Indirect Proofs

Announcements

- Review session tonight in **370-370** from 7–8PM
  - **Note the room change!**
- Problem Set 1 out, due Friday, October 7 at 2:15PM (right before class).
  - **Start early!**
  - Later problems are harder than earlier ones; don't extrapolate from the first few problems.
  - Email us at **cs103@cs.stanford.edu** or stop by OH with questions.

Outline for Today

- Logical Implication
  - What does “If P, then Q” mean?
- Proof by Contradiction
  - The basic method.
  - Contradictions and implication.
  - Contradictions and quantifiers.
- Proof by Contrapositive
  - The basic method.
  - An interesting application.

Logical Implication

Implications

- An **implication** is a statement of the form “If P, then Q.”
- Equivalent formulations:
  - “If P, then Q.”
  - “P only if Q.”
  - “Q whenever P.”
  - “P implies Q”
- **Notation**: We write “If P, then Q” as **P** → **Q**.
  - Read: “P implies Q.”

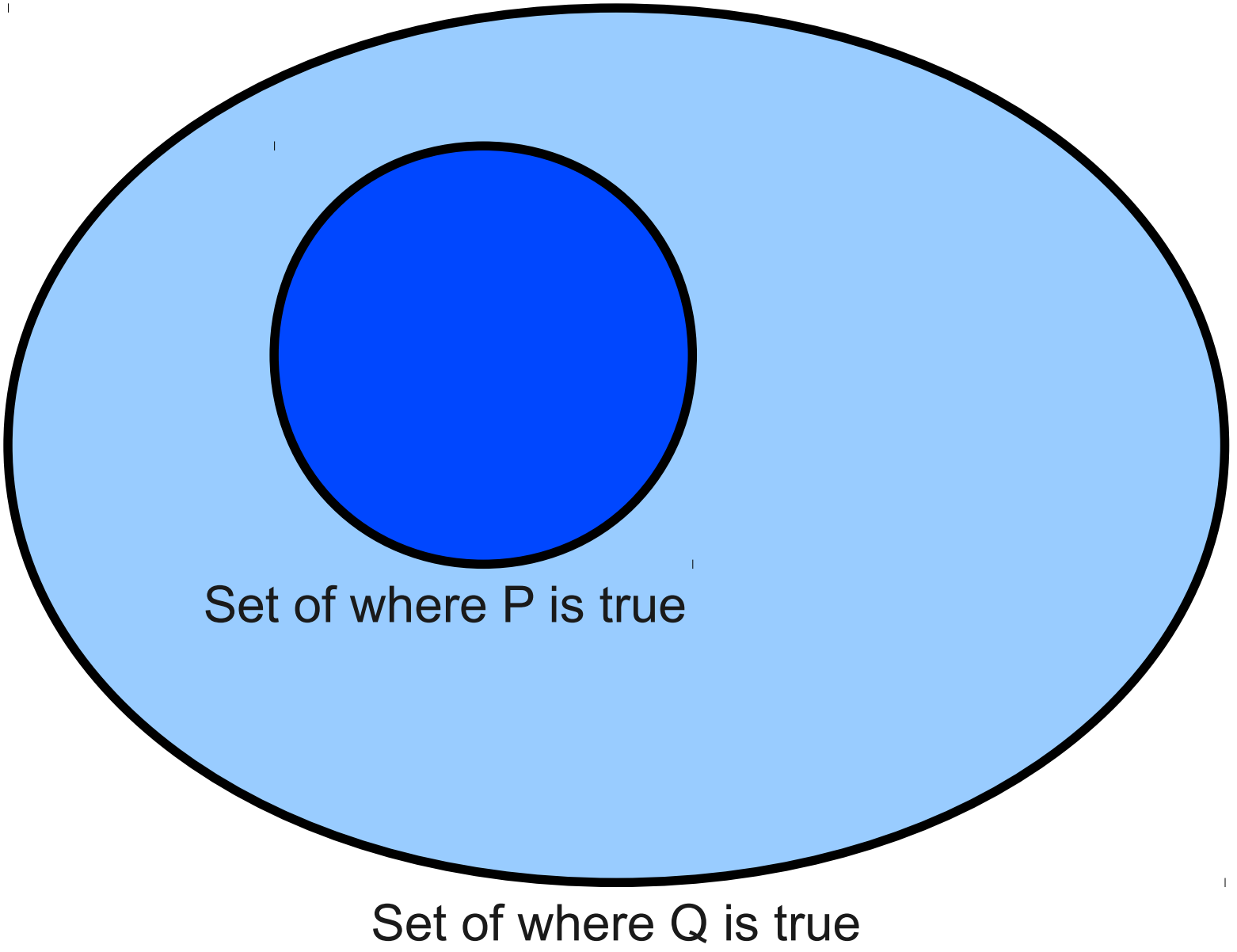
What does Implication Mean?

- What does it mean for the statement $P \to Q$ to be true?
- In any case where P is true, Q must be true as well.
- For example:
  - n is even $\to$ n^2 is even.
  - $x \in S \to x \in T$.
  - $xRy \to yRx$.
- If $P \to Q$, we call P the **antecedent** and Q the **consequent**.

What does Implication **Not** Mean?

- P → Q does **not** mean that whenever Q is true, P is true.
  - “If you are a Stanford student, you wear cardinal” does **not** mean that if you wear cardinal, you are a Stanford student.
  - (That would overcrowd the dorms)
- P → Q does **not** say anything about what happens if P is false.
  - “If you are a wide receiver, you catch the football” does not mean that if you're not a wide receiver, you don't catch the football.
  - (You might be Andrew Luck)
  - **Vacuous truth:** If P is never true, then P → Q is automatically true.
- P → Q does **not** say anything about causality.
  - “If I will it to be true, 2 + 2 = 4” is true because any time that I want it to be true, 2 + 2 = 4 already was true.
  - “If I will it to be false, 2 + 2 = 4” is also (vacuously) true.

Implication, Diagrammatically



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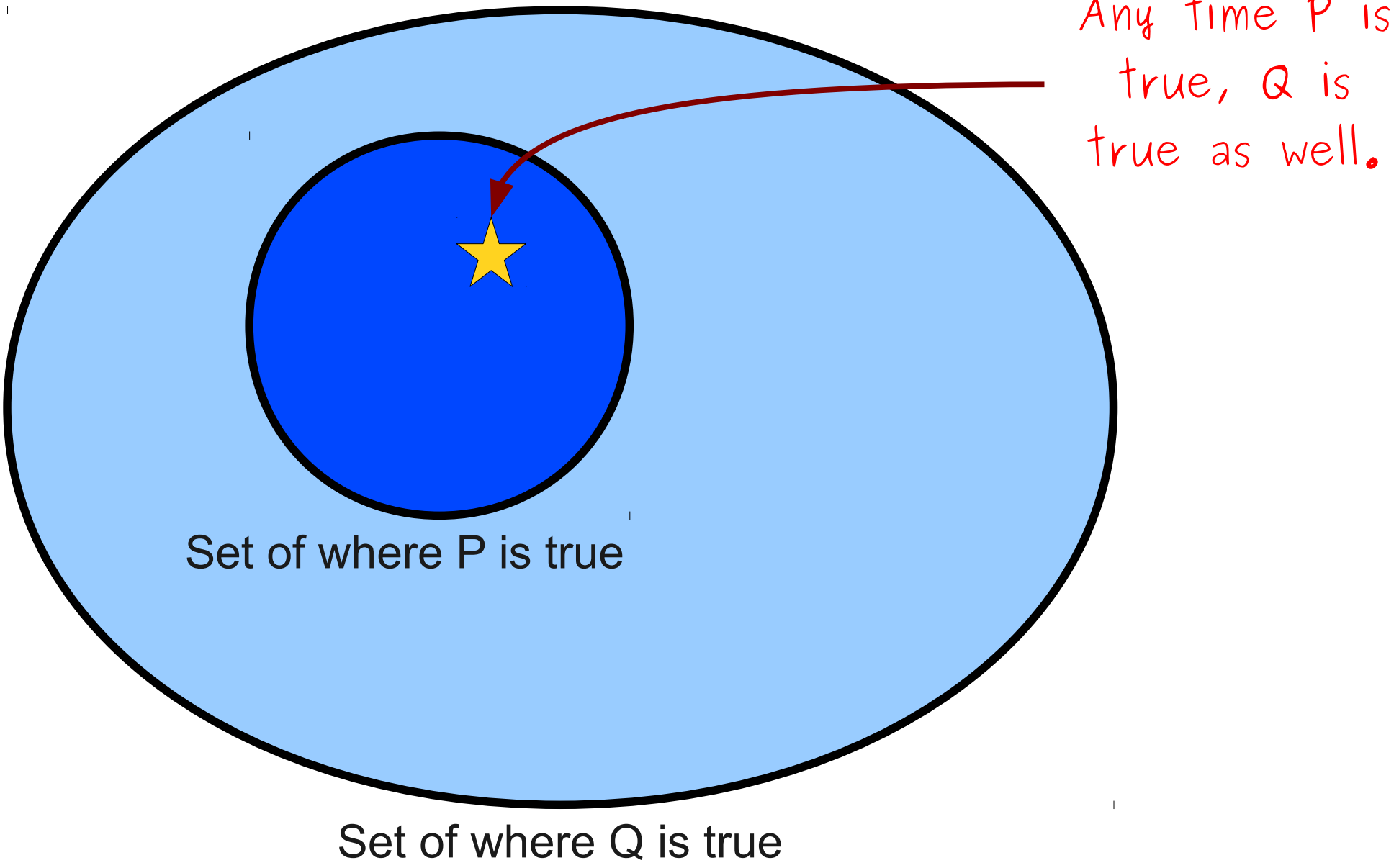

