Mathematical Logic

Announcements

- Problem Set 1 graded, will be returned at the end of class.
- Problem Set 2 due Friday at 2:15PM.
  - Drop by office hours with questions!
  - Email us at **cs103@cs.stanford.edu** with questions!

An Important Question

How do we formalize the logic we've been using in our proofs?

Where We're Going

- **Propositional Logic** (Today)
  - Basic logical connectives.
  - Truth tables.
  - Logical equivalences.
  - A simple proof calculus.
- **First-Order Logic** (Friday/Monday)
  - Reasoning about properties of multiple objects.

Propositional Logic

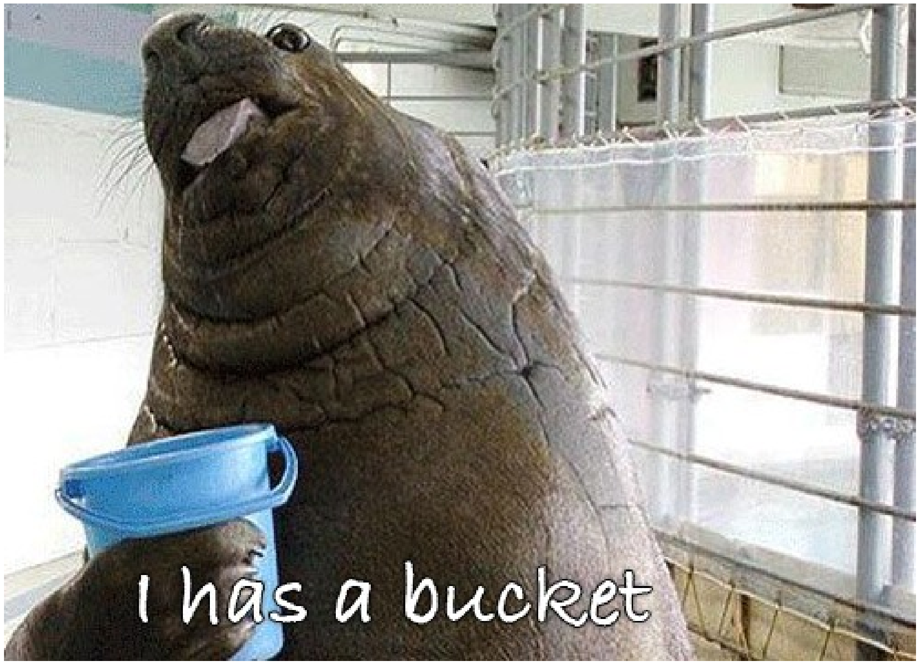
A **proposition** is a statement that is, by itself, either true or false.

Some Sample Propositions

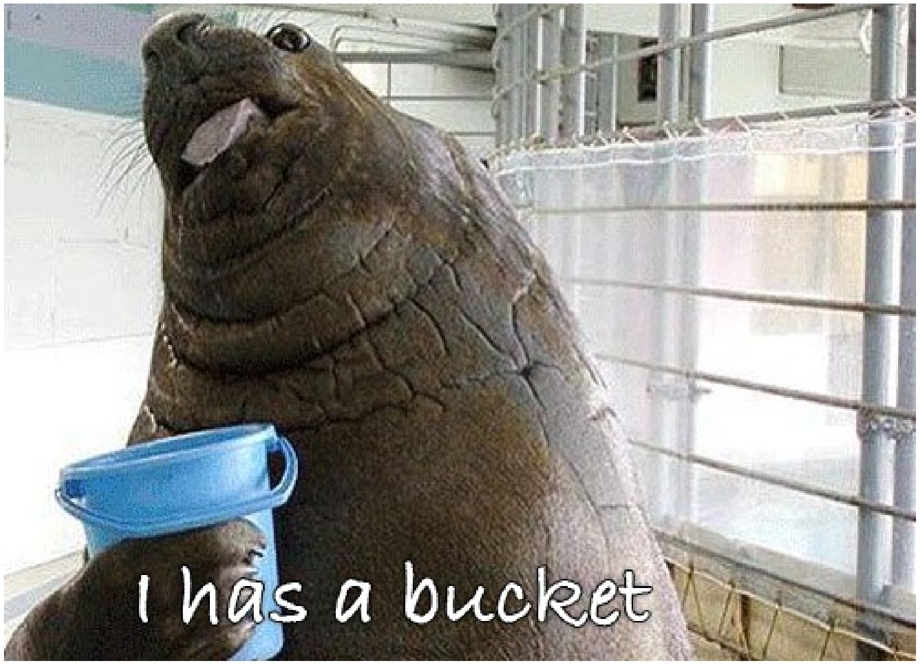
- There is a velociraptor outside my apartment.
- I don't like velociraptors.
- Velociraptors can open windows.
- Velociraptors can run faster than me.
- I am in my apartment right now.
- My apartment has windows.
- Velociraptors think I am tasty.

Some More Propositions

Some More Propositions



Some More Propositions



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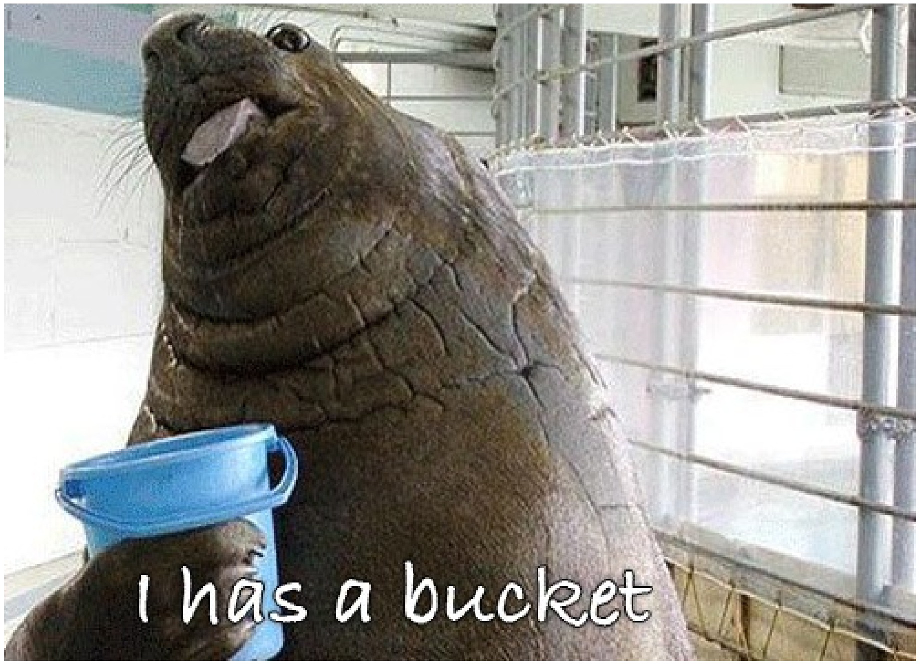

