Mathematical Induction, Part Two

Announcements

- Problem Set 1 due now.
- Problem Set 2 out, due Friday, October 14.
  - **Start early!**
  - Stop by office hours or email us at **cs103@cs.stanford.edu** if you have any questions.
- Friday Four Square today at 4:15 in front of Gates.

A Correction from Last Time

A (Flawed) Inductive Proof

Theorem: The sum of the first n natural numbers is n(n + 1)/2.
Proof: By induction. Let P(n) be “the sum of the first n natural numbers is n(n + 1) / 2.” We show that P(n) is true for all natural numbers n.

For our base case, we need to show P(0) is true, meaning that the sum of the first zero natural numbers is 0(0 + 1)/2. Since the sum of the first zero natural numbers is 0 = 0(0 + 1)/2, P(0) is true.

For the inductive step, assume that for some n, P(n) holds, so 1 + 2 + … + n = n(n + 1) / 2. We need to show that P(n + 1) holds, meaning that the sum of the first n + 1 natural numbers is (n + 1)(n + 2)/2. Consider the sum of the first n + 1 natural numbers. This is the sum of the first n natural numbers, plus n + 1. By the inductive hypothesis, this is given by

$$0+1+\ldots+n+n+1=\frac{n(n+1)}{2}+n+1=\frac{n(n+1)+2(n+1)}{2}=\frac{(n+1)(n+2)}{2}$$

Thus P(n + 1) holds when P(n) is true, so P(n) is true for all natural numbers n. ■

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WRONG: This is n + 2 numbers!

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The Takeaway Point

- Prefer **mathematical notation** to natural language when possible.
  - Explicit sums are much harder to get wrong.
  - Formalizing the beginning and end of the sum makes it less likely to include too many or too few elements.

Slimming Down Induction Proofs

Induction in Practice

- Typically, a proof by induction will not explicitly state P(n).
- Rather, the proof will describe P(n) implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what P(n) is,
  - that P(0) is true, and that
  - whenever P(n) is true, P(n + 1) is true,

  the proof is usually valid.

Theorem: For any natural number n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: By induction on n. For our base case, if n = 0, note that

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2} = 0$$

and the theorem is true for 0.

For the inductive step, assume that for some n the theorem is true. Then we have that

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

so the theorem is true for n + 1, completing the induction. ■

Some Variants of Induction

n^2 versus 2^n

- $0^2 = 0$
- $1^2 = 1$
- $2^2 = 4$
- $3^2 = 9$
- $4^2 = 16$
- $5^2 = 25$
- $6^2 = 36$
- $7^2 = 49$
- $8^2 = 64$
- $9^2 = 81$
- $10^2 = 100$

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 4$
- $2^3 = 8$
- $2^4 = 16$
- $2^5 = 32$
- $2^6 = 64$
- $2^7 = 128$
- $2^8 = 256$
- $2^9 = 512$
- $2^{10} = 1024$

n^2 versus 2^n

$0^2 = 0$	$<$	$2^0 = 1$
$1^2 = 1$	$<$	$2^1 = 2$
$2^2 = 4$	$=$	$2^2 = 4$
$3^2 = 9$	$>$	$2^3 = 8$
$4^2 = 16$	$=$	$2^4 = 16$
$5^2 = 25$	$<$	$2^5 = 32$
$6^2 = 36$	$<$	$2^6 = 64$
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2^n is <u>much</u> bigger here. Does the trend continue?

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$$(n + 1)^2 = n^2 + 2n + 1$$

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Since $n \geq 5$,

$$(n + 1)^2 = n^2 + 2n + 1 < n^2 + 2n + n = n^2 + 3n < n^2 + n^2 = 2n^2.$$

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Why is this allowed?

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Remember: $A \rightarrow B$ means "whenever A is true, B is true." If B is always true, $A \rightarrow B$ is true for any A.

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Again, $A \to B$ is automatically true if B is always true.

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- We explicitly proved P(5), so $P(4) \to P(5)$
- For any $n \geq 5$, we explicitly proved that $P(n) \to P(n + 1)$.

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- We explicitly proved P(5), so $P(4) \rightarrow P(5)$
- For any $n \geq 5$, we explicitly proved that $P(n) \rightarrow P(n + 1)$.
- Thus P(0) and for any n, $P(n) \rightarrow P(n + 1)$, so by induction P(n) is true for all natural numbers n.

Induction Starting at k

- To prove that $P(n)$ is true for all natural numbers greater than or equal to k:
  - Show that $P(k)$ is true.
  - Show that for any $n \geq k$, that $P(n) \rightarrow P(n + 1)$.
  - Conclude $P(k)$ holds for all natural numbers greater than or equal to k.

Double Induction

Factorials

- n! (read *n factorial*) is the product of all nonzero natural numbers less than or equal to n:
  - 0! = 1 (the *empty product*)
  - 1! = 1
  - $2! = 2 \cdot 1 = 2$
  - $3! = 3 \cdot 2 \cdot 1 = 6$
  - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

A Recursive Function Definition

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n - 1)! & \text{otherwise} \end{cases}$$

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How do we know that this is well-defined?

$$
n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n - 1)! & \text{otherwise} \end{cases}
$$

Theorem: n! is defined for any natural number n.

Proof: By induction. As a base case, n! is defined to be 1 if n = 0.

For the inductive step, assume that for some n, n! is defined and consider (n + 1)!. This is defined as (n + 1)(n!). Since by the inductive hypothesis n! is defined, this product is defined as well. Thus (n + 1)! is defined, completing the proof. ■

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$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ f(2n) & \text{otherwise} \end{cases}$$

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$$f(1) = f(2) = f(4) = f(8) = f(16) = \ldots = f(2^n) = \ldots$$

A Recursive Function Definition?

This is not a function!

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ f(2n) & \text{otherwise} \end{cases}$$

$$f(1) = f(2) = f(4) = f(8) = f(16) = \ldots = f(2^n) = \ldots$$

A Recursive Function Definition?

$$f(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + f(3n + 1) & \text{if } n \text{ is odd} \\ 1 + f(n / 2) & \text{if } n \text{ is even} \end{cases}$$

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No one knows if this is a function or not!

A Recursive Function Definition?

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)) & \text{otherwise} \end{cases}$$

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Is this a function?

Proving A(m, n) is a Function

- To prove that n! was a function, we proved, by induction, that n! was defined for any natural number n.
- How do we prove that A(m, n) is a function for any **pair** of natural numbers m and n?

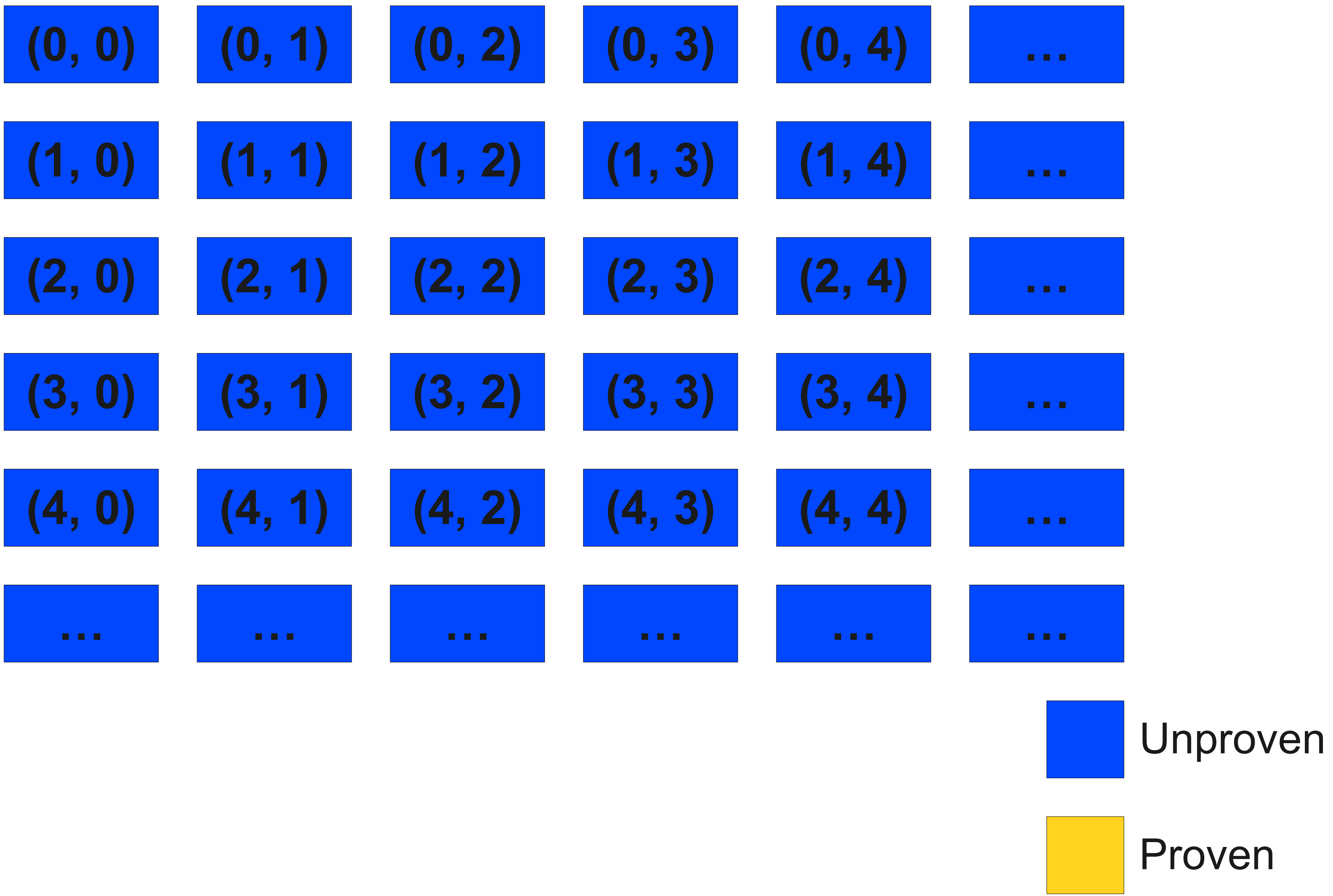
Proving A(m, n) is a Function

- To prove that n! was a function, we proved, by induction, that n! was defined for any natural number n.
- How do we prove that A(m, n) is a function for any **pair** of natural numbers m and n?
- Use a **double induction**:
  - Prove, by induction, that for all m, A(m, n) is defined for all n.
  - To prove that it's defined for all n, use a **second** induction, just on n.