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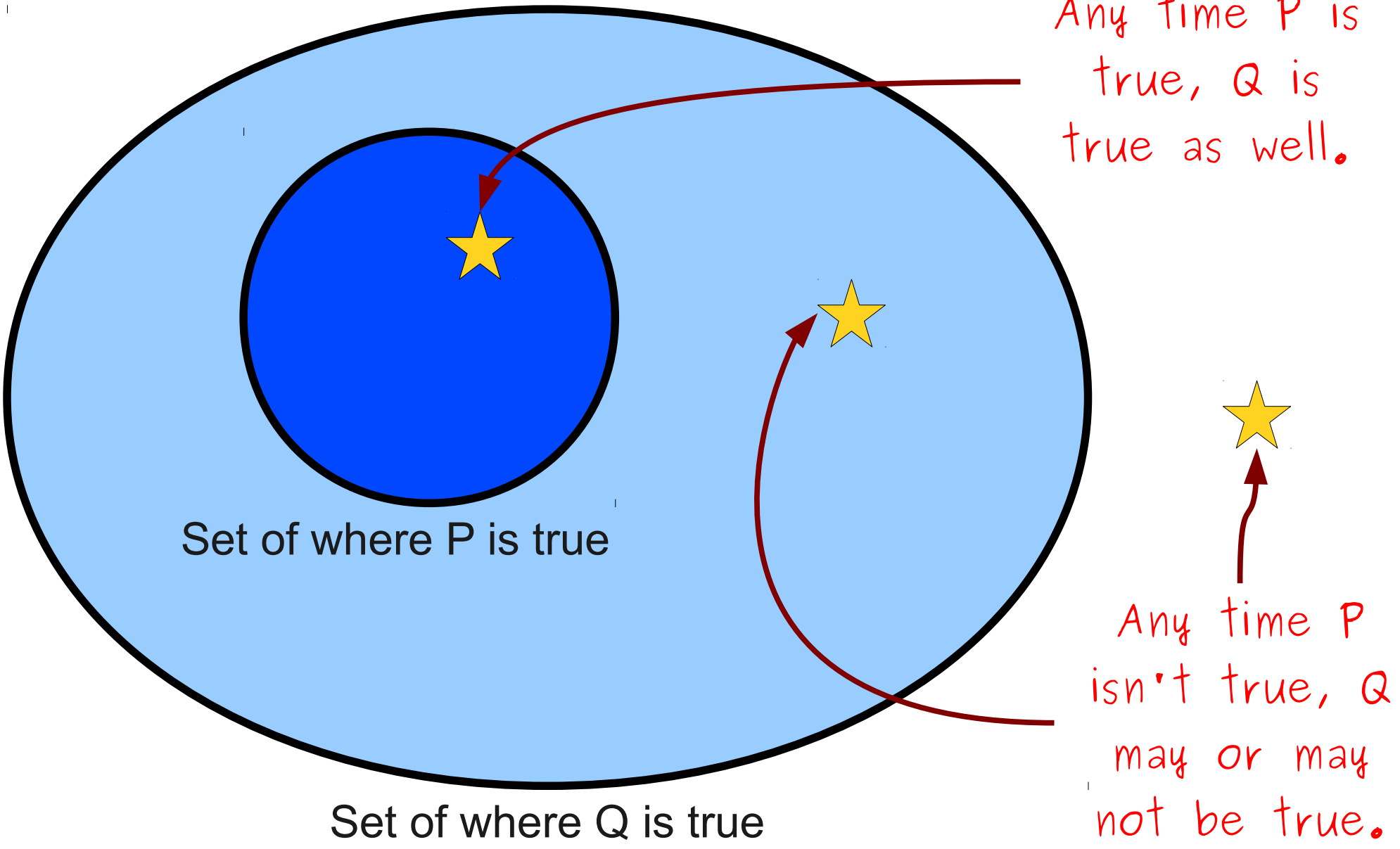

When P does not imply Q

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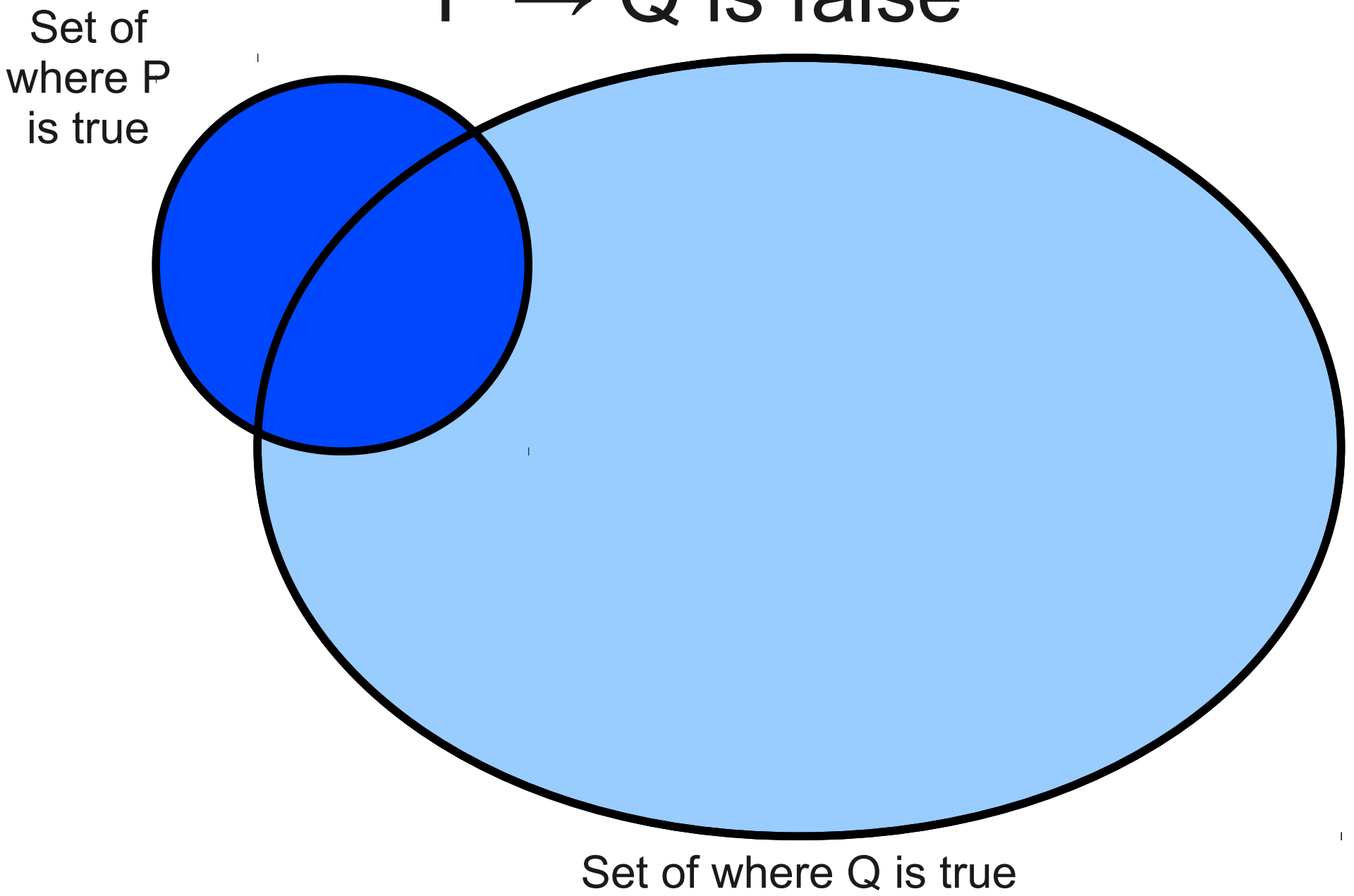
- Why?