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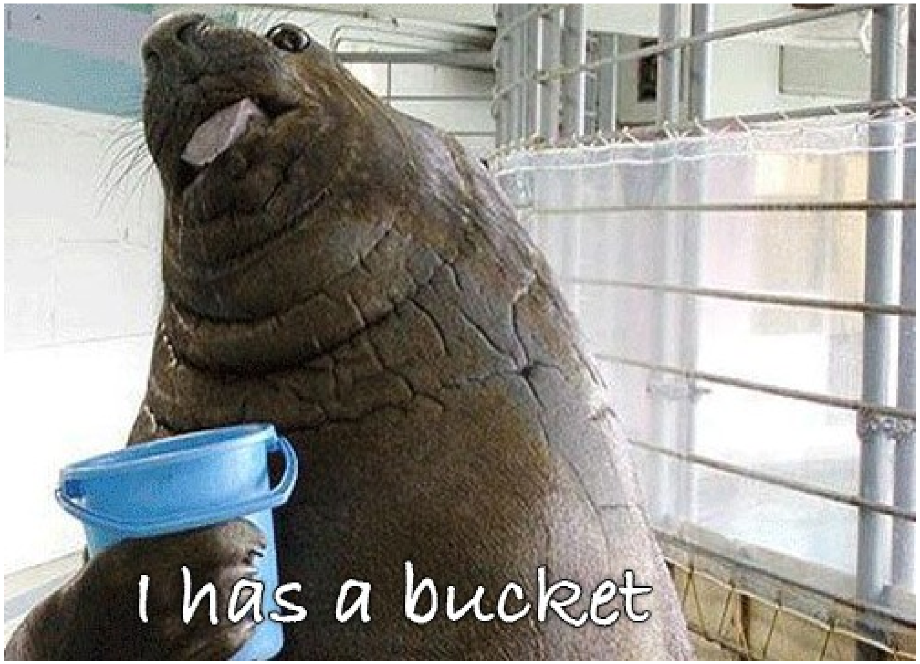

Things That Aren't Propositions

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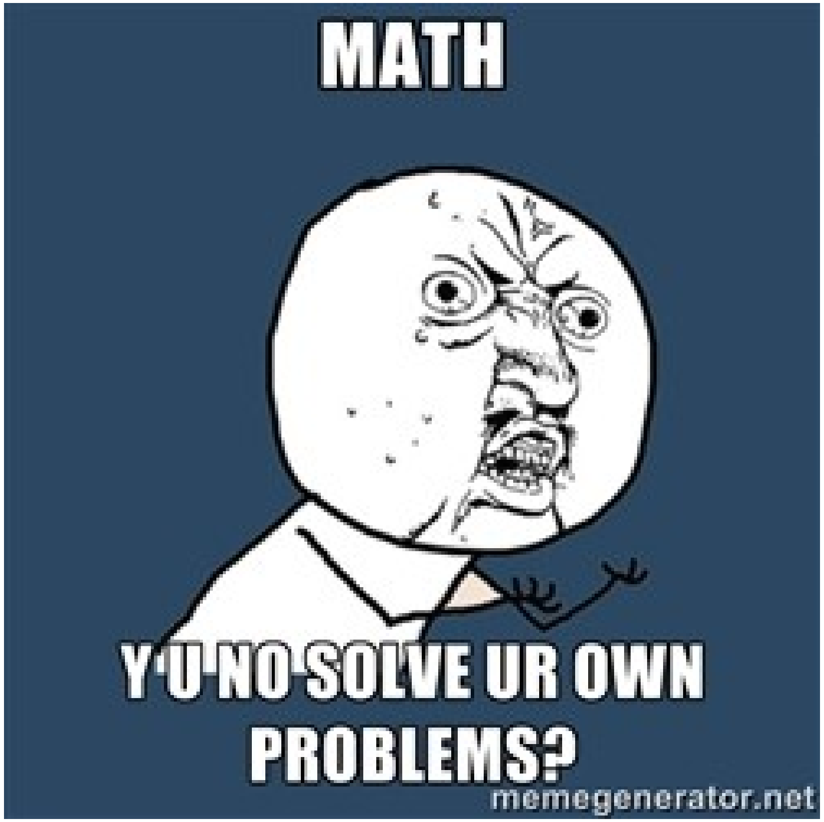
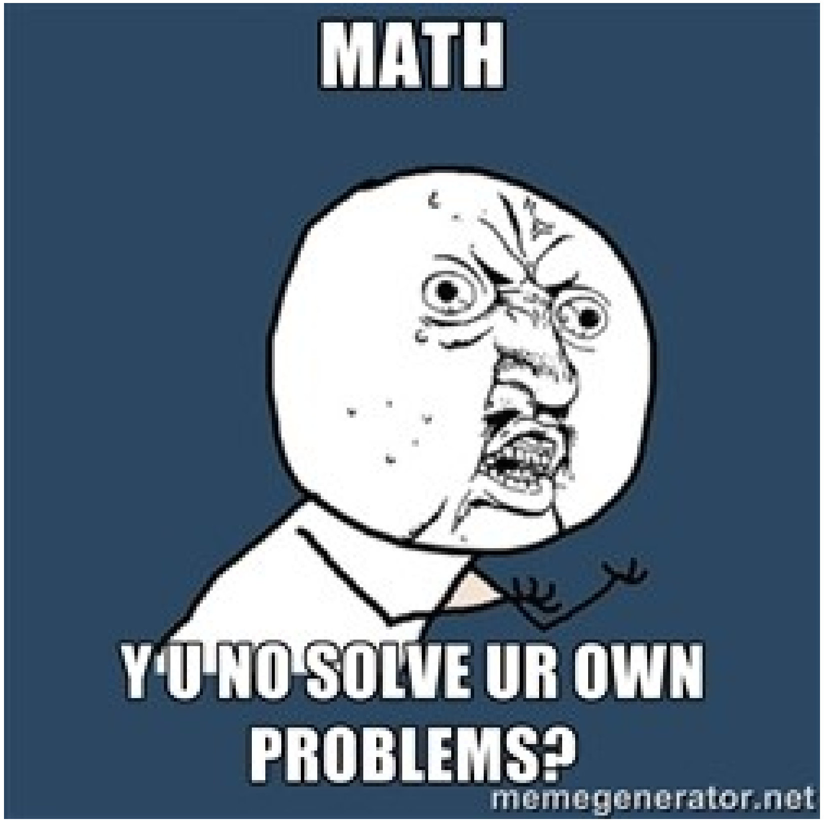
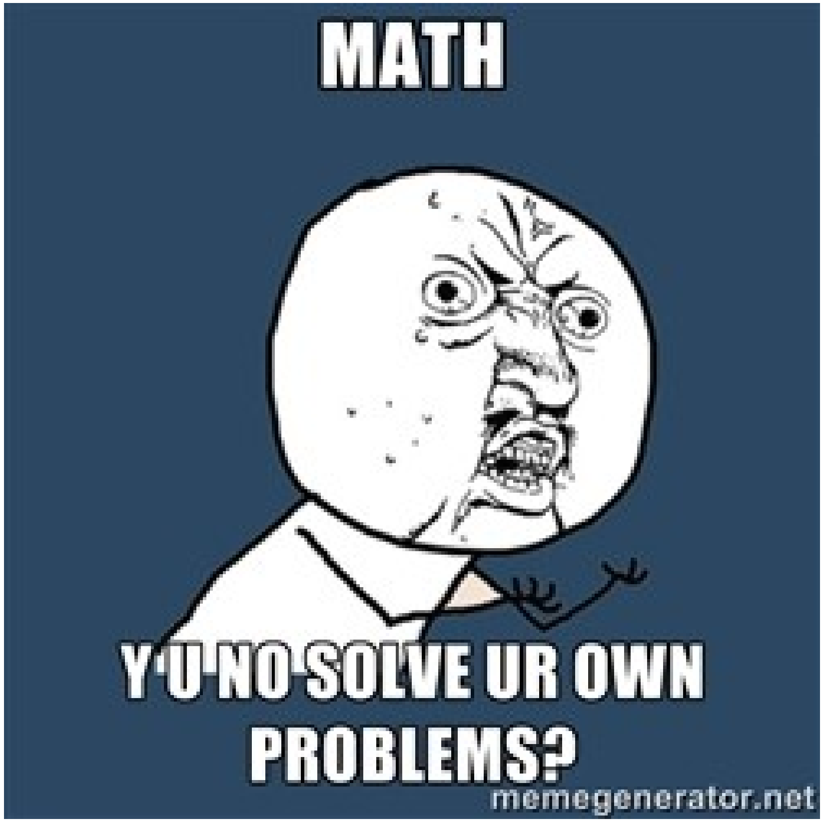