Double Induction

(0, 0)	**(0, 1)**	**(0, 2)**	**(0, 3)**	**(0, 4)**	**…**
(1, 0)	**(1, 1)**	**(1, 2)**	**(1, 3)**	**(1, 4)**	**…**
(2, 0)	**(2, 1)**	**(2, 2)**	**(2, 3)**	**(2, 4)**	**…**
(3, 0)	**(3, 1)**	**(3, 2)**	**(3, 3)**	**(3, 4)**	**…**
(4, 0)	**(4, 1)**	**(4, 2)**	**(4, 3)**	**(4, 4)**	**…**
…	**…**	**…**	**…**	**…**	**…**

Double Induction



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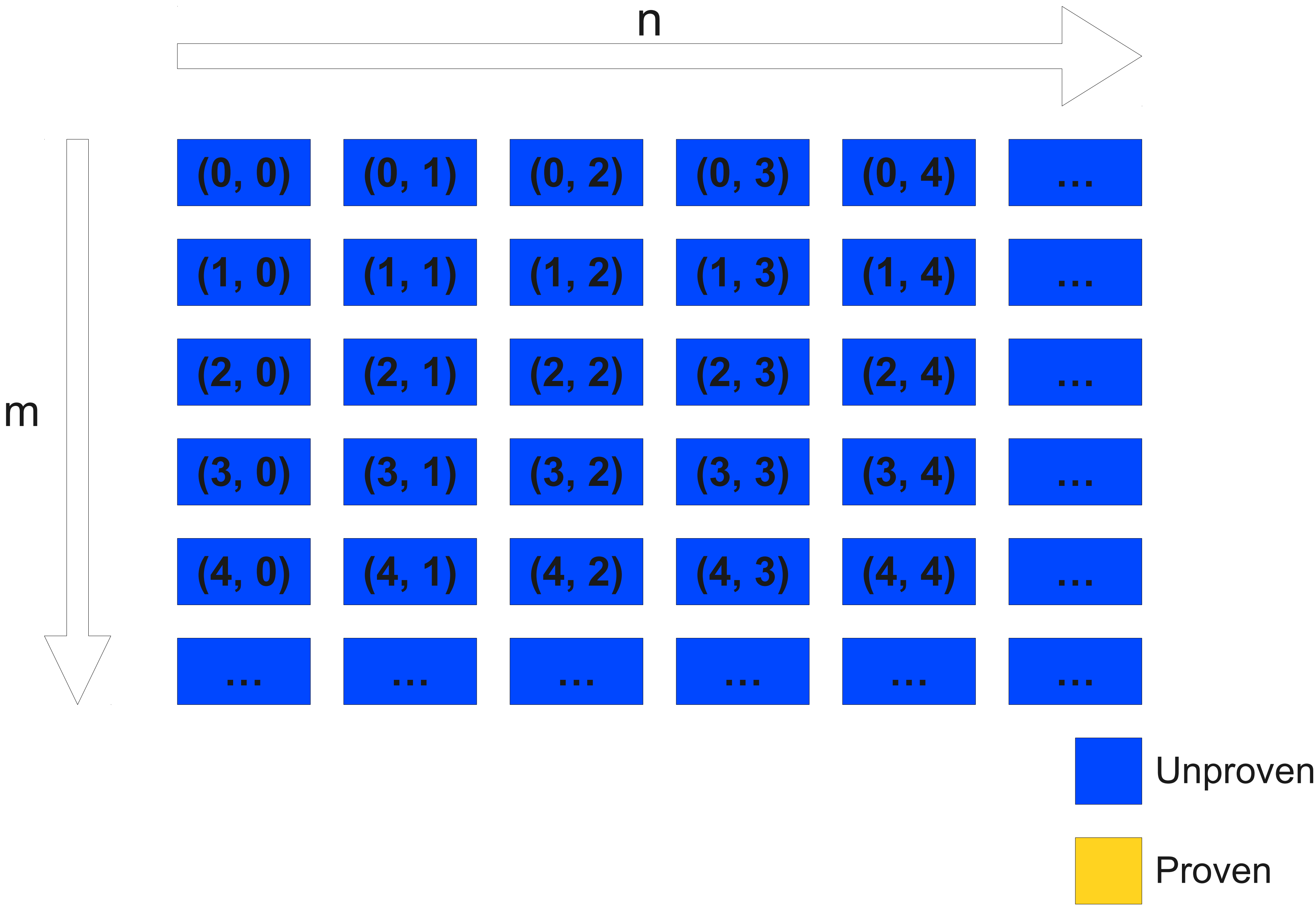

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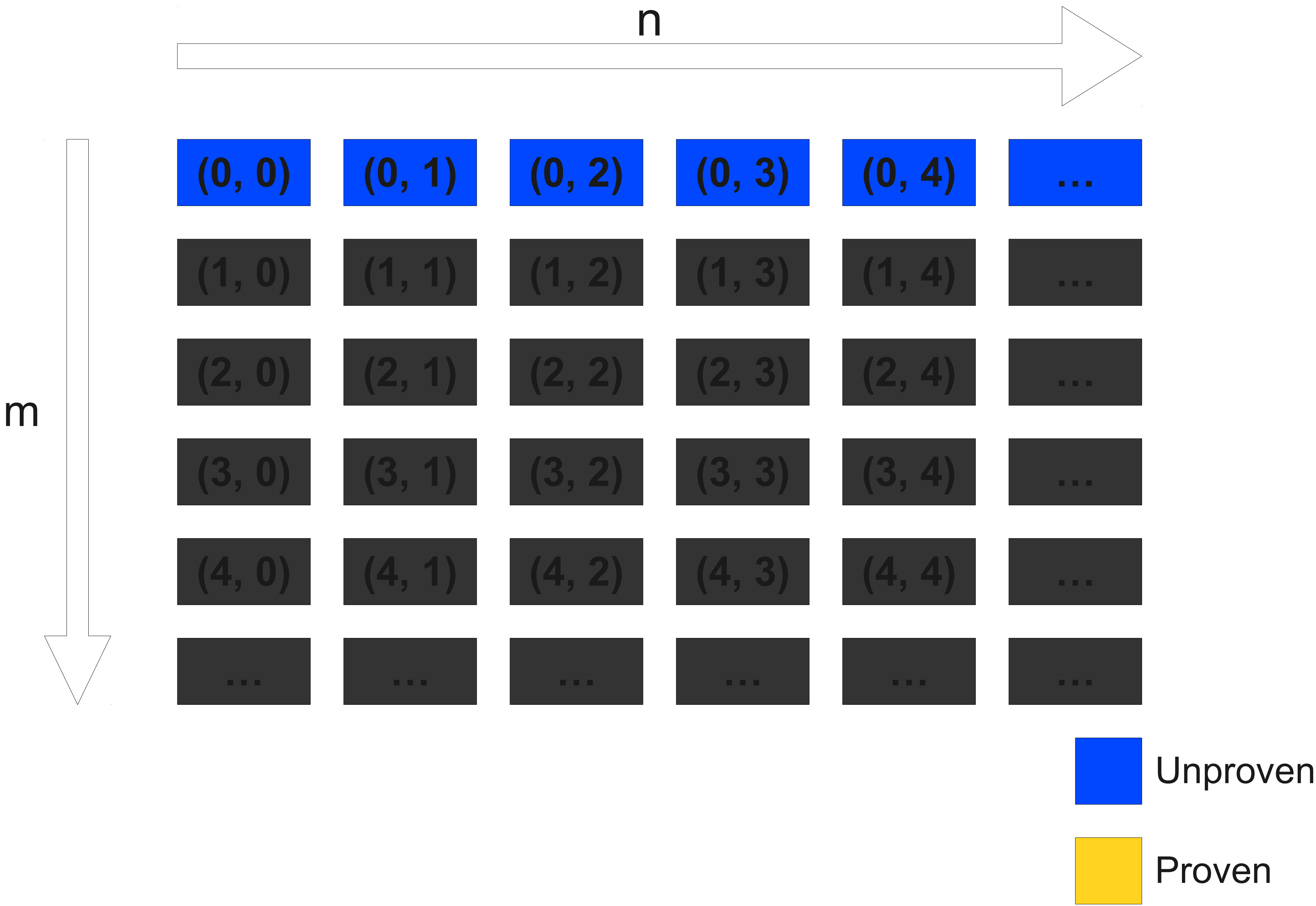