When P does not imply Q

- What would it mean for P → Q to be false?
- **Answer**: There must be some way for P to be true and Q to be false.
- Why?
  - P → Q means “any time P is true, Q is true.”
  - The only way to disprove this is to show that there is some way for P to be true and Q to be false.
- To prove that P → Q is false, find an example of where P is true and Q is false.

$P \rightarrow Q$ is false



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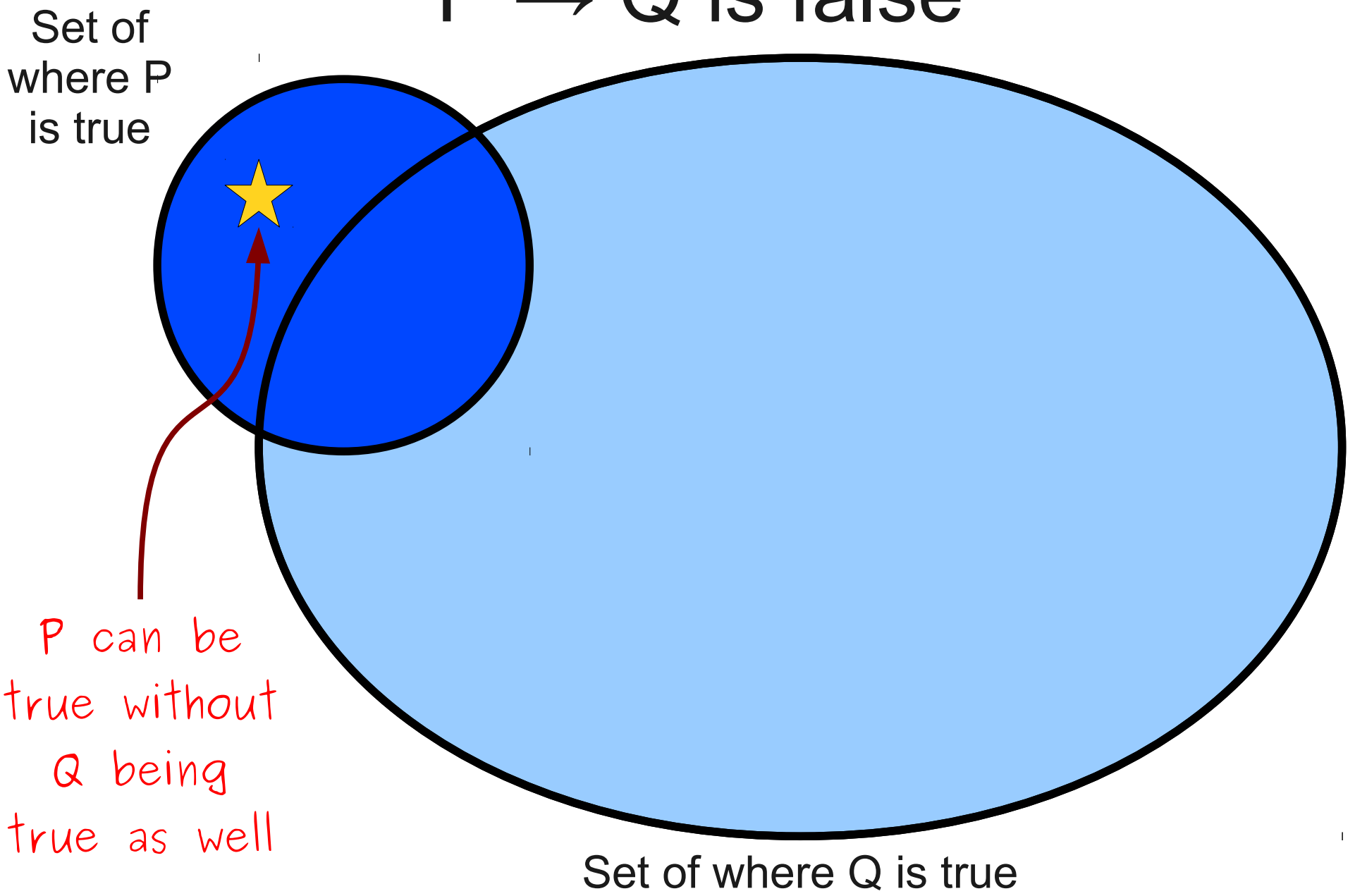

A Common Mistake

- To show that $P \rightarrow Q$ is false, it is **not** sufficient to find a case where P is false and Q is false.

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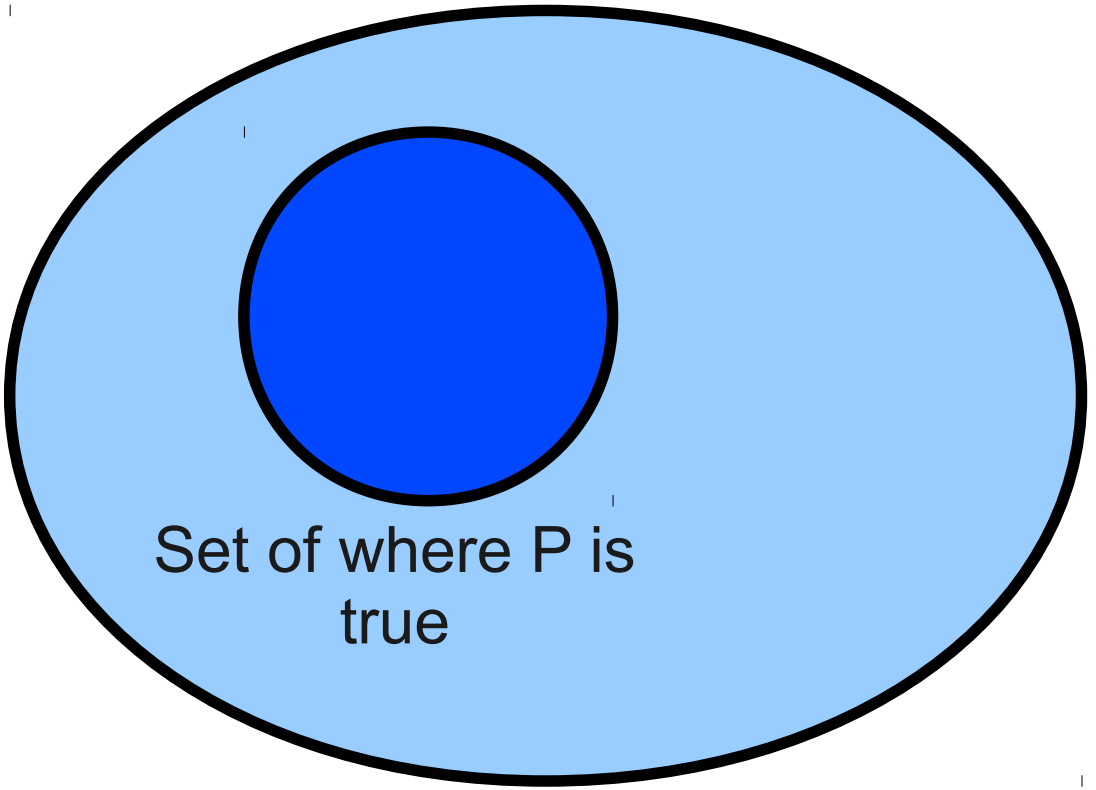