Things That Aren't Propositions

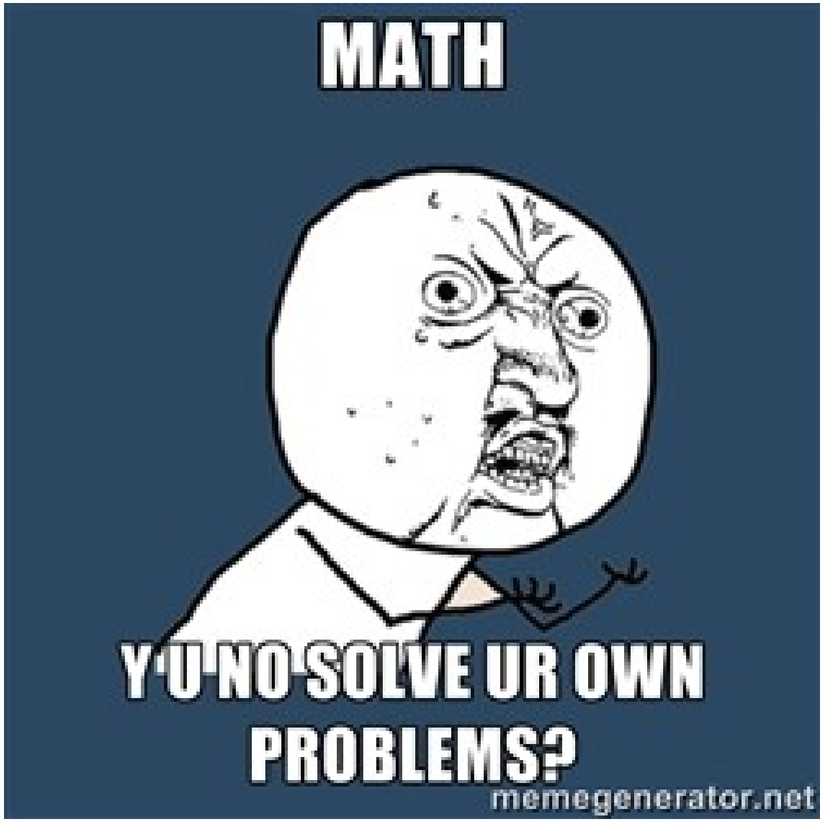


Questions are not true or false

Things That Aren't Propositions



Things That Aren't Propositions



Commands cannot be true or false

Propositional Logic

- **Propositional logic** is a mathematical system for reasoning about propositions and how they relate to one another.
- Propositional logic enables us to
  - Formally encode how the truth of various propositions influences the truth of other propositions.
  - Determine if certain combinations of propositions are always, sometimes, or never true.
  - Determine whether certain combinations of propositions logically entail other combinations.

Variables and Connectives

- Propositional logic is a **formal mathematical system** whose syntax is rigidly specified.
- Every statement in propositional logic consists of **propositional variables** combined via **logical connectives**.
  - Each variable represents some proposition, such as “I am near a velociraptor” or “I will be eaten by a velociraptor.”
  - Connectives encode how propositions are related, such as “If I am near a velociraptor, I will be eaten by a velociraptor.”

Propositional Variables

- Each proposition will be represented by a **propositional variable**.
- Propositional variables are usually represented as lower-case letters, such as p, q, r, s, etc.
  - If we need more, we can use subscripts: p_1, p_2, etc.
- Each variable can take one one of two values – true and false.

Logical Connectives

- **Logical NOT:** $\neg p$
  - Read "**not** p"
  - $\neg p$ is true if and only if p is false.
  - Also called **logical negation**.
- **Logical AND:** $p \land q$
  - Read "p **and** q."
  - $p \land q$ is true if both p and q are true.
  - Also called **logical conjunction**.
- **Logical OR:** $p \lor q$
  - Read "p **or** q."
  - $p \lor q$ is true if at least one of p or q are true.
  - Also called **logical disjunction**.

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

If p is false and q is false, then "both p and q" is false.

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

"both p and q" is true only when both p and q are true.

Truth Tables

Truth Tables

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Truth Tables

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

"Either p or q" is true even if both p and q are true. Remember that there are three ways for "either p or q" to be true!