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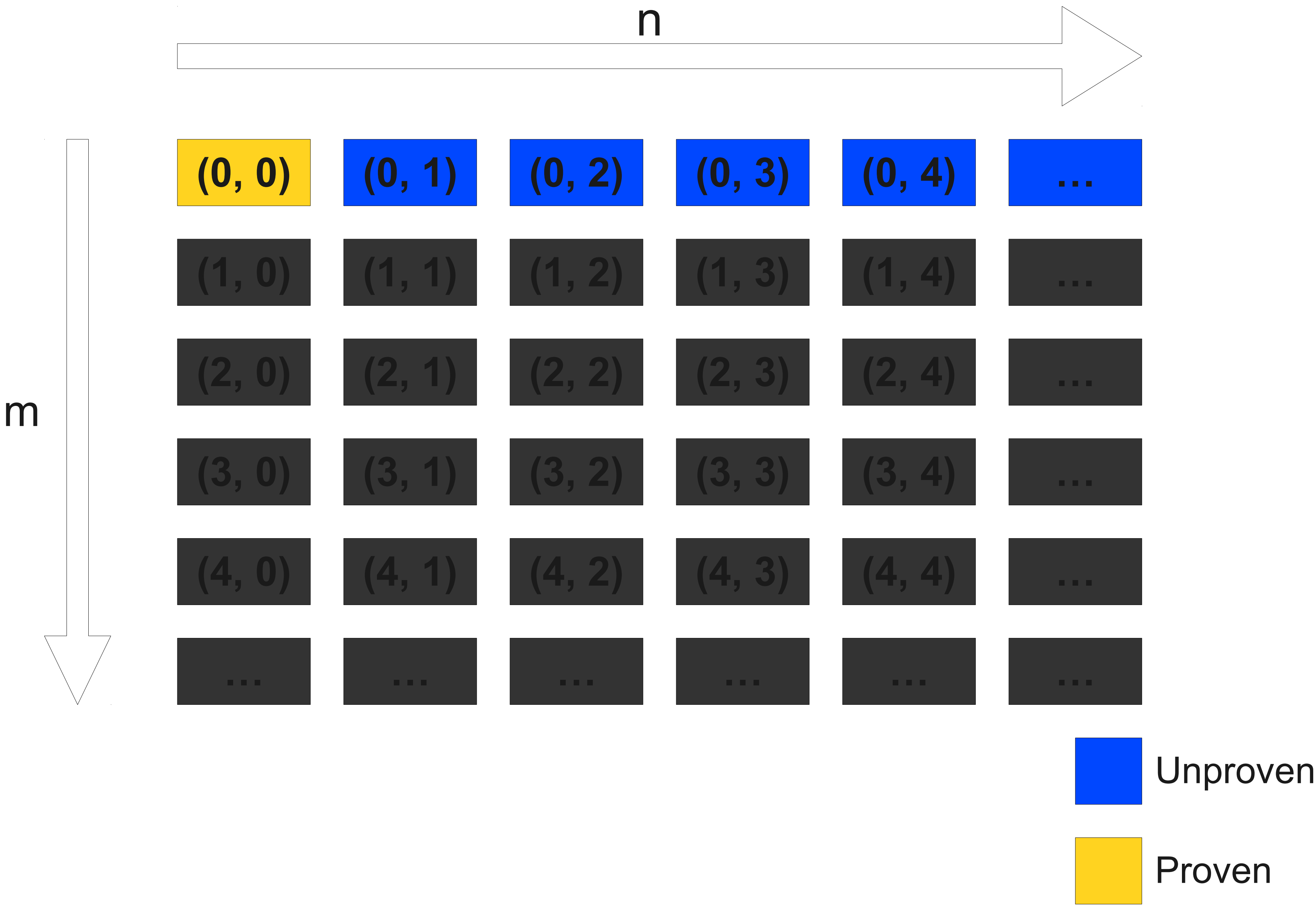

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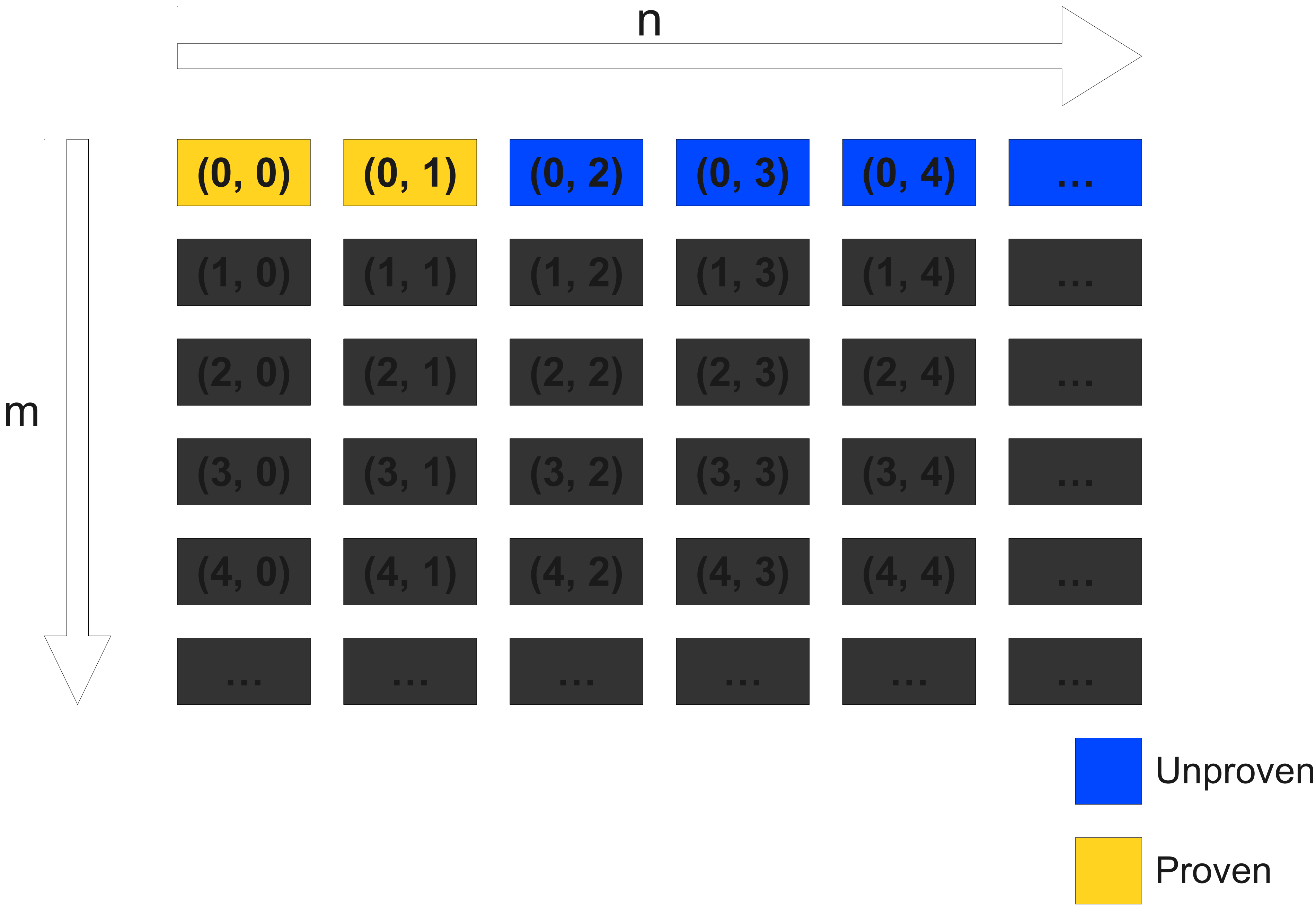

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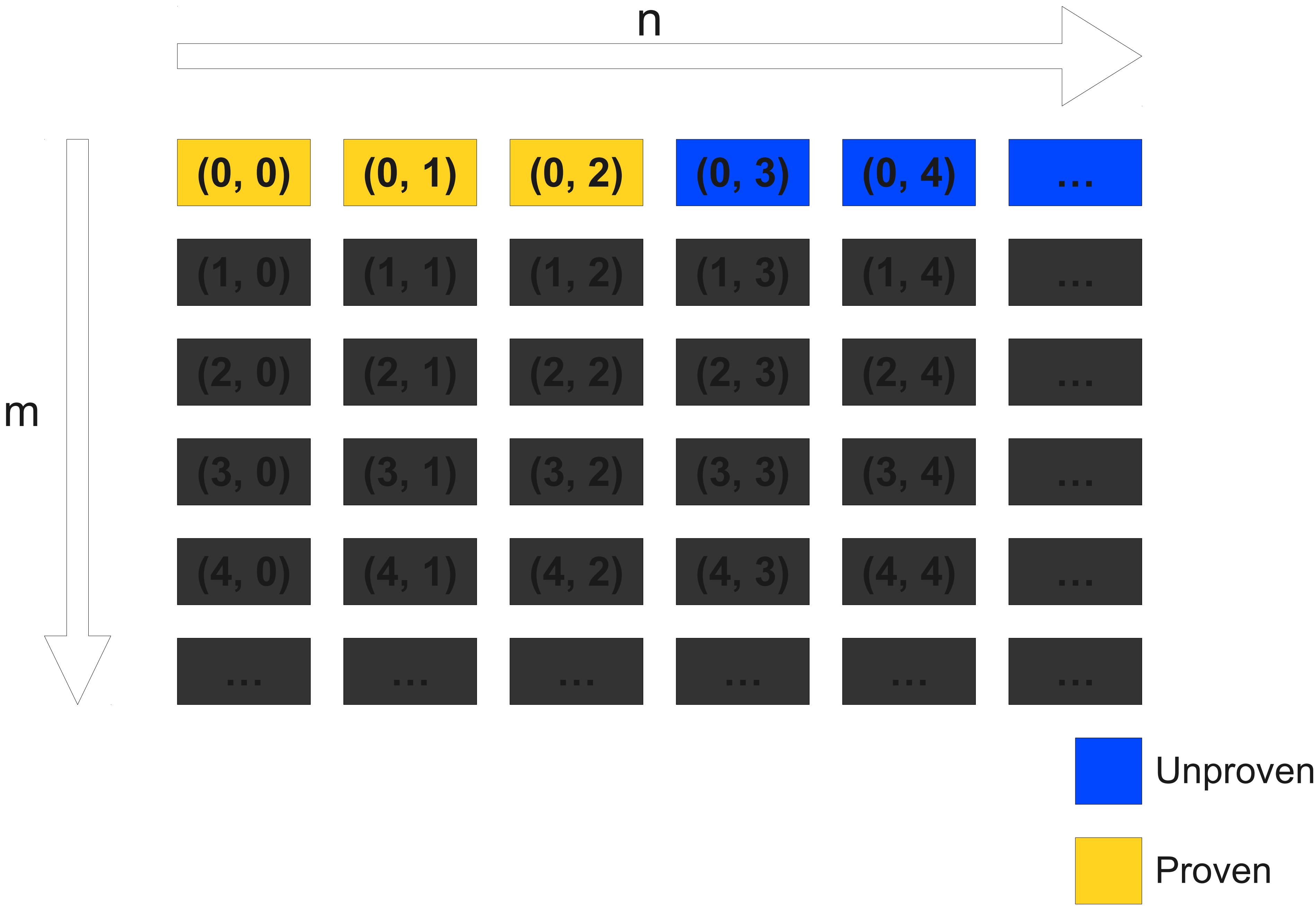