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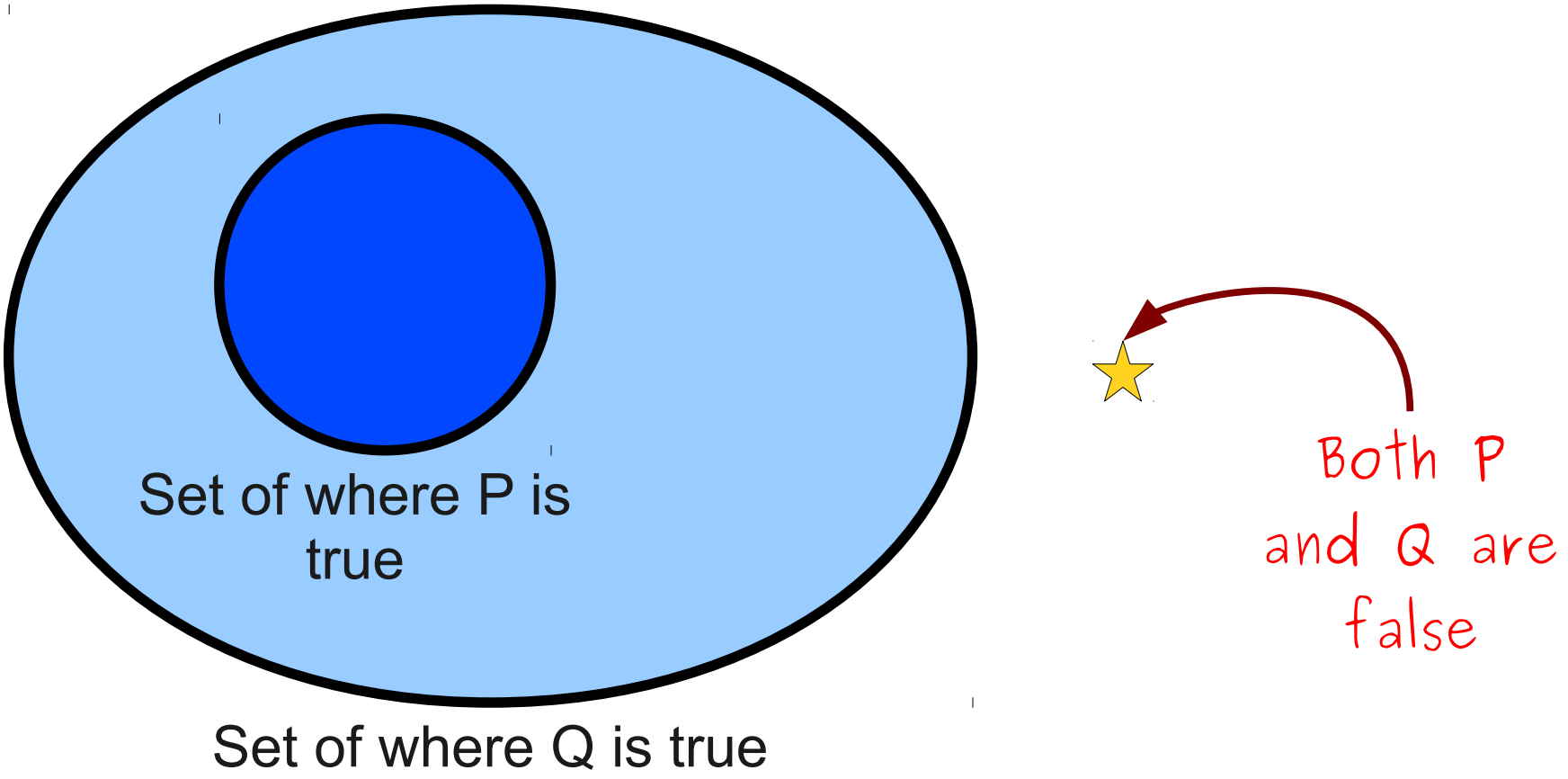

Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A **proof by contradiction** is a proof that works as follows:
  - To prove that P is true, assume that P is not true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only option is that the assumption that P is not true is incorrect.
  - Conclude, therefore, that P is true.

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that $P \to Q$ is false.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

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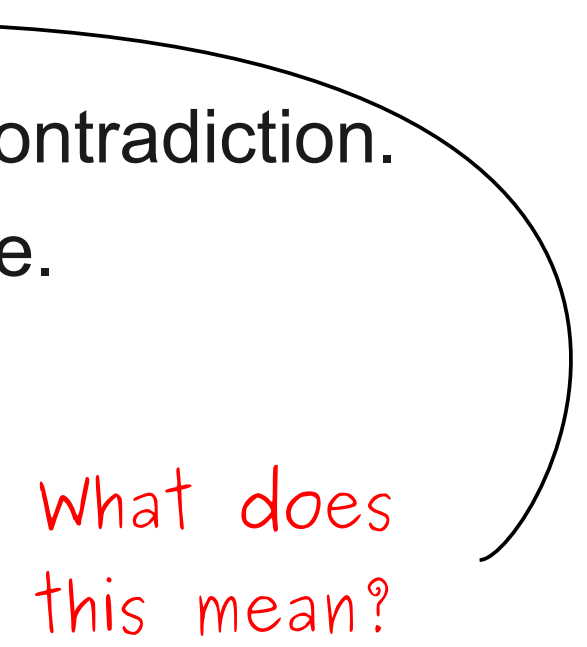

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that **P is true and Q is false**.
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A Simple Proof by Contradiction

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Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

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Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is clearly impossible, since n^2 is even.

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Last time, I accidentally wrote "n^2 is even." That's incorrect; n^2 really is odd here.

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Biconditionals

- Combined with what we saw on Friday, we have proven
  - If n is even, n^2 is even.
  - If n^2 is even, n is even.
- We sometimes write this as

  n is even **if and only if** n^2 is even.
- This is often abbreviated

  n is even **iff** n^2 is even.

  or as

  n is even $\leftrightarrow$ n^2 is even
- This is called a **biconditional**.

$$P \leftrightarrow Q$$

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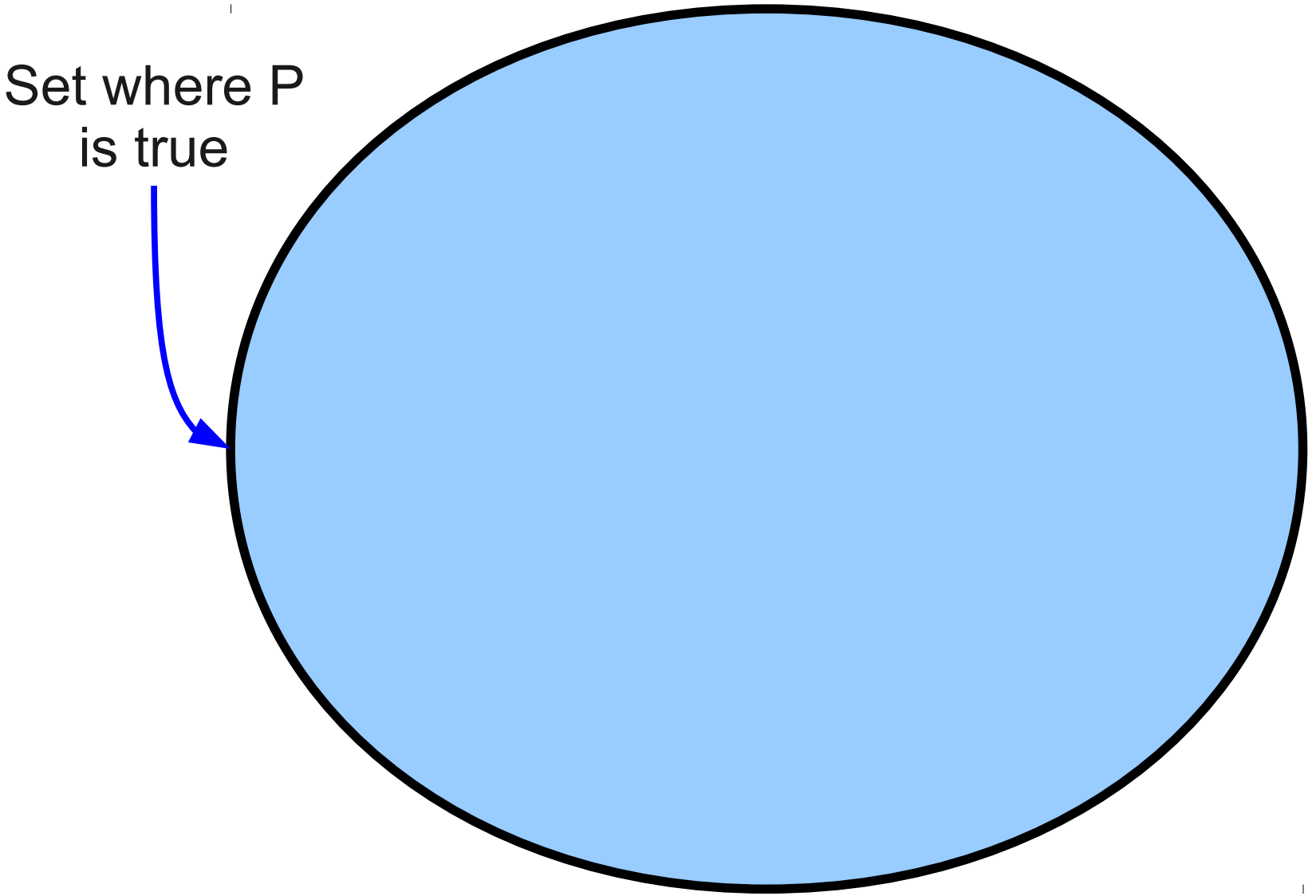

