $\to$ here.

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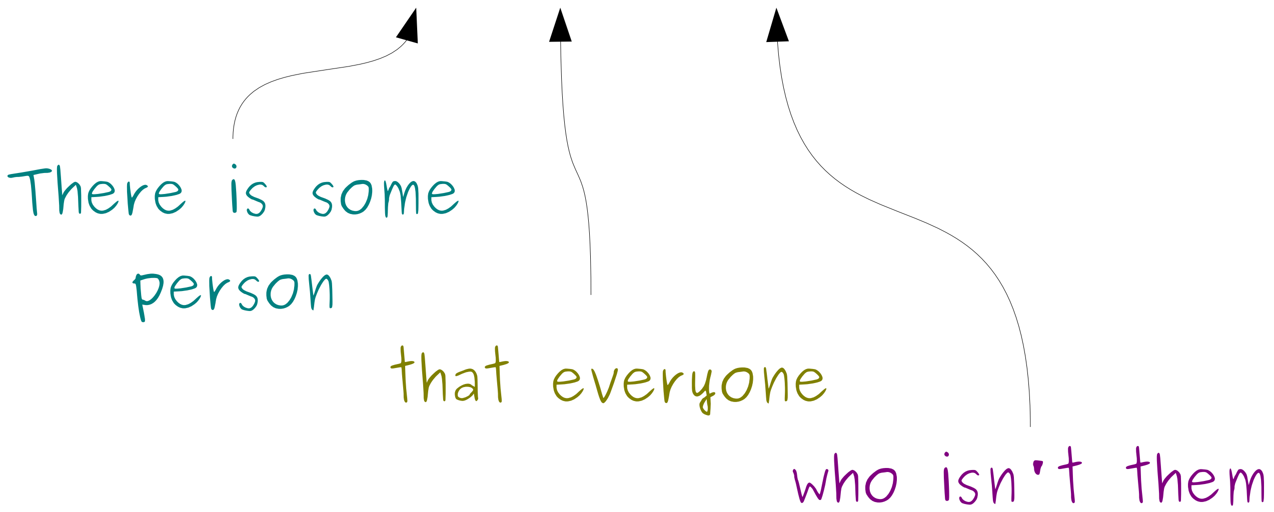