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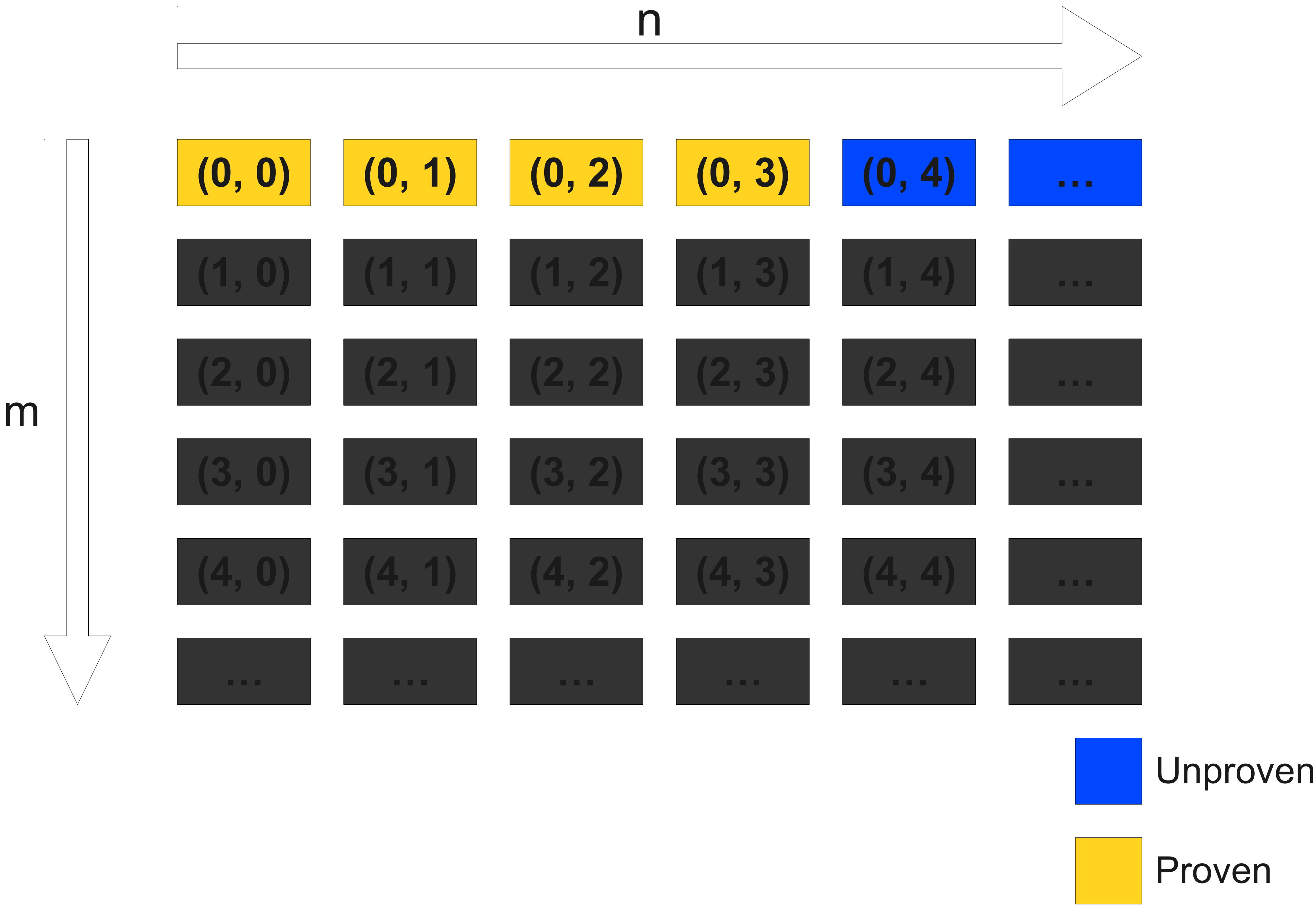

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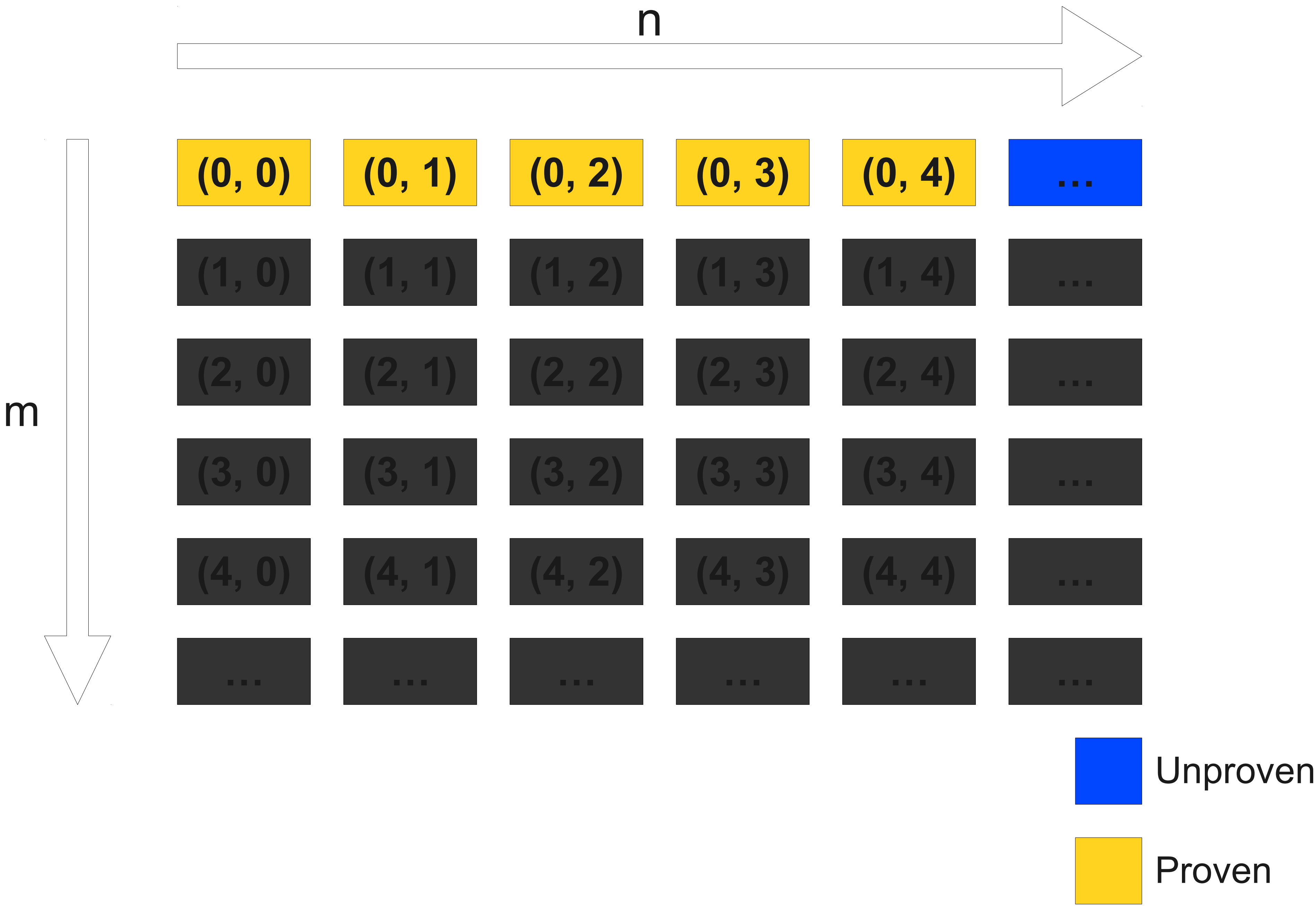