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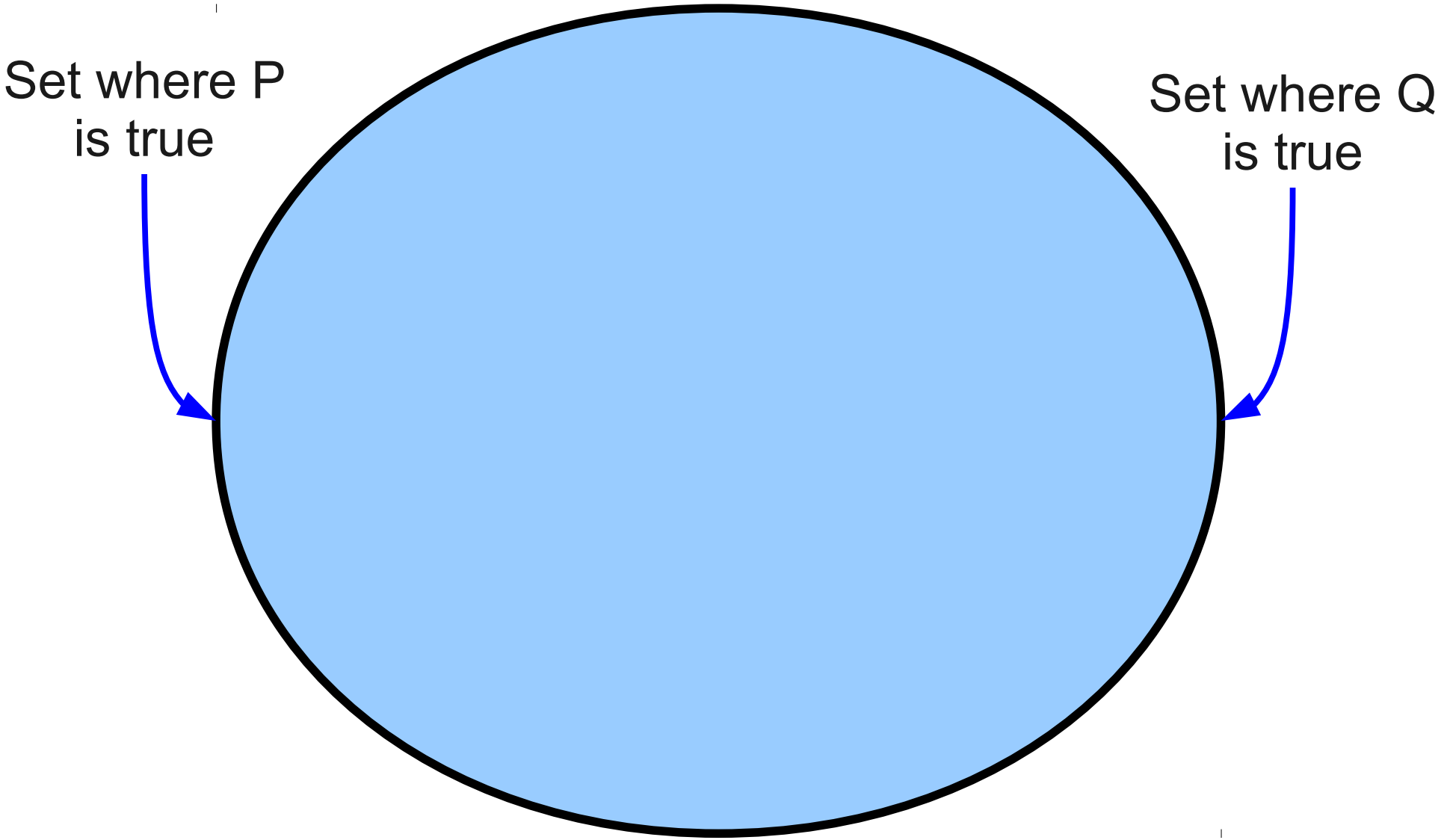

Proving Biconditionals

- To prove P iff Q, you need to prove that
  - **P** → **Q**, and
  - **Q** → **P**.
- You may use any proof techniques you'd like when doing so.
  - In our case, we used a direct proof and a proof by contradiction.
- **Make sure to prove both directions of implication**.

A Word of Warning

- To attempt a proof by contradiction, make sure that what you're assuming actually is the opposite of what you want to prove!
- Otherwise, your **entire proof is invalid**.

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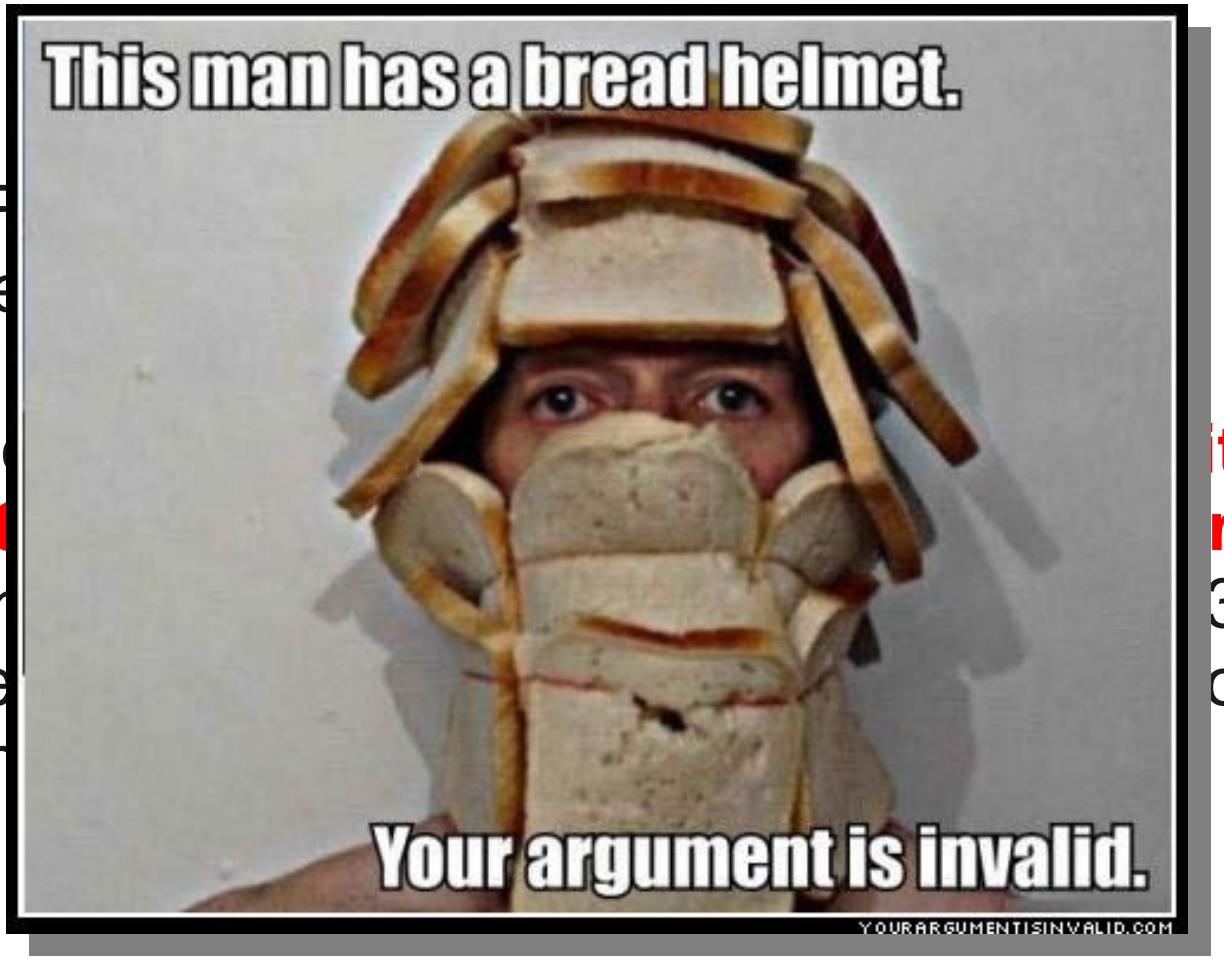

The contradiction of

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"for **some choice of x**, P(x) is false."