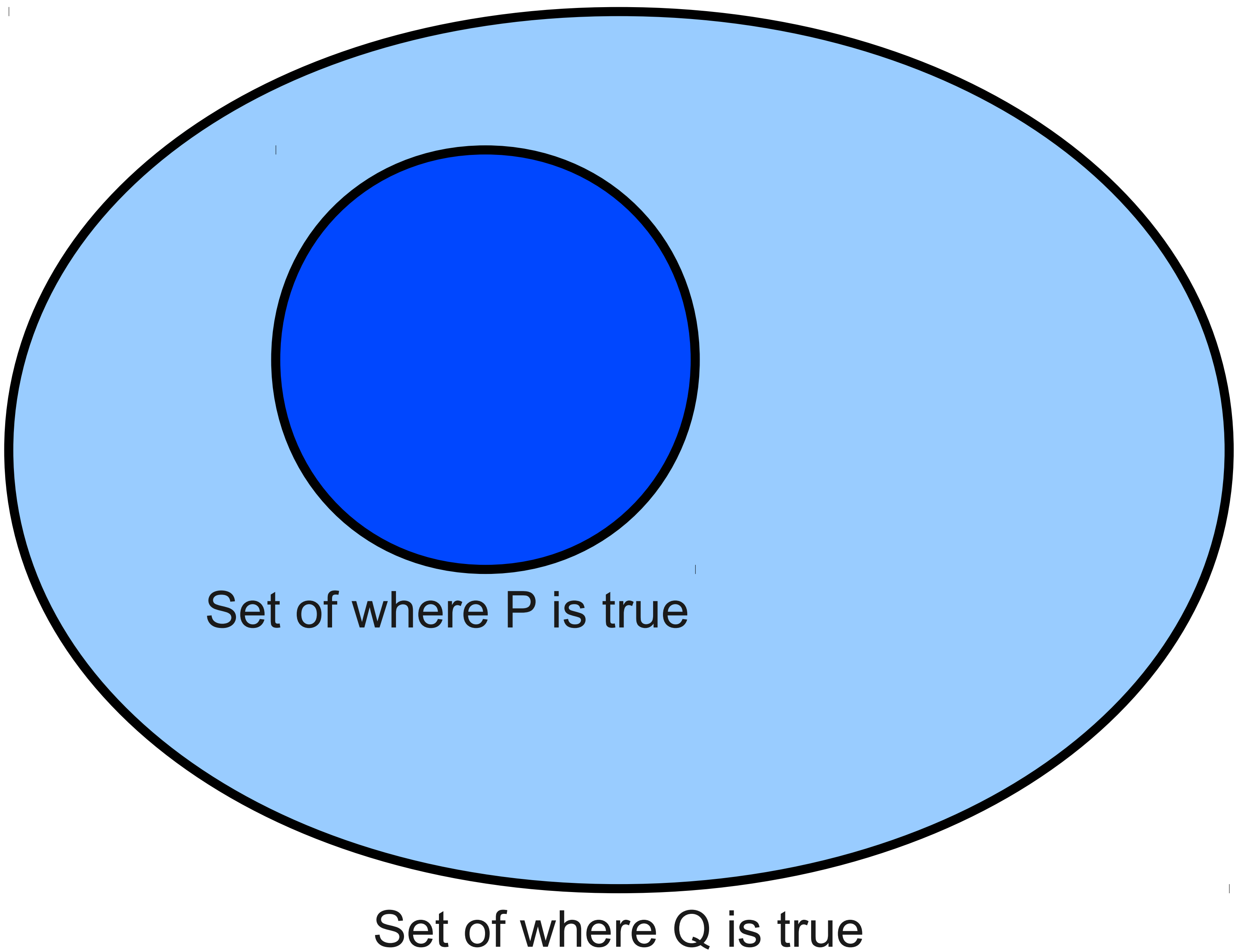
Truth Tables

p	$\neg p$
F	T
T	F

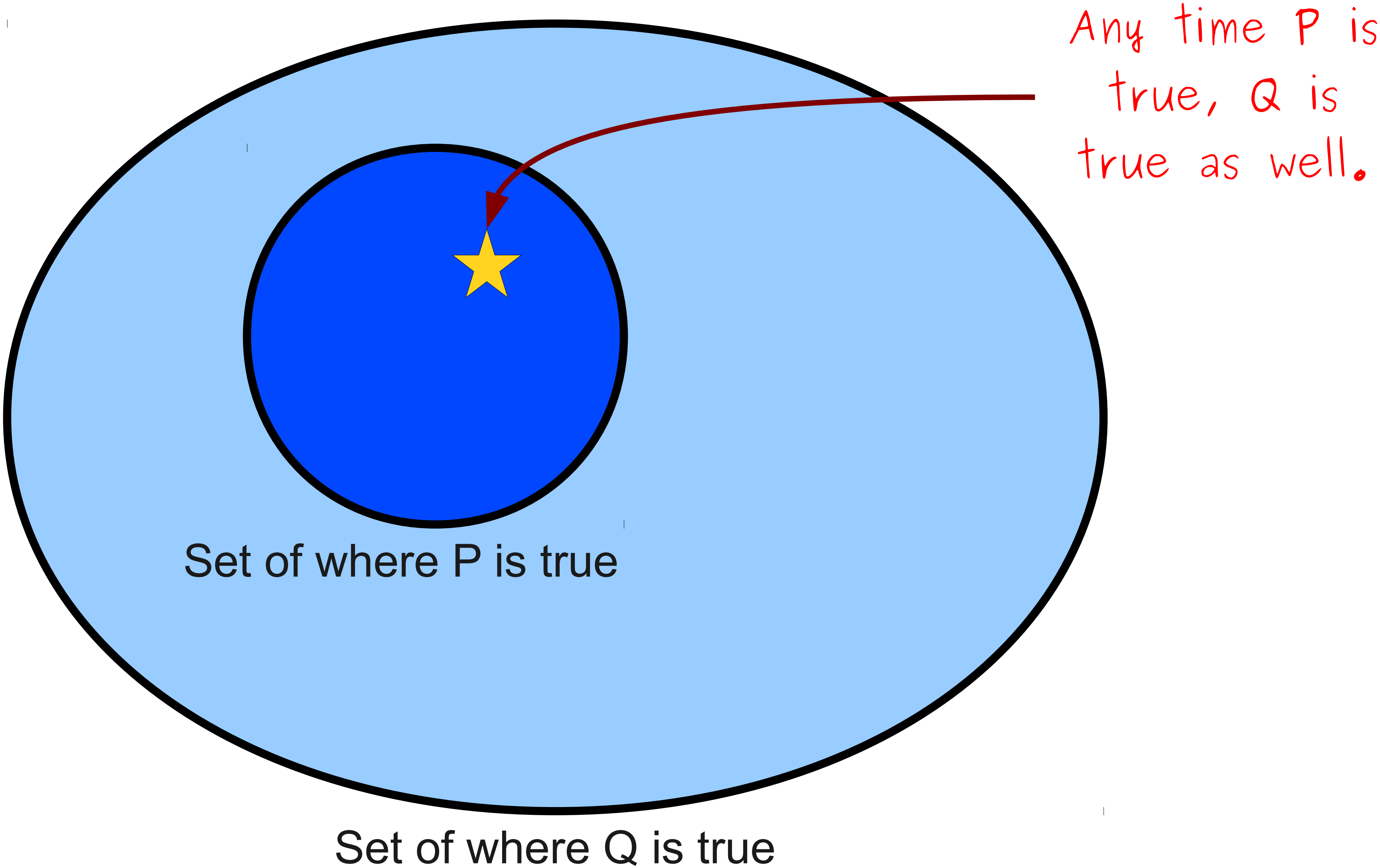
Implication

- An important connective is logical implication: $p \rightarrow q$.
- Recall: $p \rightarrow q$ means "if p is true, q is true as well."
- Recall: p → q says **nothing** about what happens if p is false.
- Recall: p → q says **nothing** about causality; it just says that whenever p is true, q will be true as well.