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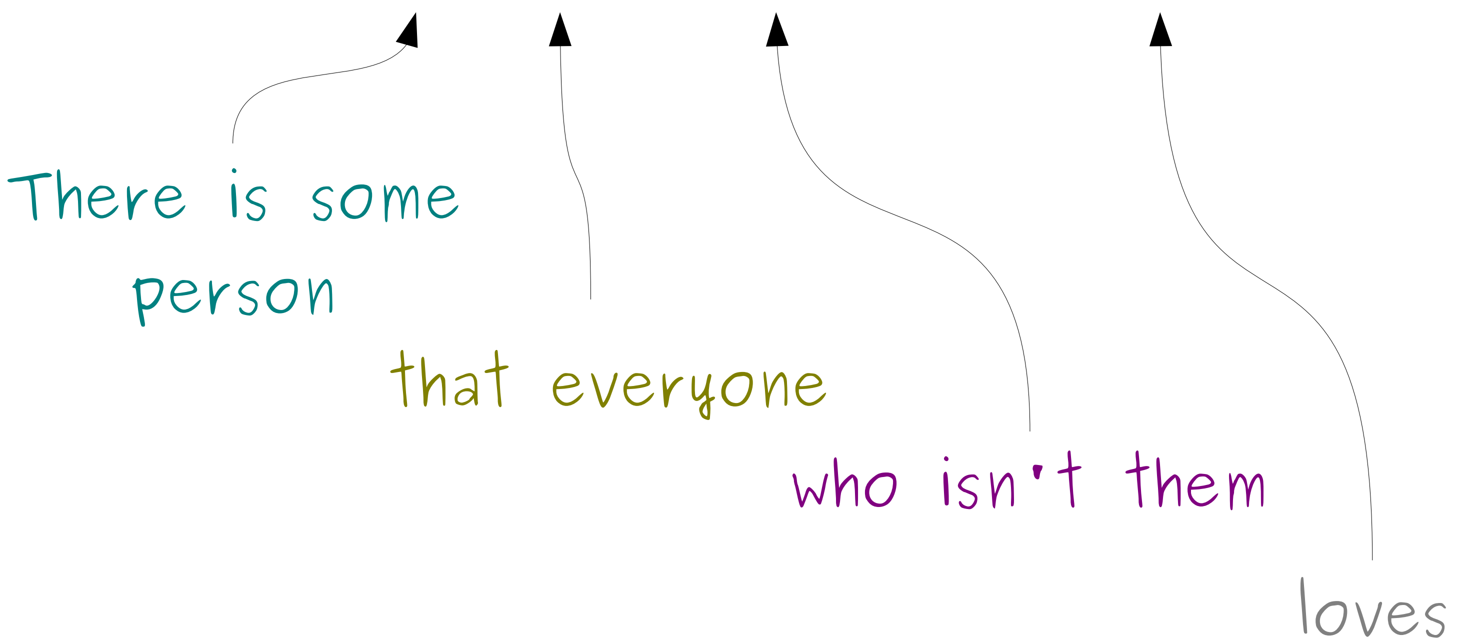
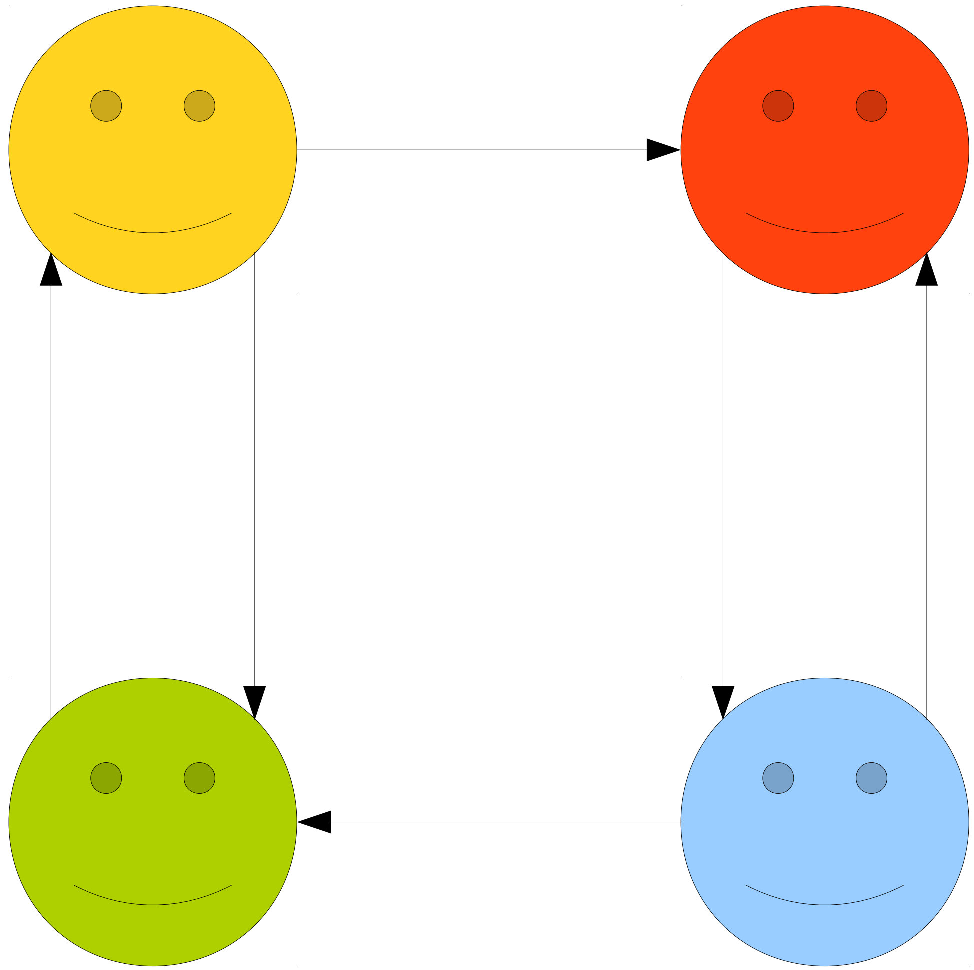


$\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))$



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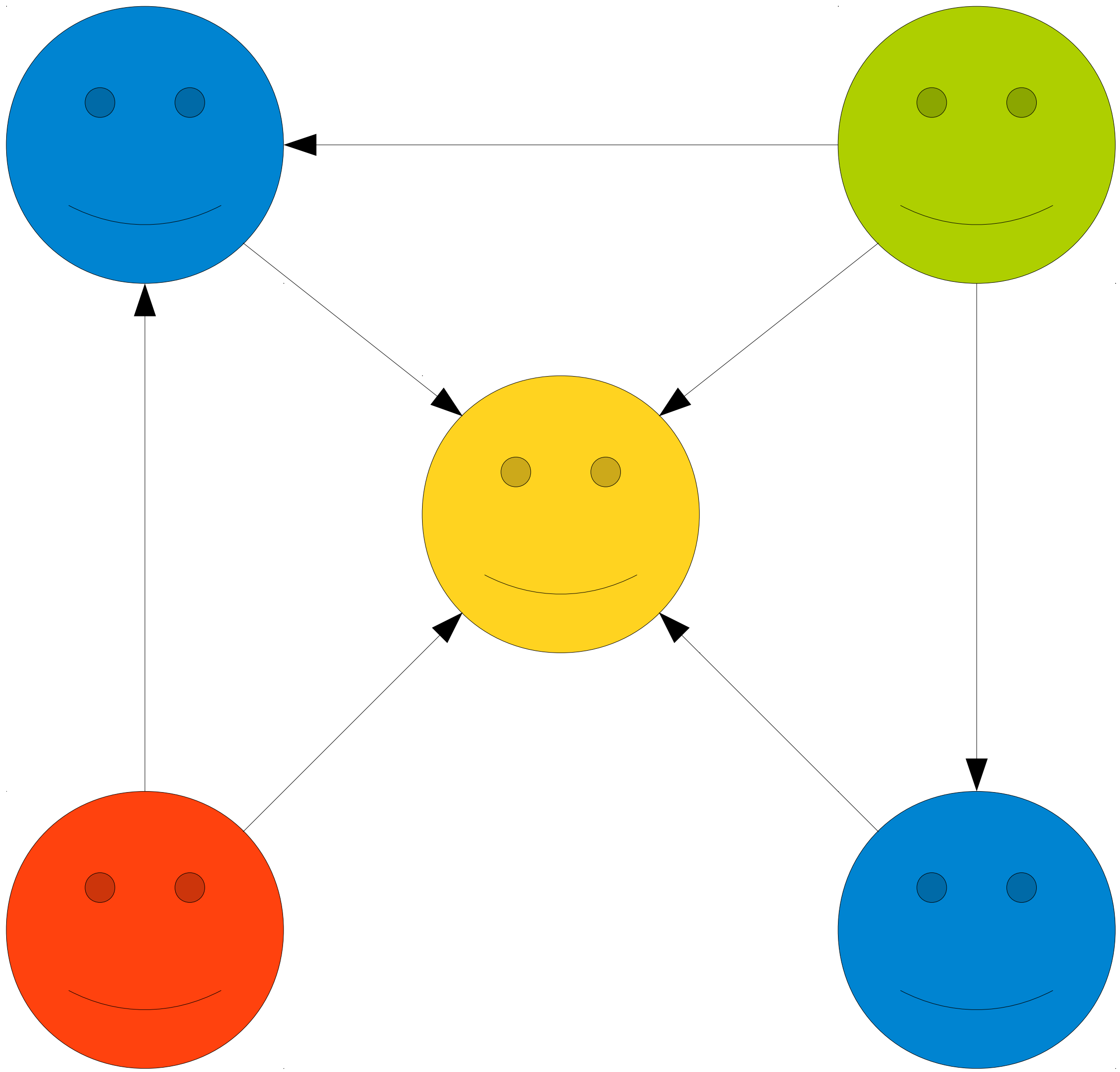