“All My Friends Are Taller Than Me”

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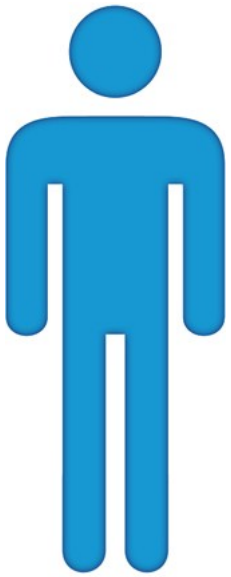

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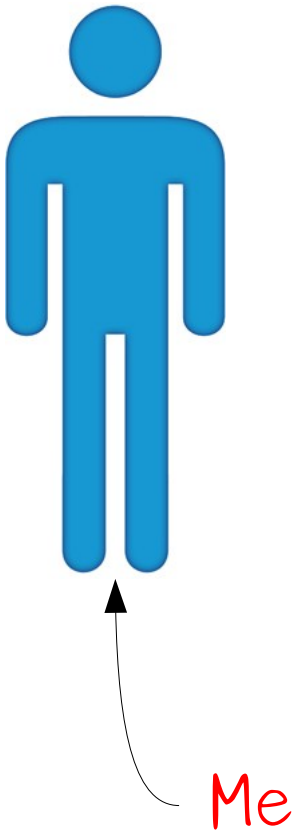

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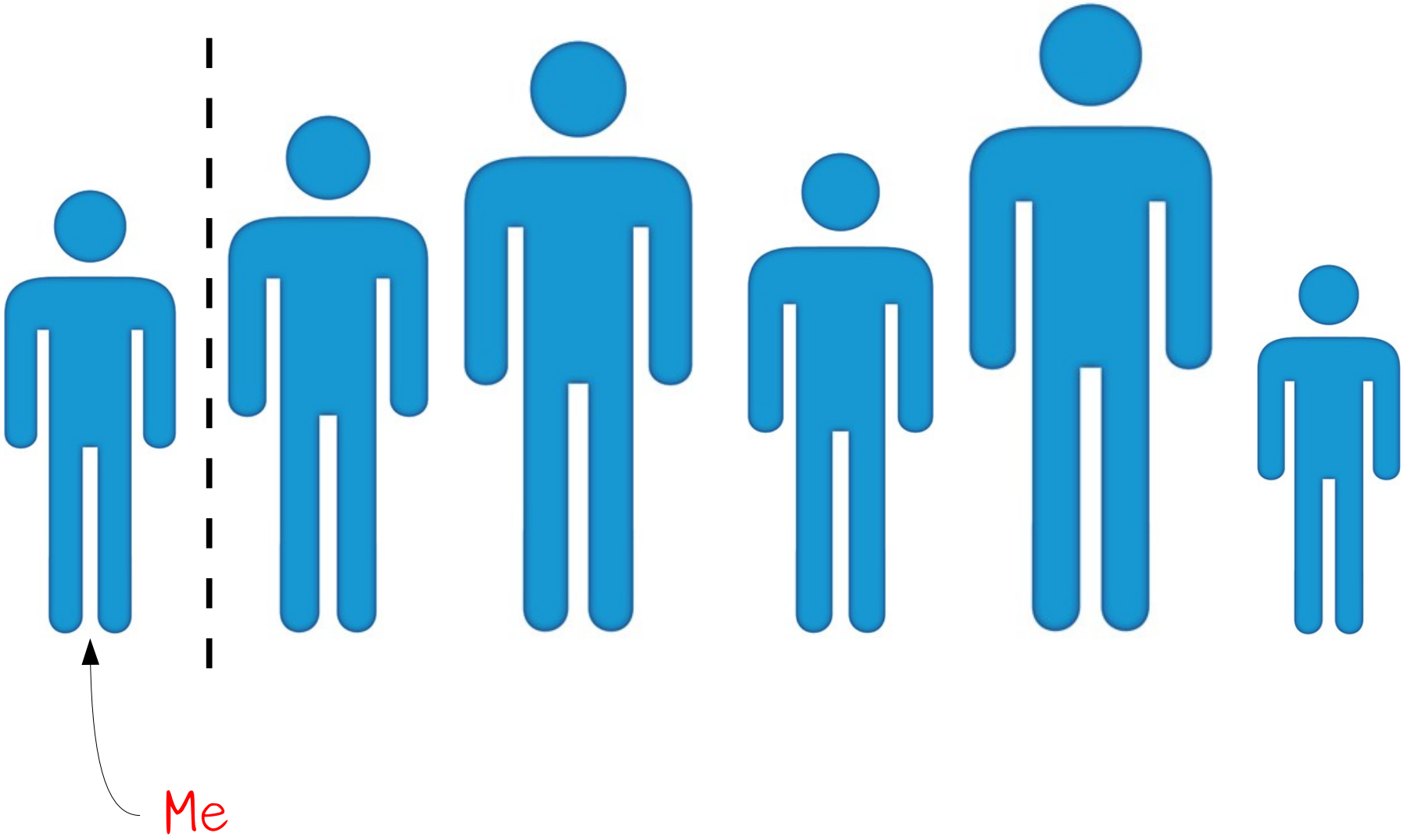

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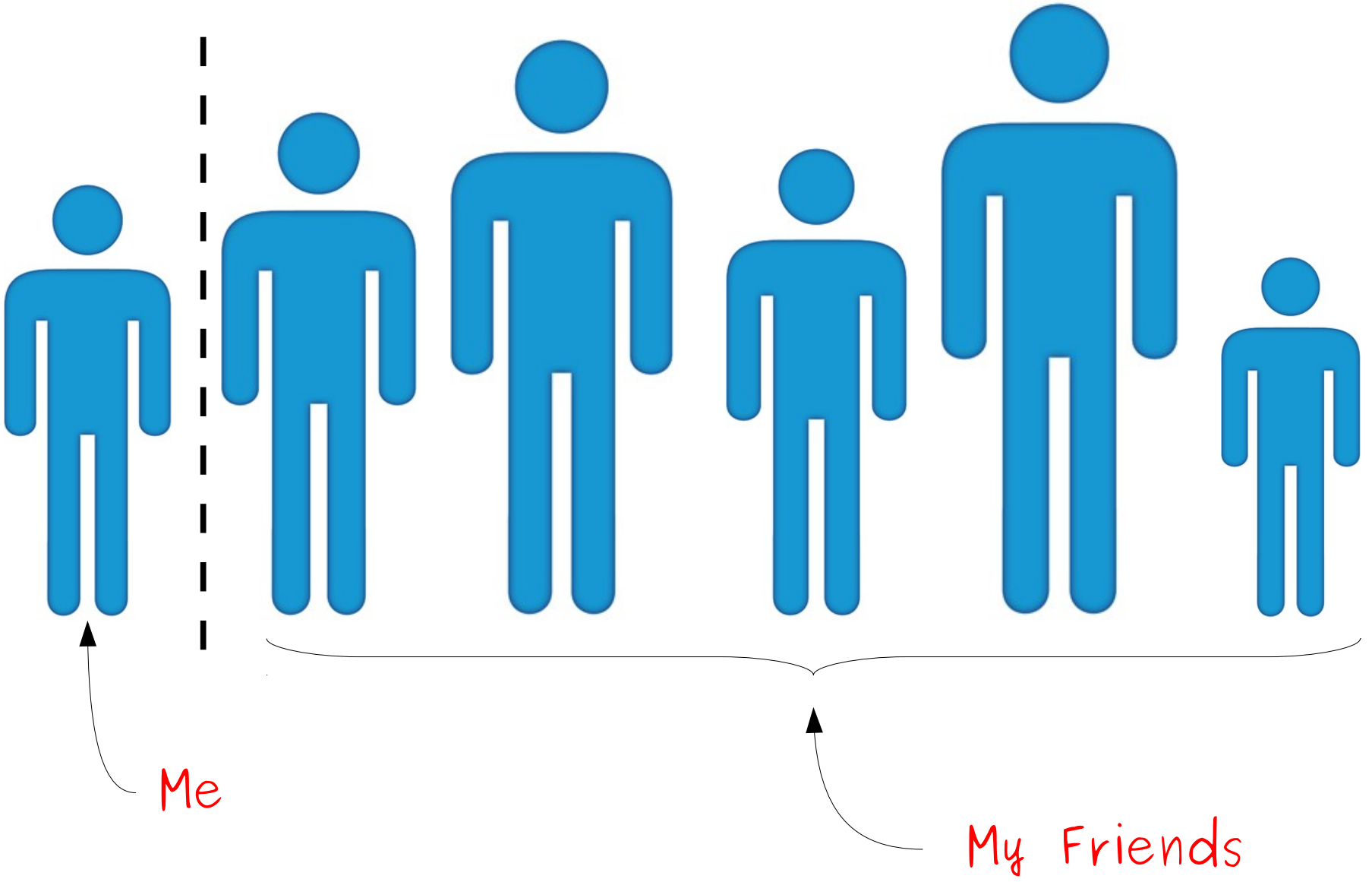