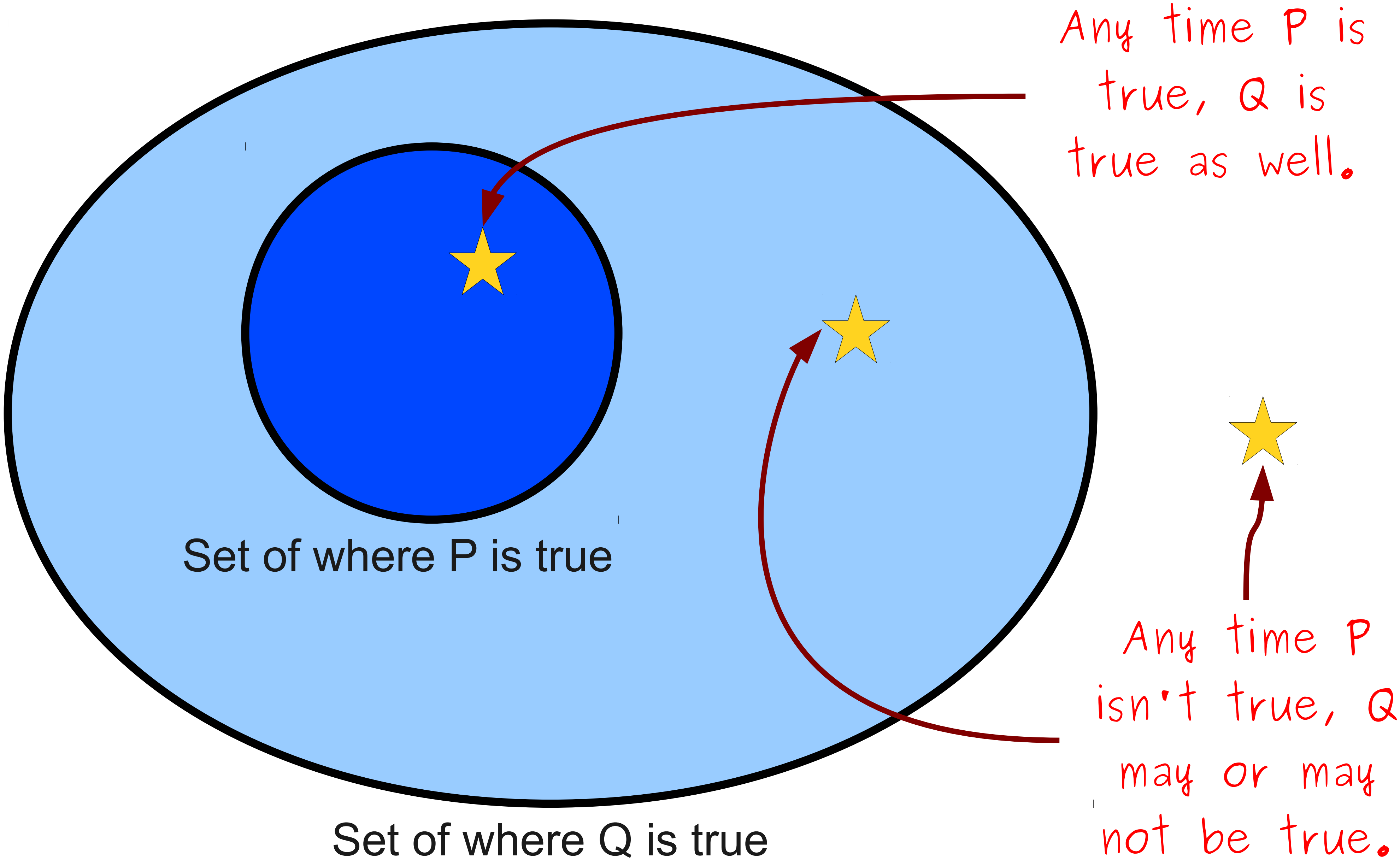
Implication, Diagrammatically



Implication, Diagrammatically



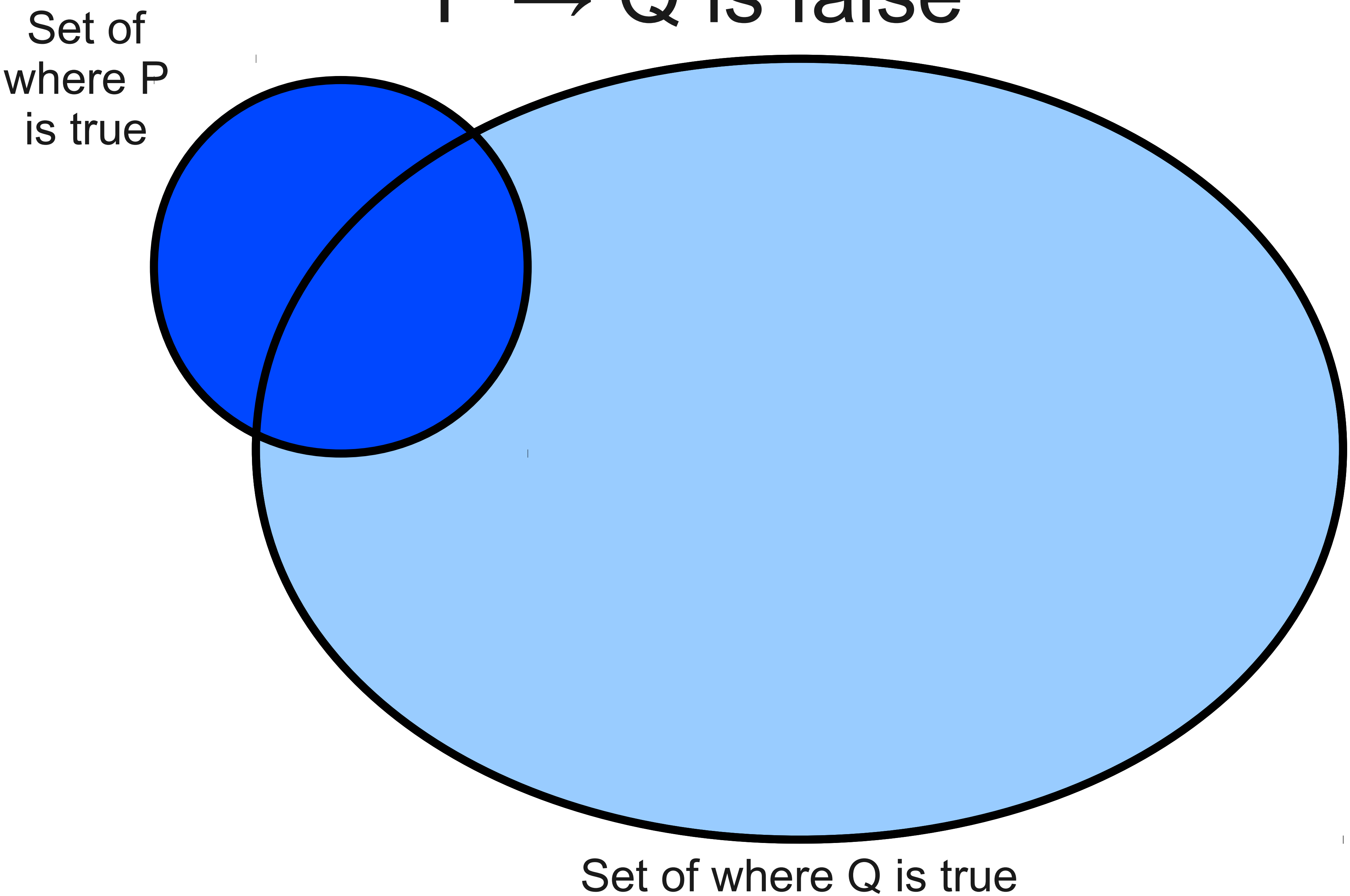
Implication, Diagrammatically

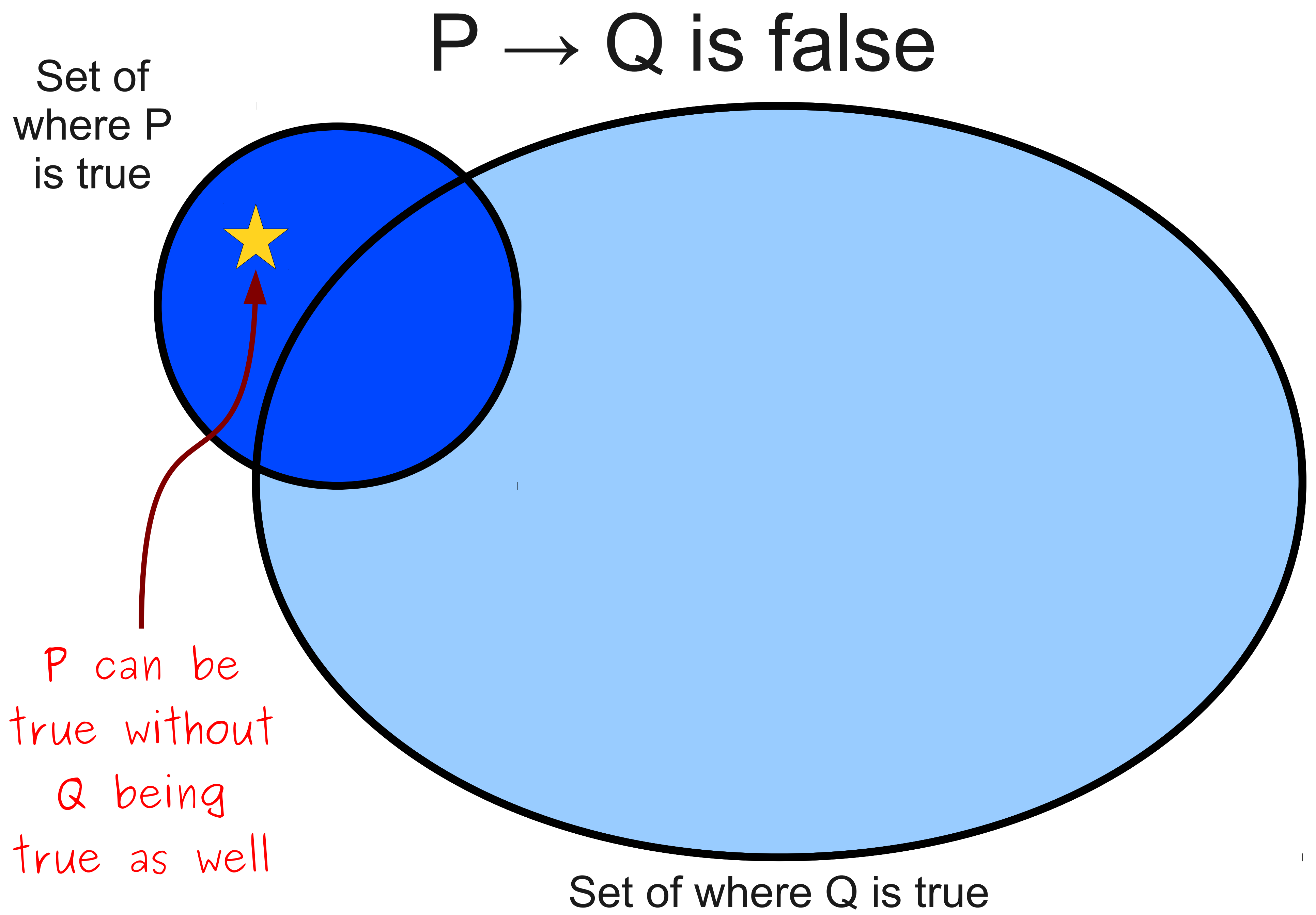


When p Does Not Imply q

- $p \rightarrow q$ means “whenever p is true, q is true as well.”
- Recall: The **only way** for $p \rightarrow q$ to be false is if we know that p is true but q is false.
- Rationale:
  - If p is false, $p \rightarrow q$ doesn't guarantee anything. It's true, but it's not **meaningful**.
  - If p is true and q is true, then the statement “if p is true, then q is also true” is itself true.
  - If p is true and q is false, then the statement “if p is true, q is also true” is false.