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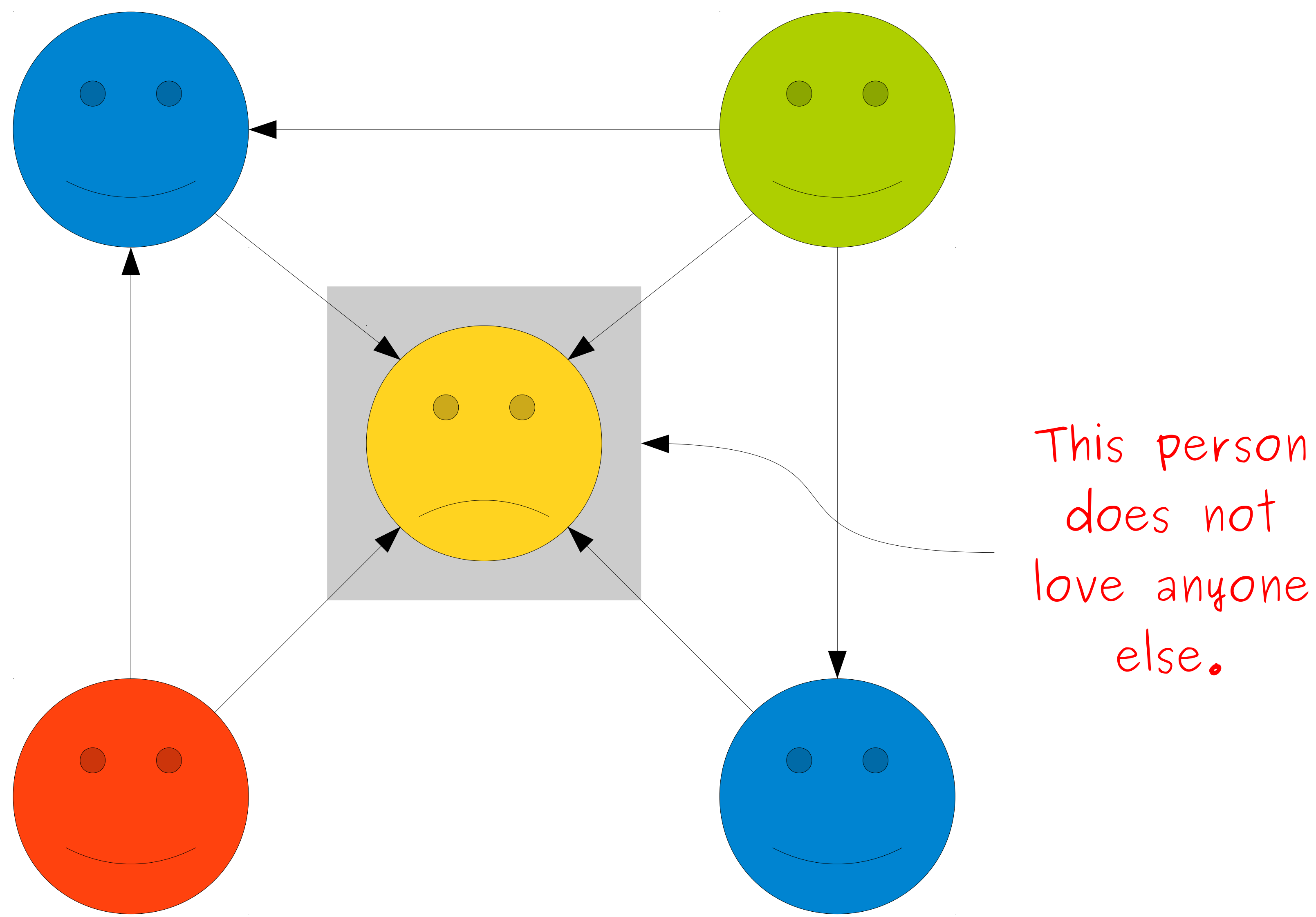