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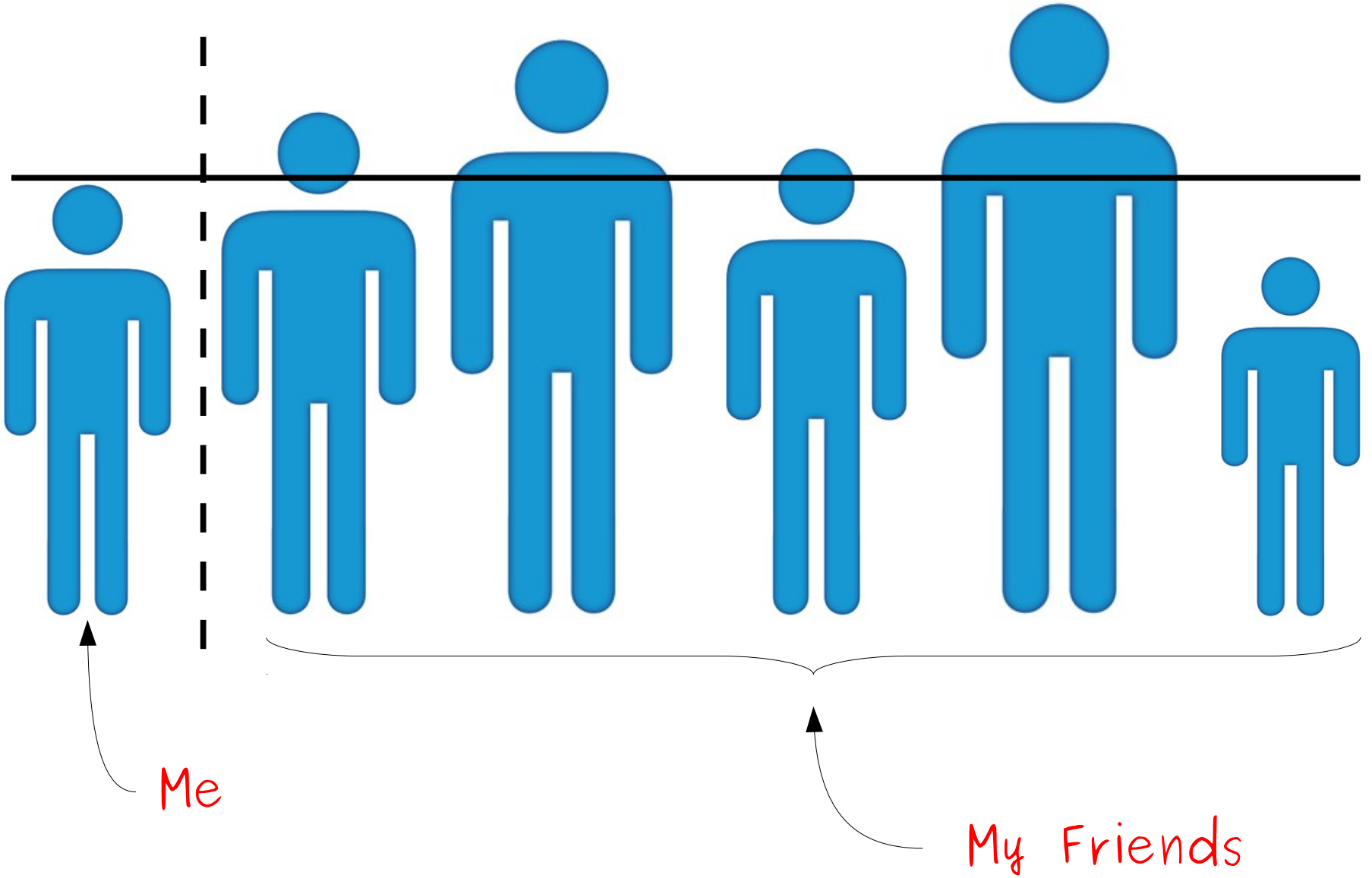

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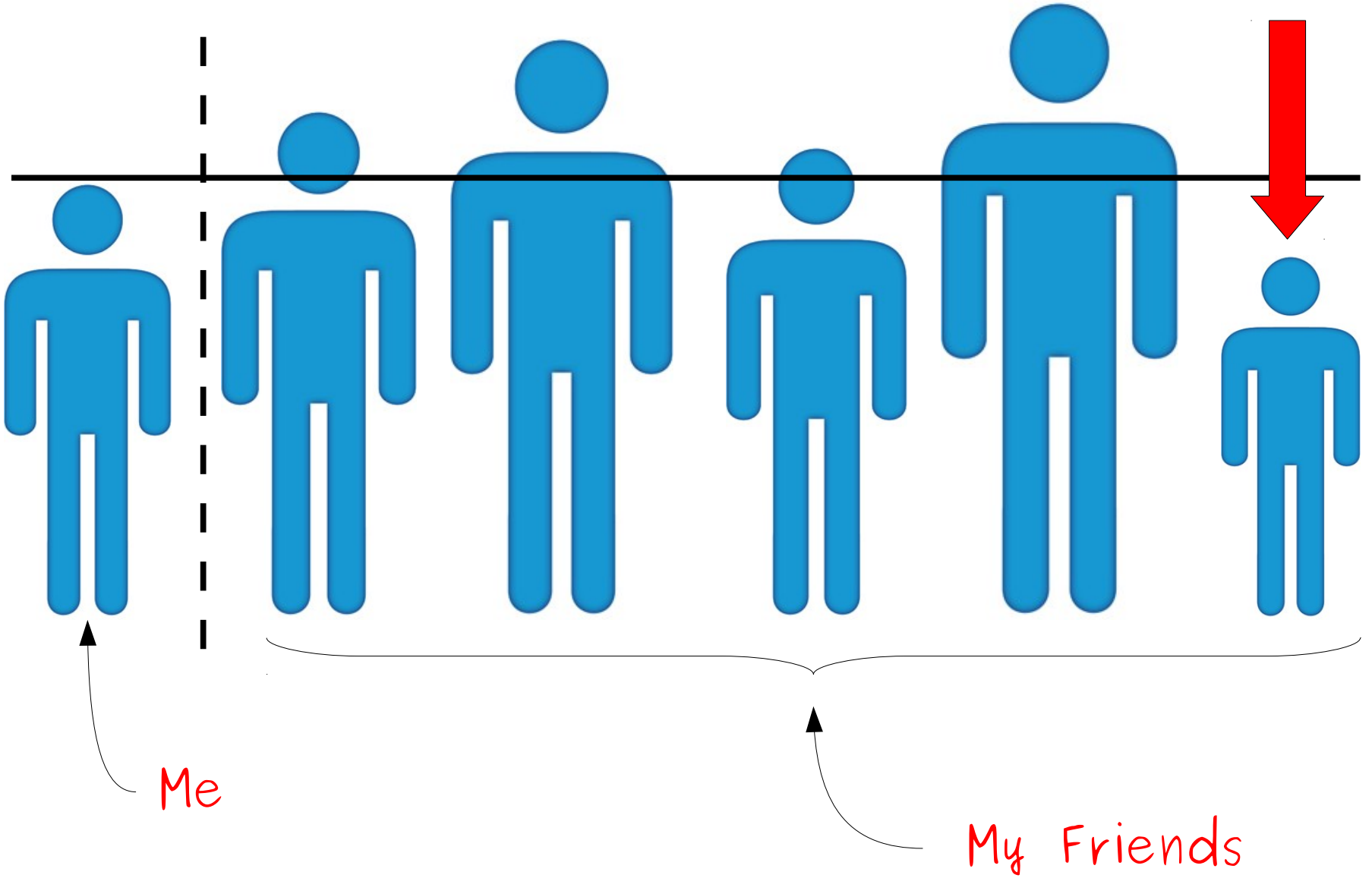

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"For any positive integer n, the sum of all smaller positive integers is not equal to n."

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Answer:

"For **some** positive integer n, the sum of all smaller positive integers **is** equal to n."

Another Incorrect Proof

Theorem: There is an integer that is smaller than all other integers.