P → Q is false




P → Q is false
Set of where P is true
P can be true without Q being true as well
Set of where Q is true

Truth Table for Implication

p	q	$p \rightarrow q$
F	F	
F	T	
T	F	
T	T	

Truth Table for Implication

p	q	$p \rightarrow q$
F	F	
F	T	
T	F	
T	T	

Truth Table for Implication

p	q	$p \to q$
F	F	
F	T	
T	F	
T	T	

In both of these cases, p is false, so the statement "if p, then q" is vacuously true.

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	
T	T	

In both of these cases, p is false, so the statement "if p, then q" is vacuously true.

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Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	
T	T	

$p \to q$ should mean when p is true, q is true as well. But here p is true and q is false!

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	

$p \to q$ should mean when p is true, q is true as well. But here p is true and q is false!

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Truth Table for Implication

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F	F	T
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$p \to q$ means that if we ever find that p is true, we'll find that q is true as well.

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
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$p \to q$ means that if we ever find that p is true, we'll find that q is true as well.

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Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

The <u>only way</u> for $p \to q$ to be false is for p to be true and q to be false.

The Biconditional

- The **biconditional** connective $p \leftrightarrow q$ is read "p if and only if q."
- Intuitively, either both p and q are true, or neither of them are.

p	q	$p \leftrightarrow q$
F	F	
F	T	
T	F	
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The Biconditional

- The **biconditional** connective p ↔ q is read "p if and only if q."
- Intuitively, either both p and q are true, or neither of them are.

p	q	$p \leftrightarrow q$
F	F	
F	T	
T	F	
T	T	

One of p or q is true without the other.

The Biconditional

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p	q	$p \leftrightarrow q$
F	F	T
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T	F	F
T	T	T

One interpretation of ↔ is to think of it as equality: the two propositions must have equal truth values.

True and False

- There are two more “connectives” to speak of: true and false.
- The symbol $\top$ is a value that is always true.
- The symbol $\bot$ is value that is always false.
- These are often called connectives, though they don't connect anything.
  - (Or rather, they connect zero things.)

Operator Precedence

- How do we parse this statement?

$$\neg x \rightarrow y \lor z \rightarrow x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

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$$\leftrightarrow$$

- All operators are right-associative.
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Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
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Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow y \lor z \rightarrow x \lor (y \land z)$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow (y \lor z) \rightarrow (x \lor (y \land z))$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\rightarrow$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow (y \lor z) \rightarrow (x \lor (y \land z))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

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- All operators are right-associative.
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Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow ((y \vee z) \rightarrow (x \vee (y \wedge z)))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Recap So Far

- A **propositional variable** is a variable that is either true or false.
- The logical connectives are
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

I won't be eaten by a velociraptor if there isn't a velociraptor outside my apartment.

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

I won't be eaten by a velociraptor if there isn't a velociraptor outside my apartment.

$$\neg a \rightarrow \neg e$$

“p if q”

translates to

$$q \rightarrow p$$

It does **not** translate to

$$p \rightarrow q$$

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

If there is a velociraptor outside my apartment, but it can't open windows, I am not going to be eaten by a velociraptor.

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

If there is a velociraptor outside my apartment, but it can't open windows, I am not going to be eaten by a velociraptor.

$$a \land \neg b \to \neg e$$

"p, but q"

translates to

$$p \wedge q$$

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

I am only in my apartment when
there are no velociraptors outside.

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

I am only in my apartment when
there are no velociraptors outside.

$$c \rightarrow \neg a$$

“p only when q”

translates to

$$p \rightarrow q$$

The Takeaway Point

- When translating into or out of propositional logic, be very careful not to get tripped up by the nuances of the English language.
- Many prepositions lead to counterintuitive translations; make sure to double-check yourself!

Logical Equivalence

More Elaborate Truth Tables

p	q	$p \wedge (p \rightarrow q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

We can't evaluate this until we have a value for $p \to q$.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	
T	T	

More Elaborate Truth Tables

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p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

p	q	$p \wedge (p \rightarrow q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	F T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	p	$\land$	$(p \to q)$
F	F		F	T
F	T			T
T	F			F
T	T			T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	p	$\wedge$	$(p$	$\to$	$q)$
F	F		F		T	
F	T		F		T	
T	F				F	
T	T				T	

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	p	$\land$	$(p$	$\rightarrow$	$q)$
F	F		F		T	
F	T		F		T	
T	F				F	
T	T				T	

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \wedge (p \rightarrow q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T		T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \wedge (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T		T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	F T
F	T	F T
T	F	F F
T	T	T T

More Elaborate Truth Tables

p	q	$p \land (p \to q)$
F	F	F T
F	T	F T
T	F	F F
T	T	T T

More Elaborate Truth Tables

This gives the final truth value for the expression.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

Negations

- $p \land q$ is false if and only if $\neg(p \land q)$ is true.
- Intuitively, this is only possible if either p is false or q is false (or both!)
- In propositional logic, we can write this as $\neg p \lor \neg q$.
- How would we prove that $\neg(p \land q)$ and $\neg p \lor \neg q$ are equivalent?
- **Idea**: Build truth tables for both expressions and confirm that they always agree.

Negating AND

p	q	$\neg(p \land q)$
F	F	
F	T	
T	F	
T	T	

Negating AND

p	q	$\neg(p \wedge q)$
F	F	F
F	T	F
T	F	F
T	T	T

Negating AND

p	q	$\neg($	$p \land q)$
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

p	q	$\neg p$	$\vee$	$\neg q$
F	F			
F	T			
T	F			
T	T			

Negating AND

p	*q*	¬(*p* ∧ *q*)
F	F	T F
F	T	T F
T	F	T F
T	T	F T

p	*q*	¬*p* ∨ ¬*q*
F	F	T
F	T	T
T	F	F
T	T	F

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T		T
F	T	T		F
T	F	F		T
T	T	F		F

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

Negating AND

p	q	$\neg(p \land q)$	
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p$	$\lor$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

These two statements are always the same!

Logical Equivalence

- If two propositional logic statements φ and ψ always have the same truth values as one another, they are called **logically equivalent**.
- We denote this by $\varphi \equiv \psi$
- ≡ is **not** a connective. Connectives are a part of logic statements; ≡ is something used to describe logic statements.
  - It is part of the **metalanguage** rather than the **language**.
- If $\varphi \equiv \psi$, we can modify any propositional logic formula containing φ by replacing it with ψ.
  - This is not true when we talk about first-order logic; we'll see why later.