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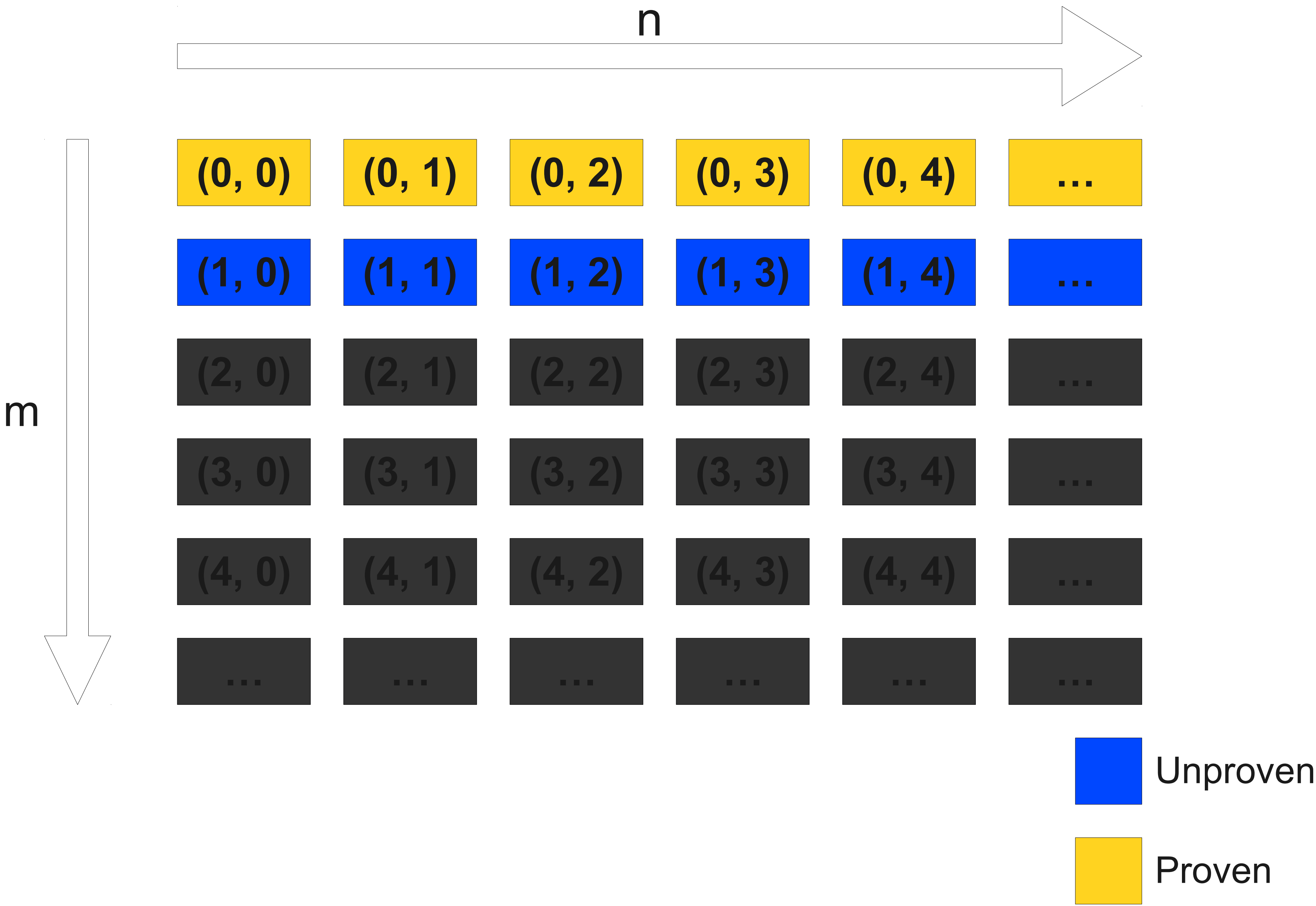

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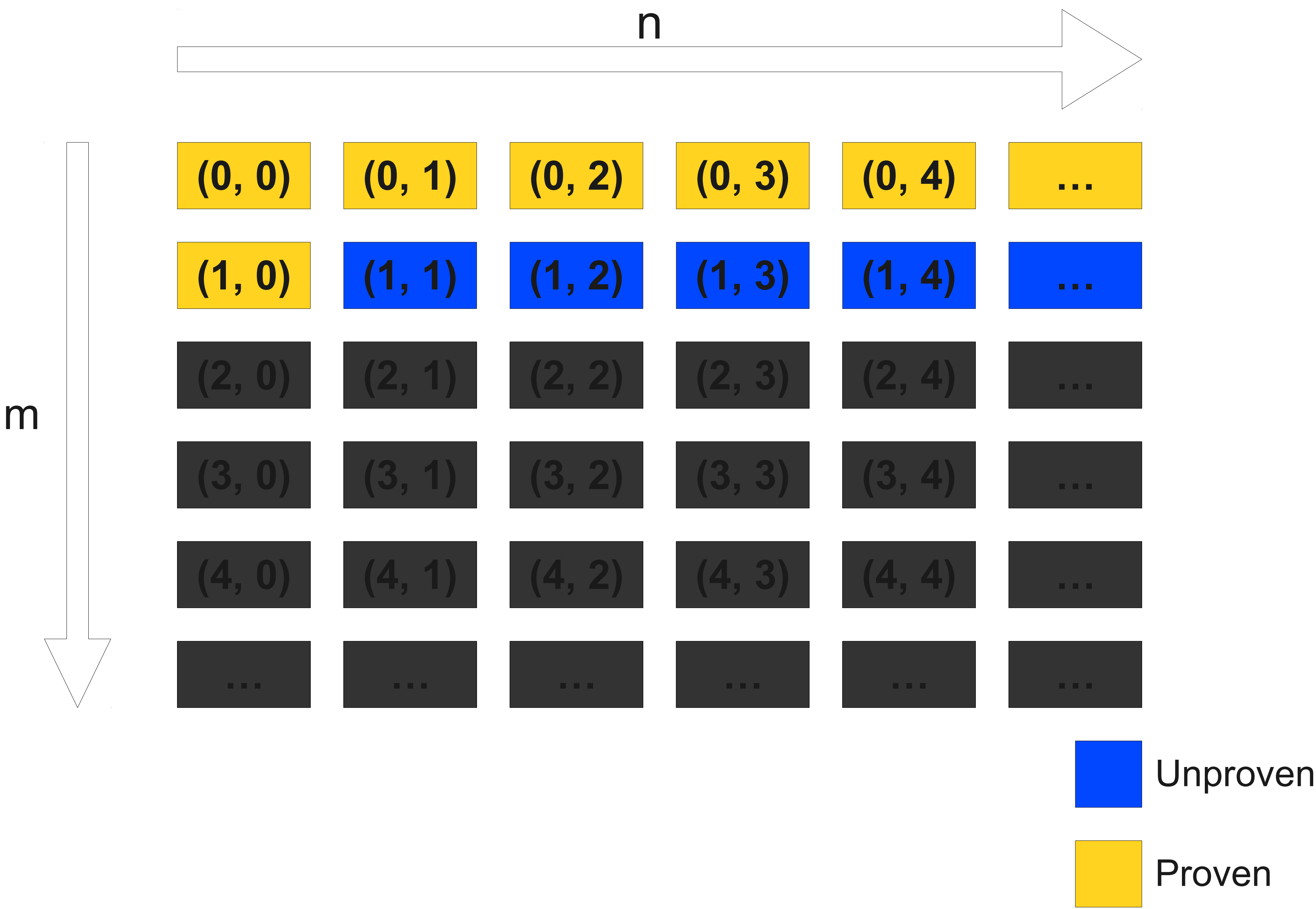

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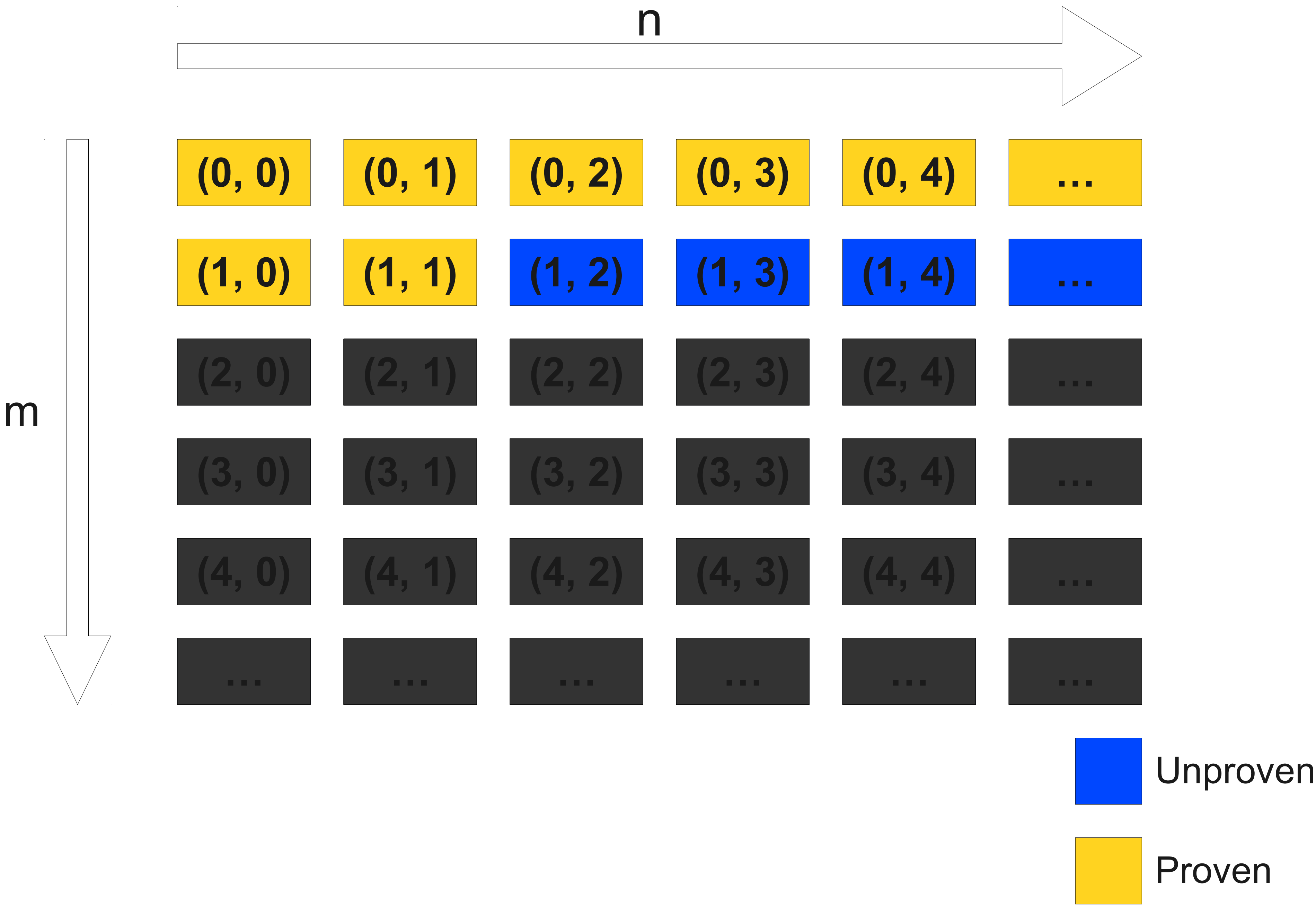