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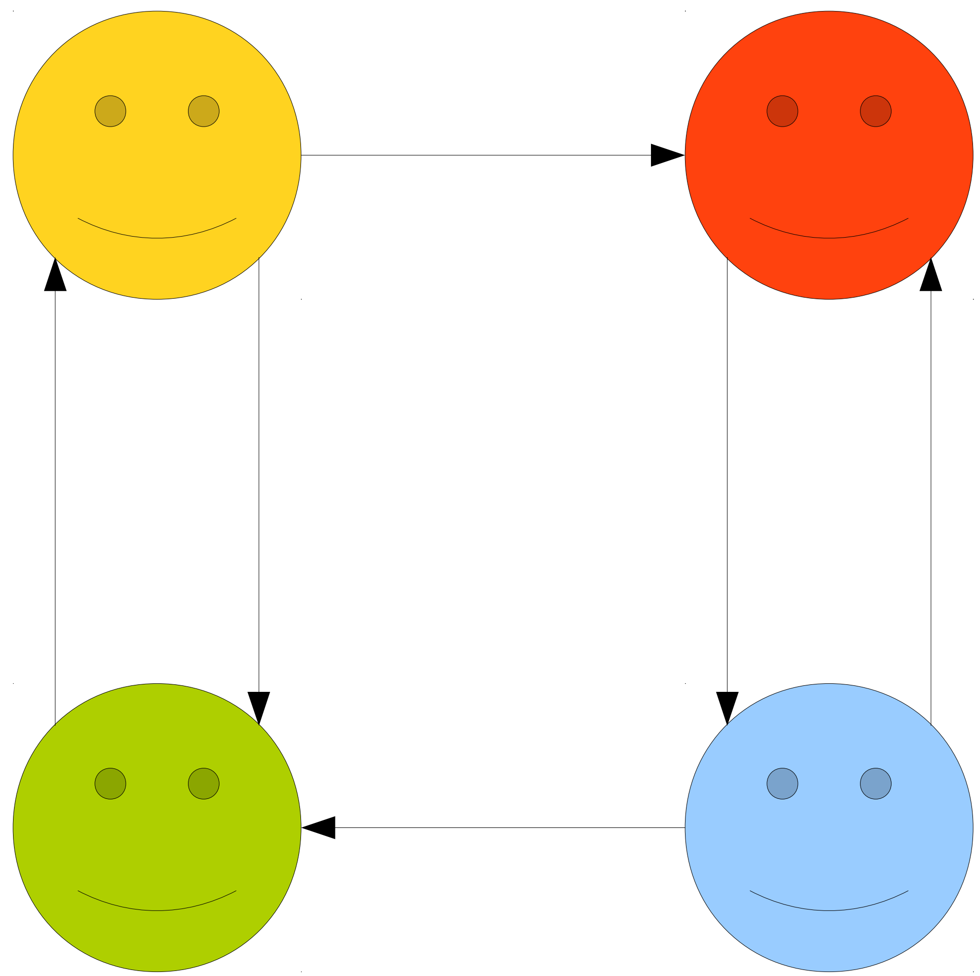