De Morgan's Laws

- Using truth tables, we concluded that

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

- We can also use truth tables to show that

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

- These two equivalences are called **De Morgan's Laws**.

More Negations

- $p \rightarrow q$ if and only if $\neg(p \rightarrow q)$ is false.
- As mentioned earlier, this only happens if p is true and q is false.
- In propositional logic, this is $p \land \neg q$.
- Can we prove this?

Negating Implications

Negating Implications

p	q	$\neg(p \longrightarrow q)$
F	F	
F	T	
T	F	
T	T	

Negating Implications

p	q	$\neg(p \rightarrow q)$
F	F	T
F	T	T
T	F	F
T	T	T

Negating Implications

p	q	$\neg(p \rightarrow q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

Negating Implications

p	q	$\neg(p$	$\rightarrow$	$q)$
F	F	F	T	
F	T	F	T	
T	F	T	F	
T	T	F	T	

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$
F	F	
F	T	
T	F	
T	T	

Negating Implications

p	q	$\neg(p \rightarrow q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$
F	F	F
F	T	F
T	F	T
T	T	T

Negating Implications

p	q	$\neg(p$	$\rightarrow q)$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\wedge \neg q$
F	F	F	T
F	T	F	F
T	F	T	T
T	T	T	F

Negating Implications

p	q	$\neg(p$	$\to$	$q)$
F	F	F	T	
F	T	F	T	
T	F	T	F	
T	T	F	T	

p	q	p	$\land$	$\neg q$
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

Negating Implications

p	q	$\neg$	$(p \to q)$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\wedge$	$\neg q$
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

Negating Implications

p	q	$\neg(p$	$\rightarrow q)$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\wedge$	$\neg q$
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

An Important Observation

- We have just proven that

$$\neg(p \to q) \equiv p \land \neg q$$

- If we negate both sides, we get that

$$p \to q \equiv \neg(p \land \neg q)$$

- By De Morgan's laws:

$$p \to q \equiv \neg(p \land \neg q)$$

$$p \to q \equiv \neg p \lor \neg\neg q$$

$$p \to q \equiv \neg p \lor q$$

- Thus $\mathbf{p \to q \equiv \neg p \lor q}$

An Important Observation

- We have just proven that

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- If we negate both sides, we get that

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- By De Morgan's laws:

$$p \to q \equiv \neg(p \land \neg q)$$

$$p \to q \equiv \neg p \lor \neg\neg q$$

$$p \to q \equiv \neg p \lor q$$

- Thus **$p \to q \equiv \neg p \lor q$**

If p is false, the whole thing is true and we gain no information. If p is true, then q has to be true for the whole expression to be true.

Another Idea

- We've just shown that $\neg(p \to q) \equiv p \land \neg q$.
- Is it also true that $\neg(p \to q) \equiv p \to \neg q$?
- Let's go check!

$\neg(p \rightarrow q)$ and $p \rightarrow \neg q$

$\neg(p \rightarrow q)$ and $p \rightarrow \neg q$

p	q	$\neg(p \rightarrow q)$
F	F	
F	T	
T	F	
T	T	

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \to \neg q$
F	F	
F	T	
T	F	
T	T	

¬(p → q) and p → ¬q

p	q	$\neg(p$	$\to$	$q)$
F	F	F	T	
F	T	F	T	
T	F	T	F	
T	T	F	T	

p	q	p	$\to$	$\neg q$
F	F	F		
F	T	F		
T	F	T		
T	T	T		

$\neg(p \rightarrow q)$ and $p \rightarrow \neg q$

p	q	$\neg(p \rightarrow q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \rightarrow \neg q$	
F	F	F	T
F	T	F	F
T	F	T	T
T	T	T	F

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p$	$\to q)$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\to$	$\neg q$
F	F	F	T	T
F	T	F		F
T	F	T		T
T	T	T		F

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \to \neg q$		
F	F	F	T	T
F	T	F	T	F
T	F	T		T
T	T	T		F

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \to \neg q$		
F	F	F	T	T
F	T	F	T	F
T	F	T	T	T
T	T	T		F

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \to \neg q$		
F	F	F	T	T
F	T	F	T	F
T	F	T	T	T
T	T	T	F	F

$\neg(p \rightarrow q)$ and $p \rightarrow \neg q$

p	q	$\neg(p \rightarrow q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\rightarrow$	$\neg q$
F	F	F	T	T
F	T	F	T	F
T	F	T	T	T
T	T	T	F	F

$\neg(p \to q)$ and $p \to \neg q$

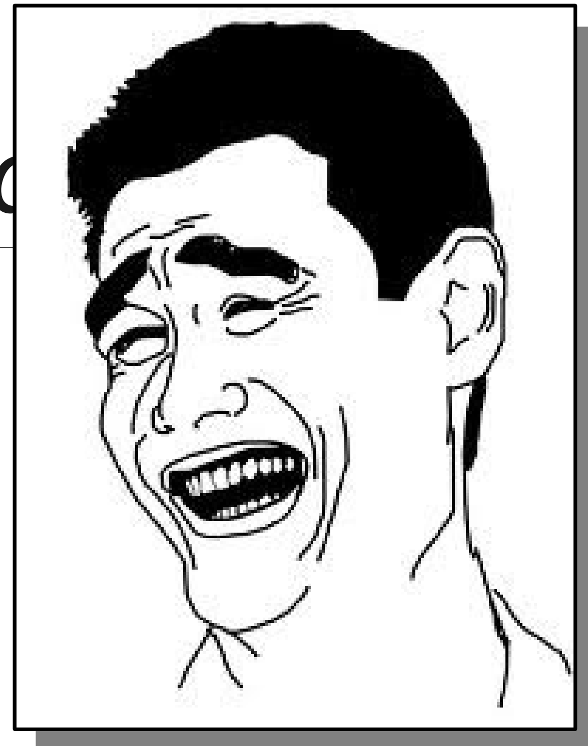
p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \to \neg q$		
F	F	F	T	T
F	T	F	T	F
T	F	T	T	T
T	T	T	F	F

These are ***not*** the same thing!

$\neg(p \to q)$ and $p \to \neg q$

p	q	$\neg(p$	$\to$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T



p	$\to$	$\neg q$
F	T	T
F	T	F
T	T	T
T	F	F

These are ***not*** the same thing!

To prove that $p \rightarrow q$ is false, do **not** prove $p \rightarrow \neg q$.

Instead, prove that $p \wedge \neg q$ is true.