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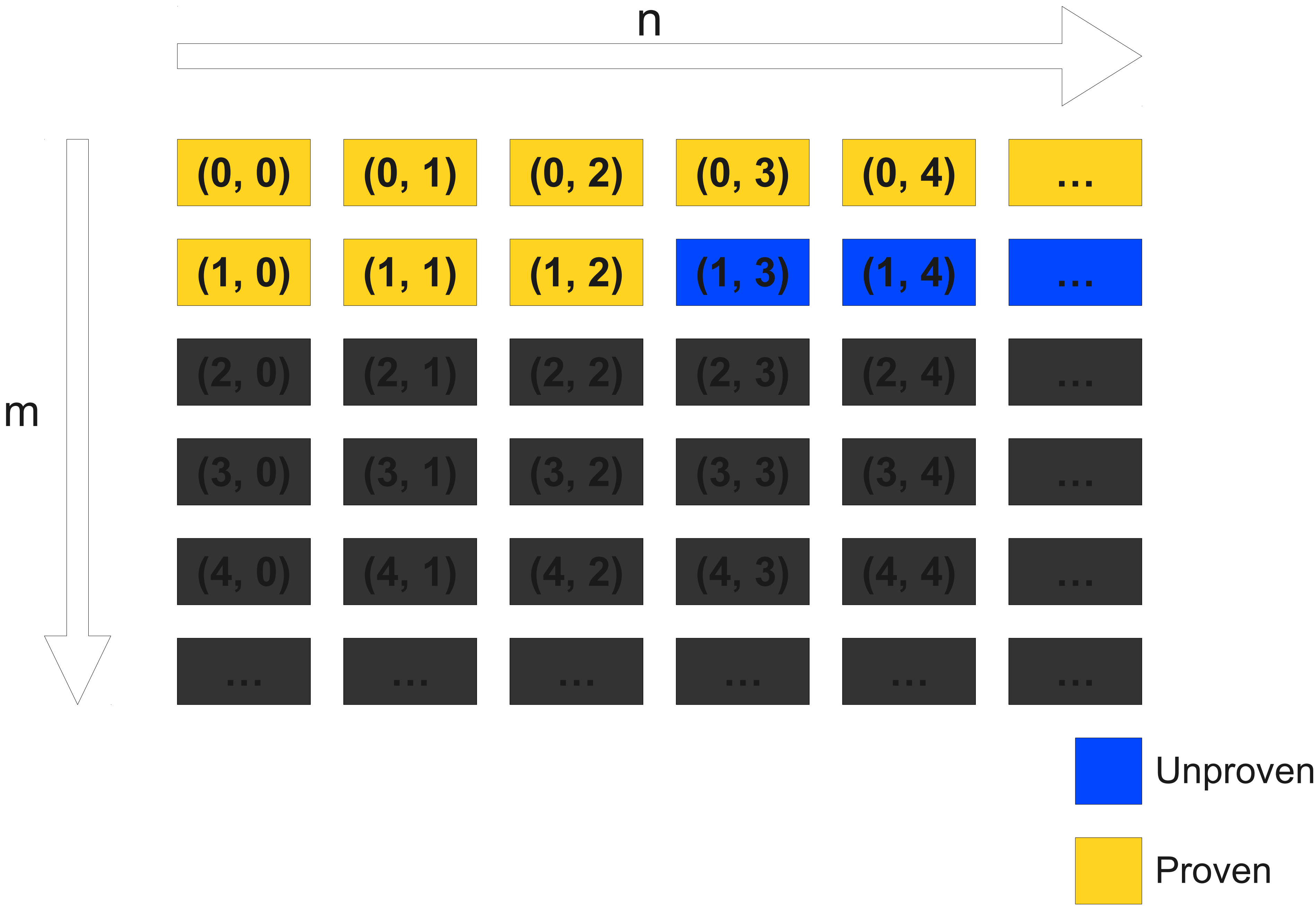

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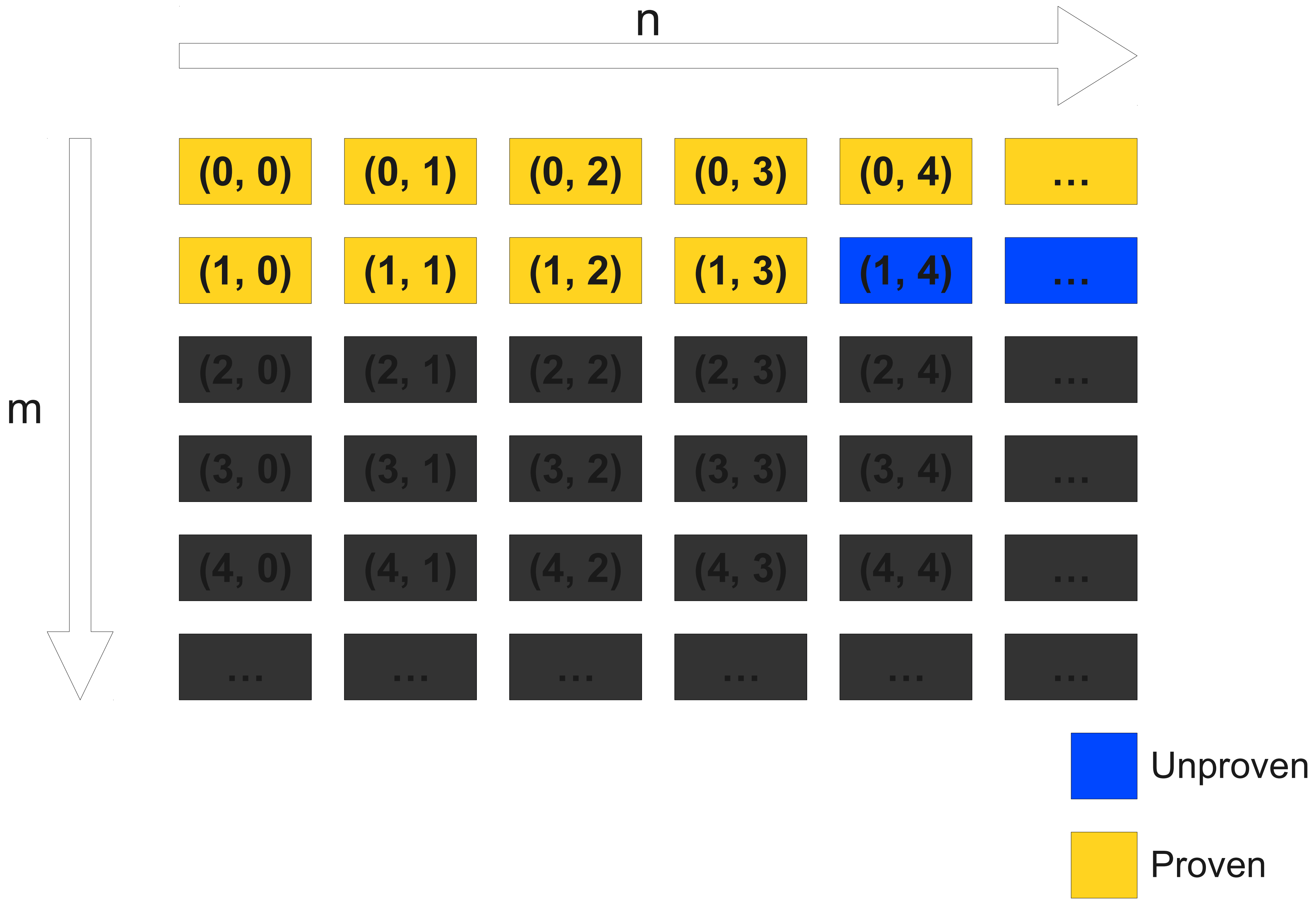