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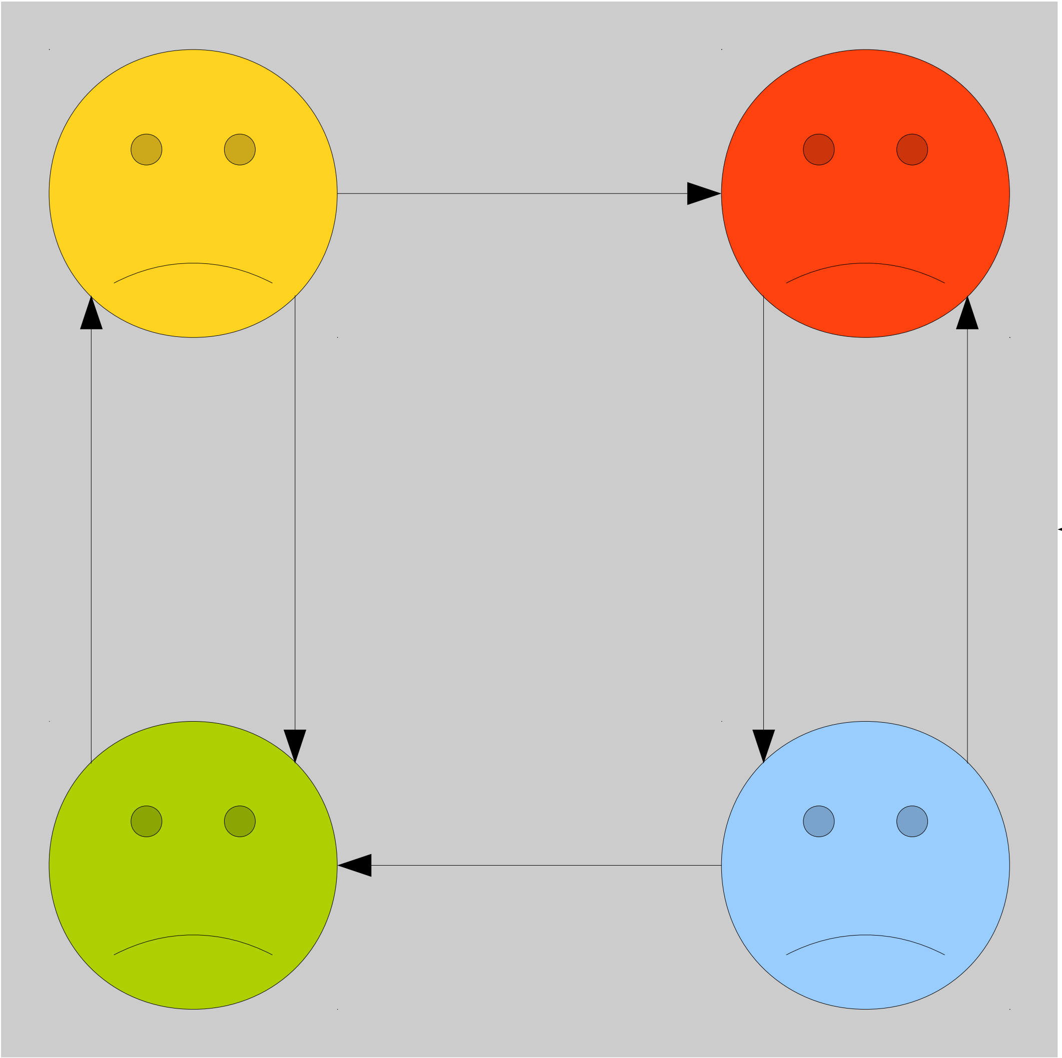
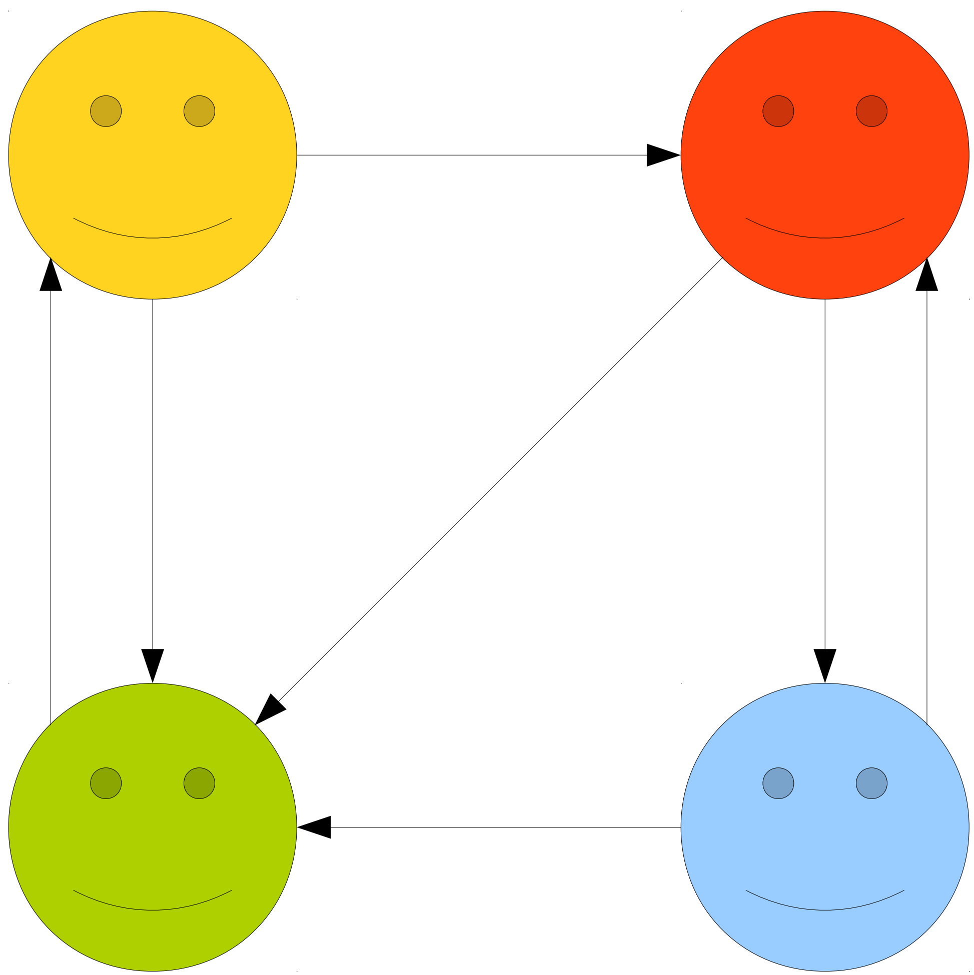


No one is universally loved here.

$$(\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))) \land$$
$$(\exists y.\ \forall x.\ (x \neq y \rightarrow Loves(x, y)))$$



The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means "For any choice of x, there is **some** choice of y where P(x, y)."

The statement

$$\exists y.\ \forall x.\ P(x, y)$$

means “There is some choice of y where for **any** choice of x, P(x, y).”