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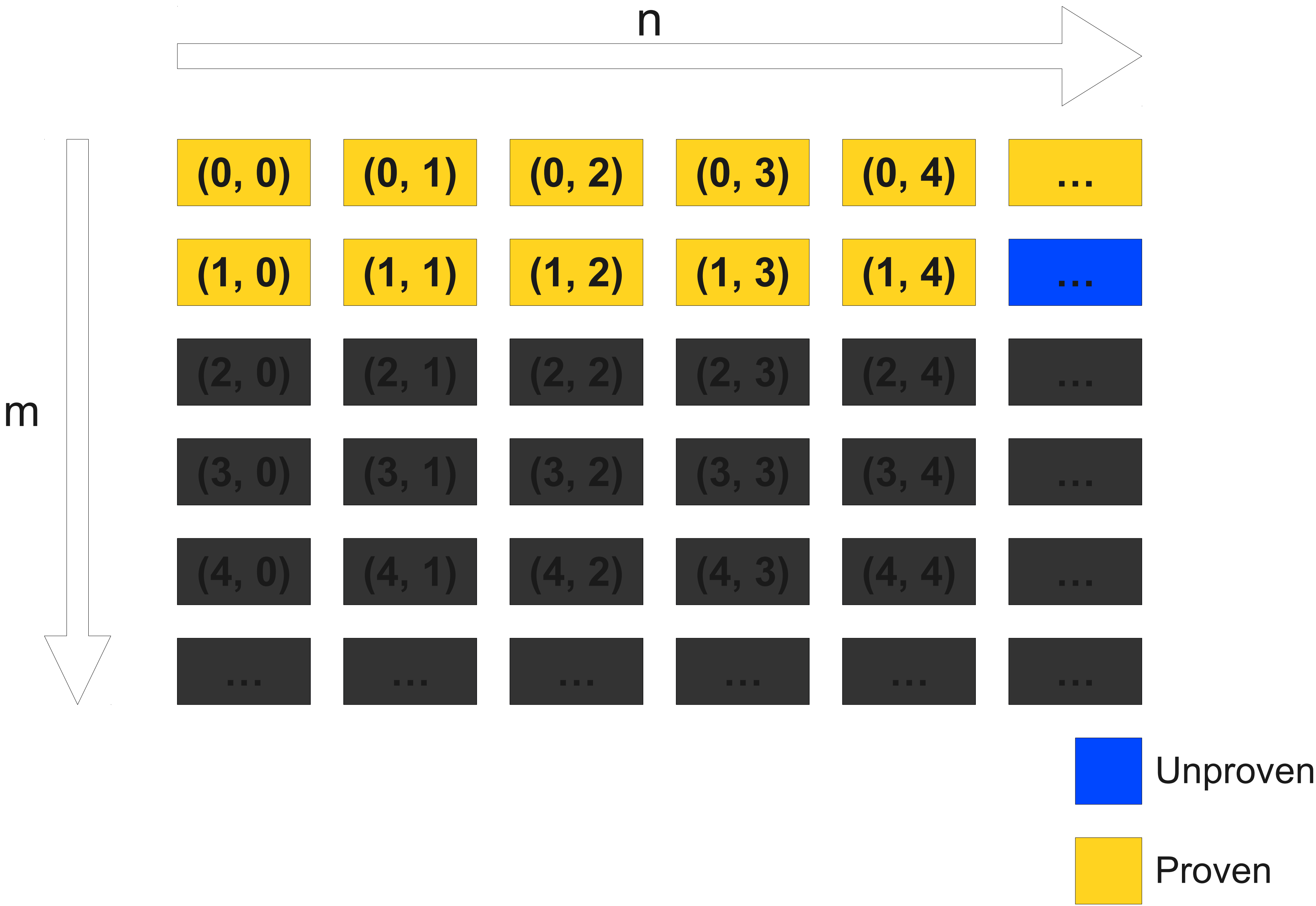

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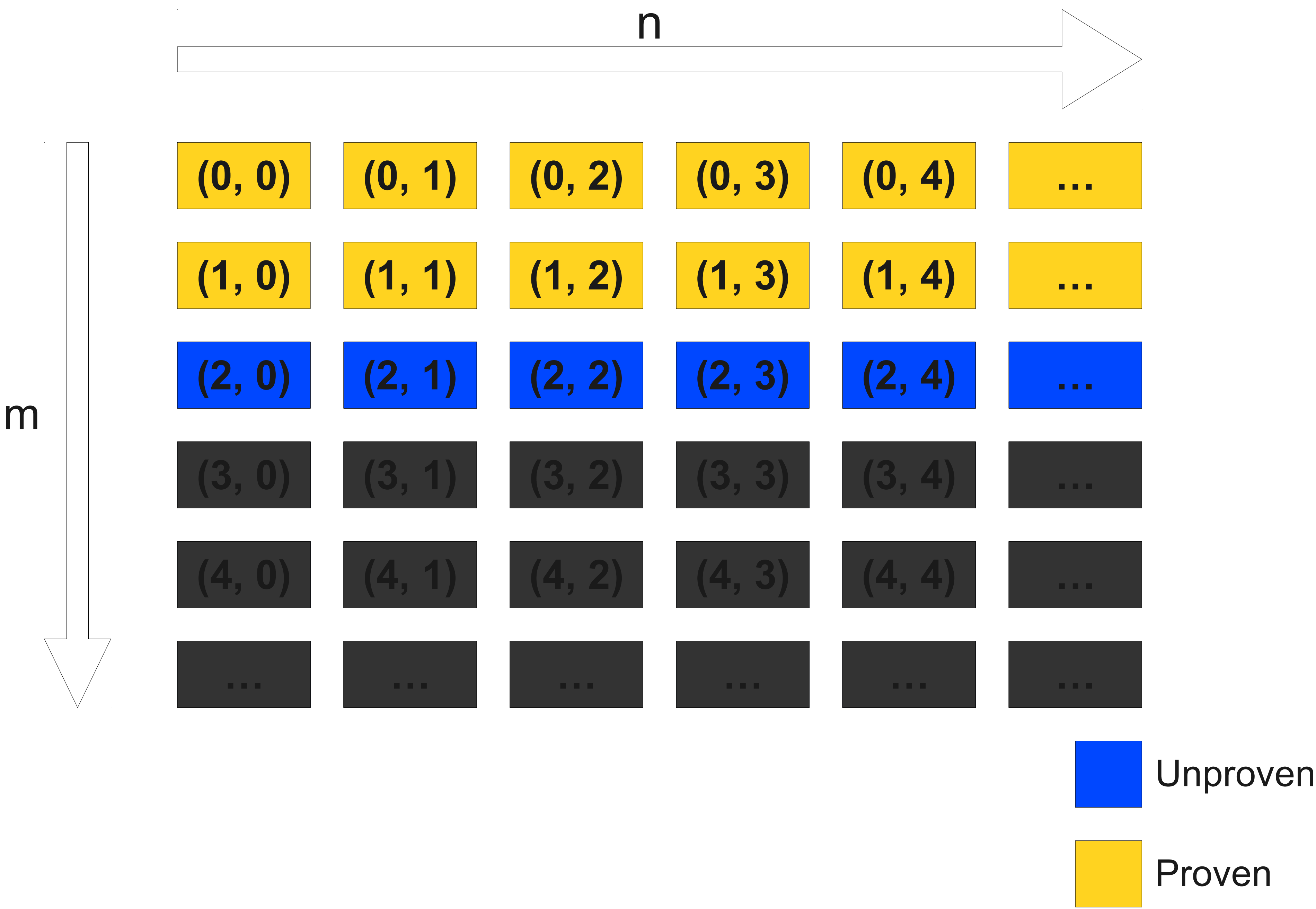

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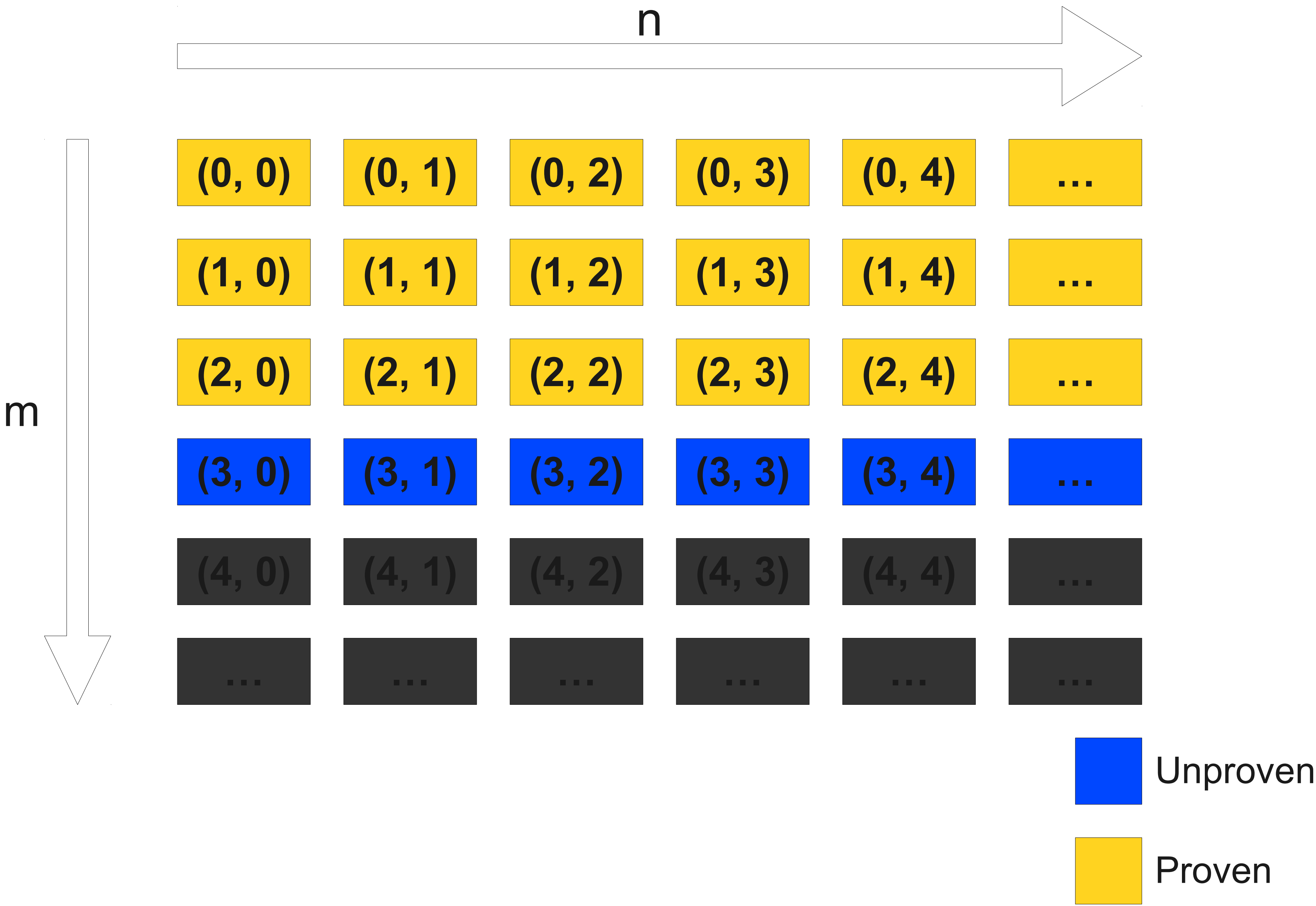

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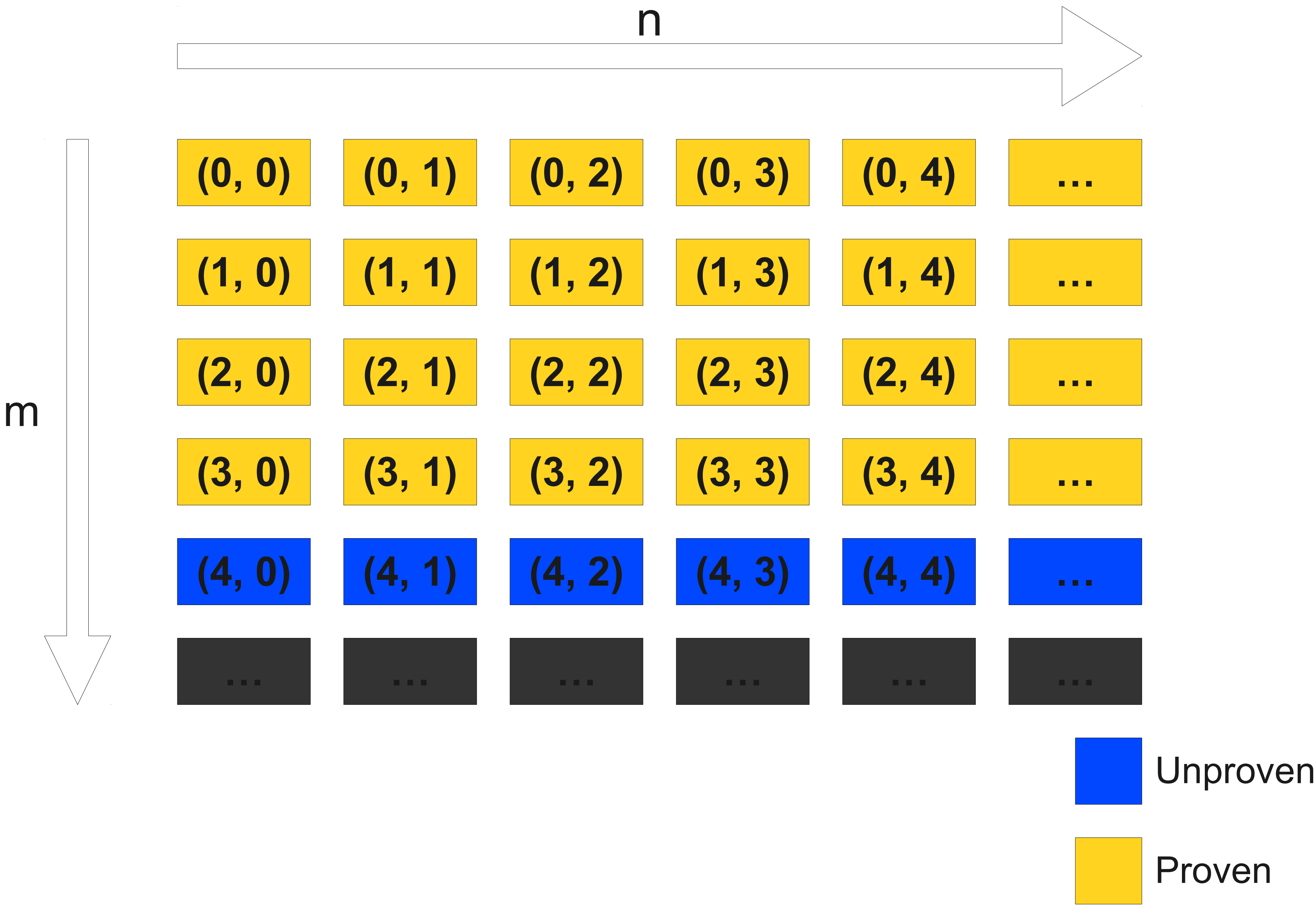

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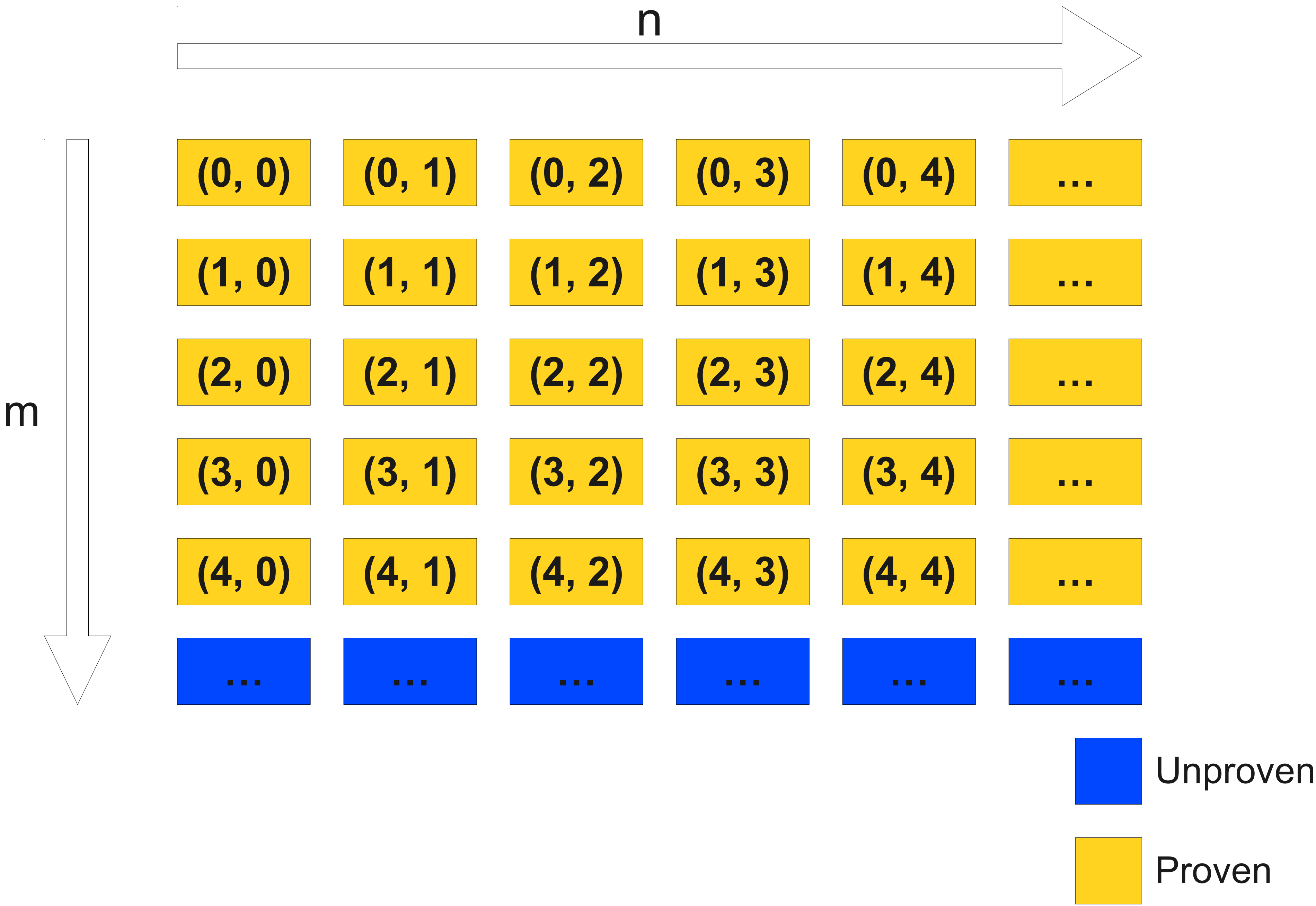

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Notice that P(m) claims something is true for all natural numbers n. In order to show this, we're going to need to use another induction later on.

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We're already doing an induction on m, and now we're about to start up a separate induction on n. This induction will prove that A(m + 1, n) is always defined, which will help prove that P(m + 1) holds.

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Note that this property references m + 1 from the induction step. If Q(n) is is true for all natural numbers n, we've just proven P(m+1).

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We've given this property a different name to distinguish it. Although you don't always have to be this explicit, it definitely helps when doing a double induction!

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This is one of the key steps of a double induction. We use the inductive hypothesis from P in order to prove something about Q. This is perfectly fine; we already assumed P(m) was true.

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Note how we use both the inductive hypotheses here!

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For the inductive step, assume that for some n, Q(n) holds and that A(m + 1, n) is defined. We need to prove that Q(n + 1) holds, that is, that A(m + 1, n + 1) is defined as well. By definition, A(m + 1, n + 1) = A(m, A(m + 1, n)). By the inductive hypothesis for Q(n), A(m + 1, n) is defined. By the inductive hypothesis for P(m), A(m, A(m + 1, n)) is defined because A(m + 1, n) is defined. Thus Q(n + 1) holds, so by induction Q(n) holds for all natural numbers.

Thus P(m + 1) holds, so by induction P(m) holds for all natural numbers.

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)) & \text{otherwise} \end{cases}$$

Theorem: A(m, n) is defined for any m, n.
Proof: By induction. Let P(m) be "A(m, n) is a defined for all natural numbers n." We prove P(m) holds for all natural numbers by induction. As our base case, to show that P(0) holds, we need to show that A(0, n) is defined for all natural numbers n. But A(0, n) is defined to be n + 1 for any choice of n, so P(0) is true.

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- A(5, 5) is too large to represent in scientific notation.