More generally, **order matters** when mixing existential and universal quantifiers!

Translating into First-Order Logic

- First-order logic has great expressive power and is often used to formally encode mathematical definitions.
- Let's go provide rigorous definitions for the terms we've been using so far.

Set Theory

"Two sets are equal iff they contain the same elements."

Set Theory

"Two sets are equal iff they contain the same elements."

$$S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T)$$

Set Theory

"Two sets are equal iff they contain the same elements."

$$S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T)$$

Every possible element is either in both S and T, or it's in neither S nor T.

Set Theory

"Two sets are equal iff they contain the same elements."

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Set Theory

"Two sets are equal iff they contain the same elements."

$$S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T)$$

Is something missing here?

Set Theory

"Two sets are equal iff they contain the same elements."

$$\forall S.\ \forall T.\ (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T))$$

Set Theory

"Two sets are equal iff they contain the same elements."

$$\forall S.\ \forall T.\ (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T))$$

These quantifiers are critical here, but they don't appear anywhere in the English. Many statements asserting a general claim is true are implicitly universally quantified.

Set Theory

"The **union** of two sets is the set containing all elements of both sets."

Set Theory

"The **union** of two sets is the set containing all elements of both sets."

$\forall S.\ \forall T.\ \forall x.\ (x \in S \cup T \leftrightarrow \mathbf{x \in S \lor x \in T})$

Set Theory

“T ... ing all

T)

Set Theory

“The **intersection** of two sets is the set containing all elements common to both sets.”

Set Theory

"The **intersection** of two sets is the set containing all elements common to both sets."

$$\forall S.\ \forall T.\ \forall x.\ (x \in S \cap T \leftrightarrow \mathbf{x \in S \land x \in T})$$

Set Theory

"The **difference** of two sets is the set of all elements in the first set but not the second set."

Set Theory

"The **difference** of two sets is the set of all elements in the first set but not the second set."

$\forall S.\ \forall T.\ \forall x.\ (x \in S - T \leftrightarrow \mathbf{x \in S \land x \notin T})$

Relations

"R is reflexive."

Relations

"R is reflexive."

$\forall a.\ aRa$

Relations

“R is symmetric.”

$\forall a.\ \forall b.\ (aRb \rightarrow bRa)$

Relations

"R is antisymmetric."

$$\forall a.\ \forall b.\ (aRb \land bRa \rightarrow a = b)$$

Relations

"R is transitive."

$$\forall a.\ \forall b.\ \forall c.\ (aRb \land bRc \rightarrow aRc)$$

Negating Quantifiers

- We spent much of last lecture discussing how to negate propositional constructs.
- How do we negate quantifiers?

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$		
$\exists x.\ P(x)$		
$\forall x.\ \neg P(x)$		
$\exists x.\ \neg P(x)$		

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, P(x)	
$\exists x.\ P(x)$		
$\forall x.\ \neg P(x)$		
$\exists x.\ \neg P(x)$		

An Extremely Important Table

	When is this true?	When is this false?
∀x. P(x)	For any choice of x, P(x)	For some choice of x, ¬P(x)
∃x. P(x)		
∀x. ¬P(x)		
∃x. ¬P(x)		

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	When is this true?	When is this false?
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∀x. ¬P(x)		
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∀x. ¬P(x)		
∃x. ¬P(x)		

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	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, P(x)	For some choice of x, ¬P(x)
$\exists x.\ P(x)$	For some choice of x, P(x)	For any choice of x, ¬P(x)
$\forall x.\ \neg P(x)$	For any choice of x, ¬P(x)	
$\exists x.\ \neg P(x)$		

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$		

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∃x. P(x)	For some choice of x, P(x)	For any choice of x, ¬P(x)
∀x. ¬P(x)	For any choice of x, ¬P(x)	For some choice of x, P(x)
∃x. ¬P(x)	For some choice of x, ¬P(x)	

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$\forall x.\ \neg P(x)$	For any choice of x, ¬P(x)	For some choice of x, P(x)
$\exists x.\ \neg P(x)$	For some choice of x, ¬P(x)	For any choice of x, P(x)

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, P(x)	**For some choice of x, ¬P(x)**
$\exists x.\ P(x)$	For some choice of x, P(x)	For any choice of x, ¬P(x)
$\forall x.\ \neg P(x)$	For any choice of x, ¬P(x)	For some choice of x, P(x)
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$\forall x.\ \neg P(x)$	For any choice of x, ¬P(x)	For some choice of x, P(x)
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	When is this true?	When is this false?
∀x. P(x)	For any choice of x, P(x)	**∃x. ¬P(x)**
∃x. P(x)	For some choice of x, P(x)	**For any choice of x, ¬P(x)**
∀x. ¬P(x)	**For any choice of x, ¬P(x)**	For some choice of x, P(x)
∃x. ¬P(x)	For some choice of x, ¬P(x)	For any choice of x, P(x)

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∀x. P(x)	For any choice of x, P(x)	**∃x. ¬P(x)**
∃x. P(x)	For some choice of x, P(x)	**∀x. ¬P(x)**
∀x. ¬P(x)	**For any choice of x, ¬P(x)**	For some choice of x, P(x)
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∀x. ¬P(x)	For any choice of x, ¬P(x)	**For some choice of x, P(x)**
∃x. ¬P(x)	For some choice of x, ¬P(x)	For any choice of x, P(x)

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$\exists x.\ P(x)$	For some choice of x, P(x)	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, ¬P(x)	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, ¬P(x)	For any choice of x, P(x)

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∀x. ¬P(x)	For any choice of x, ¬P(x)	**∃x. P(x)**
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∀x. ¬P(x)	For any choice of x, ¬P(x)	**∃x. P(x)**
∃x. ¬P(x)	For some choice of x, ¬P(x)	**∀x. P(x)**

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	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, P(x)	**$\exists x.\ \neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, P(x)	**$\forall x.\ \neg P(x)$**
$\forall x.\ \neg P(x)$	For any choice of x, ¬P(x)	**$\exists x.\ P(x)$**
$\exists x.\ \neg P(x)$	For some choice of x, ¬P(x)	**$\forall x.\ P(x)$**

Negating Quantifiers

- What is the negation of the following statement?

$$\forall x.\ \exists y.\ (P(x) \land Q(y))$$

- We can obtain it as follows:

$$\neg\forall x.\ \exists y.\ (P(x) \land Q(y))$$

$$\exists x. \neg\exists y.\ (P(x) \land Q(y))$$

$$\exists x. \forall y.\ \neg(P(x) \land Q(y))$$

$$\exists x. \forall y.\ (\neg P(x) \lor \neg Q(y))$$

Negating First-Order Statements

- To negate a first-order formula, push the negation inward as much as possible.
- Use techniques from propositional logic to negate connectives.
- Use the equivalences

$$\neg\forall x.\ \varphi \equiv \exists x.\ \neg\varphi$$

$$\neg\exists x.\ \varphi \equiv \forall x.\ \neg\varphi$$

to negate quantifiers.

Analyzing Relations

"R is not reflexive"

Analyzing Relations

"R is not reflexive"

$\neg \forall a.\ aRa$

Analyzing Relations

"R is not reflexive"

$\neg \forall a.\ aRa$

$\exists a.\ \neg aRa$

Analyzing Relations

"R is not reflexive"

$\neg\forall a.\ aRa$

$\exists a.\ \neg aRa$

"Some a is not related to itself."

Analyzing Relations

Analyzing Relations

“R is not antisymmetric”

Analyzing Relations

"R is not antisymmetric"

$$\neg\forall x.\ \forall y.\ (xRy \land yRx \rightarrow x = y)$$

Analyzing Relations

"R is not antisymmetric"

$$\neg\forall x.\ \forall y.\ (xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \neg\forall y.\ (xRy \land yRx \rightarrow x = y)$$

Analyzing Relations

"R is not antisymmetric"

$$\neg\forall x.\ \forall y.\ (xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \neg\forall y.\ (xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \exists y.\ \neg(xRy \land yRx \rightarrow x = y)$$

Analyzing Relations

"R is not antisymmetric"

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$$\exists x.\ \neg\forall y.\ (xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \exists y.\ \neg(xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \exists y.\ (xRy \land yRx \land \neg(x = y))$$

Analyzing Relations

"R is not antisymmetric"

$$\neg\forall x.\ \forall y.\ (xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \neg\forall y.\ (xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \exists y.\ \neg(xRy \land yRx \rightarrow x = y)$$
$$\exists x.\ \exists y.\ (xRy \land yRx \land \neg(x = y))$$
$$\exists x.\ \exists y.\ (xRy \land yRx \land x \neq y)$$

Analyzing Relations

"R is not antisymmetric"

$$\neg\forall x.\ \forall y.\ (xRy \land yRx \to x = y)$$
$$\exists x.\ \neg\forall y.\ (xRy \land yRx \to x = y)$$
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$$\exists x.\ \exists y.\ (xRy \land yRx \land \neg(x = y))$$
$$\exists x.\ \exists y.\ (xRy \land yRx \land x \neq y)$$

"Some x and y are related to one another, but are not equal"

Uniqueness

Uniqueness

- Often, statements have the form "there is a unique x such that …"
- Some sources use a **uniqueness quantifier** to express this:

$$\exists! n.\ P(n)$$

- However, it's possible to encode uniqueness using just the two quantifiers we've seen.

$$\exists! n.\ P(n) \equiv \exists n.\ (P(n) \land \forall m.\ (P(m) \to m = n))$$

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There is some n where P(n) is true

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$$\exists! n.\ P(n) \equiv \exists n.\ (P(n) \land \forall m.\ (P(m) \rightarrow m = n))$$

There is some n where P(n) is true

And whenever P is true, it must be for n.

Uniqueness is Tricky

"Every person is eaten by a velociraptor."

$$\forall p.\ (Person(p) \rightarrow \exists v.\ (Vel(v) \land Eats(v, p))$$

Uniqueness is Tricky

"Every person is eaten by a **unique** velociraptor."

$$\forall p.\ (Person(p) \to \exists v.\ (Vel(v) \land Eats(v, p))$$

Uniqueness is Tricky

"Every person is eaten by a **unique** velociraptor."

$$\forall p.\ (Person(p) \to \exists v.\ (Vel(v) \land Eats(v, p) \land (\forall w.\ (Vel(w) \land Eats(w, p) \to w = v)))$$

Uniqueness is Tricky

"Every person is eaten by a **unique** velociraptor."

$$\forall p.\ (Person(p) \to \exists v.\ (Vel(v) \land Eats(v, p) \land (\forall w.\ (Vel(w) \land Eats(w, p) \to w = v))))$$

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"Every person is eaten by a unique velociraptor."

$$\forall p.\ (Person(p) \to \exists v.\ (Vel(v) \land Eats(v, p) \land (\forall w.\ (Vel(w) \land Eats(w, p) \to w = v))))$$

What is the negation of this statement?

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\neg\forall p.\ (\text{Person}(p) \to \exists v.\ (\text{Vel}(v) \land \text{Eats}(v, p) \land (\forall w.\ (\text{Vel}(w) \land \text{Eats}(w, p) \to w = v)))$$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ \neg(Person(p) \rightarrow \exists v.\ (Vel(v) \land Eats(v, p) \land (\forall w.\ (Vel(w) \land Eats(w, p) \rightarrow w = v)))$$

Uniqueness is Tricky

“Every person is eaten by a unique velociraptor.”

$$\exists p.\ (Person(p) \land \neg\exists v.\ (Vel(v) \land Eats(v, p) \land (\forall w.\ (Vel(w) \land Eats(w, p) \rightarrow w = v)))$$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ (\mathrm{Person}(p) \land \forall v.\ \lnot(\mathrm{Vel}(v) \land \mathrm{Eats}(v, p) \land (\forall w.\ (\mathrm{Vel}(w) \land \mathrm{Eats}(w, p) \to w = v)))$$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (\neg Vel(v) \lor \neg Eats(v, p) \lor \neg(\forall w.\ (Vel(w) \land Eats(w, p) \rightarrow w = v)))$$

Uniqueness is Tricky

“Every person is eaten by a unique velociraptor.”

$$\exists p.\ (Person(p) \land \forall v.\ (\neg Vel(v) \lor \neg Eats(v, p) \lor (\exists w.\ \neg(Vel(w) \land Eats(w, p) \rightarrow w = v)))$$

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"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (\neg Vel(v) \lor \neg Eats(v, p) \lor (\exists w.\ (Vel(w) \land Eats(w, p) \land \neg(w = v)))))$$

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"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (\neg Vel(v) \lor \neg Eats(v, p) \lor (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)))$$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$\exists p.\ (Person(p) \land \forall v.\ (\mathbf{\neg Vel(v) \lor \neg Eats(v, p)} \lor$
$(\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)))$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$\exists p. \, (Person(p) \land \forall v. \, (\mathbf{\neg(Vel(v) \land Eats(v, p))} \lor (\exists w. \, (Vel(w) \land Eats(w, p) \land w \neq v)))$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$\exists p.\ (\text{Person}(p) \land \forall v.\ (\boldsymbol{\neg(\text{Vel}(v) \land \text{Eats}(v, p)) \lor (\exists w.\ (\text{Vel}(w) \land \text{Eats}(w, p) \land w \neq v)}))$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$\exists p.\ (Person(p) \land \forall v.\ (\neg(Vel(v) \land Eats(v, p)) \lor$
$(\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v))))$

Recall: $p \rightarrow q \equiv \neg p \lor q$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$\exists p.\ (Person(p) \land \forall v.\ (\mathbf{Vel(v) \land Eats(v, p) \rightarrow (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)}))$

Recall: $p \rightarrow q \equiv \neg p \lor q$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (Vel(v) \land Eats(v, p) \to (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)))$$

The negation of

$$\exists x.\ (P(x) \land Q(x))$$

is

$$\forall x.\ (P(x) \to \neg Q(x))$$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (Vel(v) \land Eats(v, p) \rightarrow (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)))$$

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$$\exists p.\ (Person(p) \land \forall v.\ (Vel(v) \land Eats(v, p) \rightarrow (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v))))$$

Dissecting a Statement

$$\forall v.\ (Vel(v) \land Eats(v, p) \to (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v))$$

Dissecting a Statement

$$\forall v.\ (Vel(v) \land Eats(v, p) \rightarrow (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v))$$

This whole statement is true if this is always false.

Dissecting a Statement

$$\forall v.\ (Vel(v) \land Eats(v, p)) \to (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v))$$

This whole statement is true if this is always false.

But if it isn't and some velociraptor eats person p, then some other velociraptor must as well.

Dissecting a Statement

$$\forall v.\ (Vel(v) \land Eats(v, p) \to (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v))$$

This whole statement is true if this is always false.

But if it isn't and some velociraptor eats person p, then some other velociraptor must as well.

So it's true if there are either 0 or 2 or more velociraptors that eat person p.

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (Vel(v) \land Eats(v, p) \rightarrow (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)))$$

Uniqueness is Tricky

"Every person is eaten by a unique velociraptor."

$$\exists p.\ (Person(p) \land \forall v.\ (Vel(v) \land Eats(v, p) \to (\exists w.\ (Vel(w) \land Eats(w, p) \land w \neq v)))$$

"Some person is eaten by zero velociraptors or multiple velociraptors."

Important Concepts

- Constants
- Predicates
- Functions
- Quantifiers
- Translating into FOL
- Translating from FOL
- Nested quantifiers
- Negating quantifiers
- Uniqueness

Next Time

- Functions
- Closures
- The Pigeonhole Principle