Proof: By contradiction; assume that there is an integer that is *larger* than all other integers (call this integer n). Then n + 1 is an integer, and n < n + 1. But this contradicts the fact that n is larger than all other integers. We have reached a contradiction, so our assumption was false and the theorem is true. ■

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“Some Friend is Shorter than Me”

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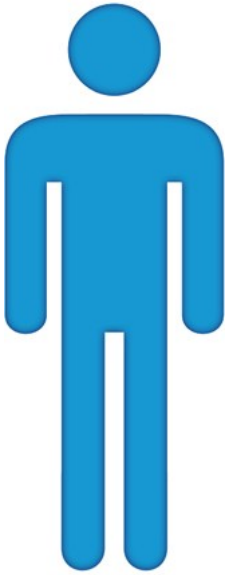

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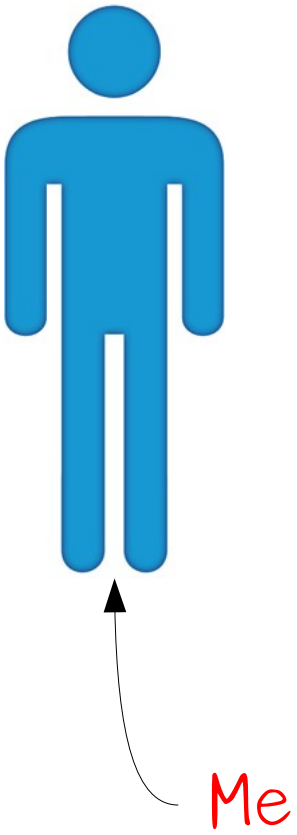

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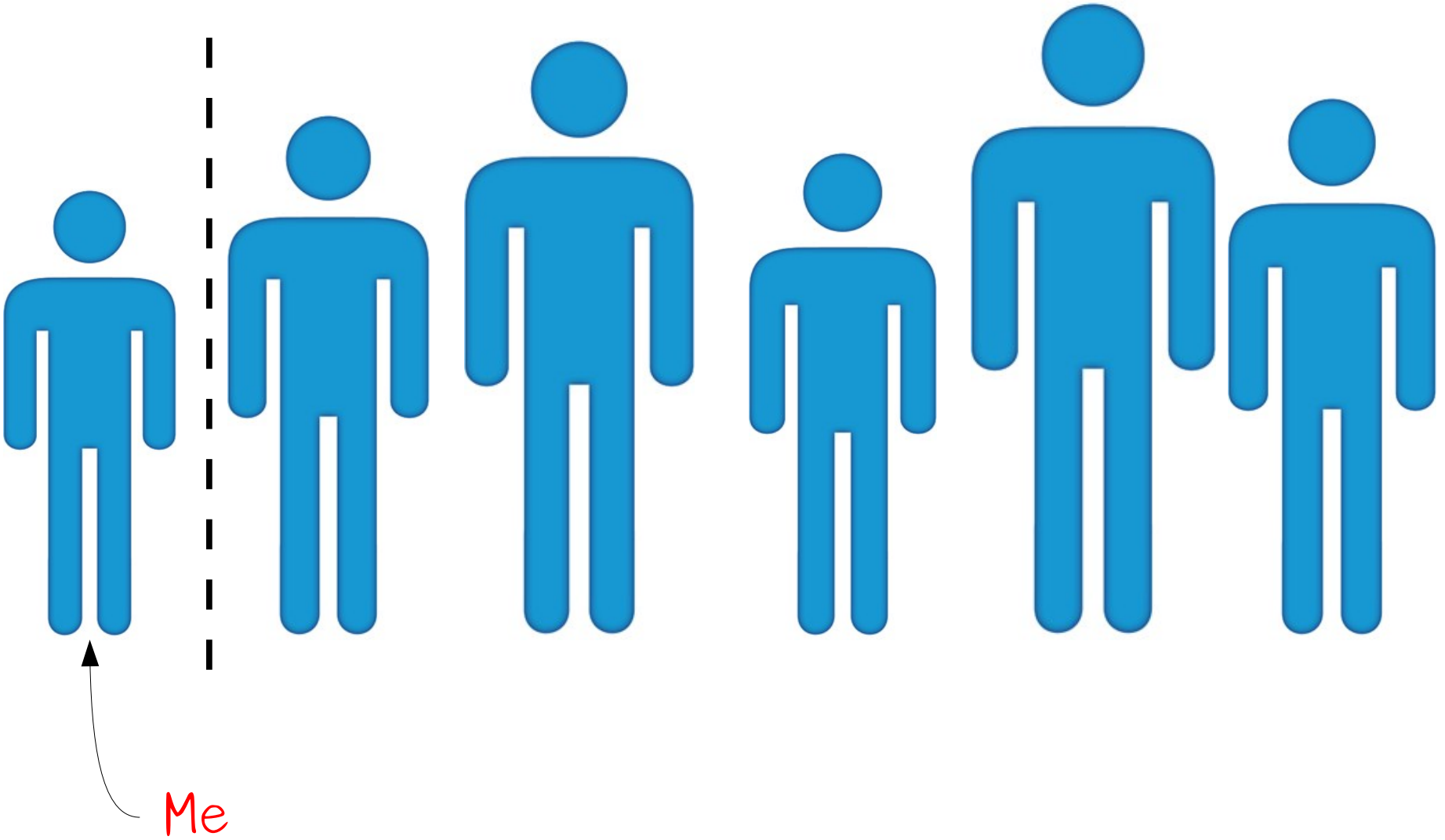

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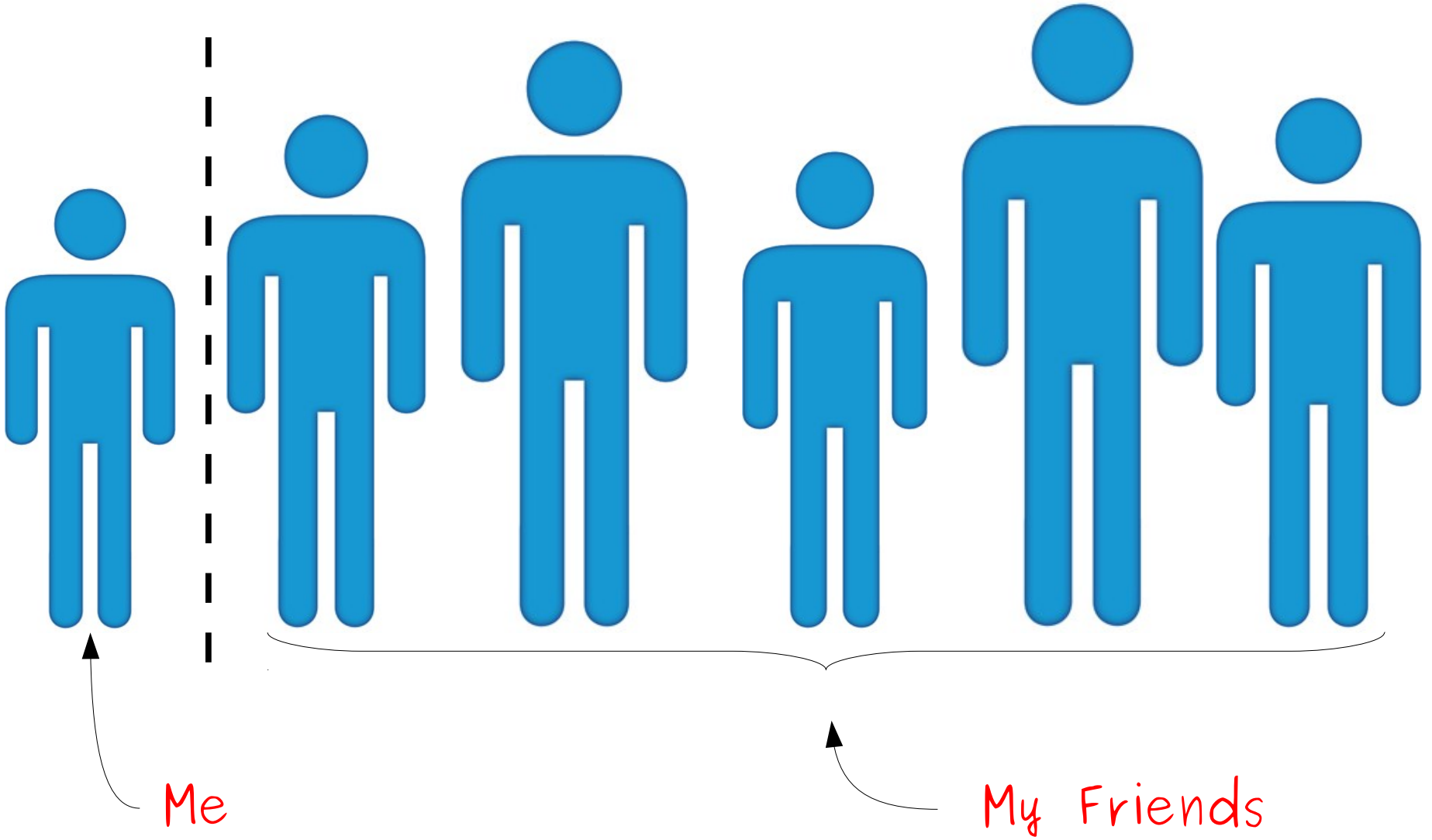

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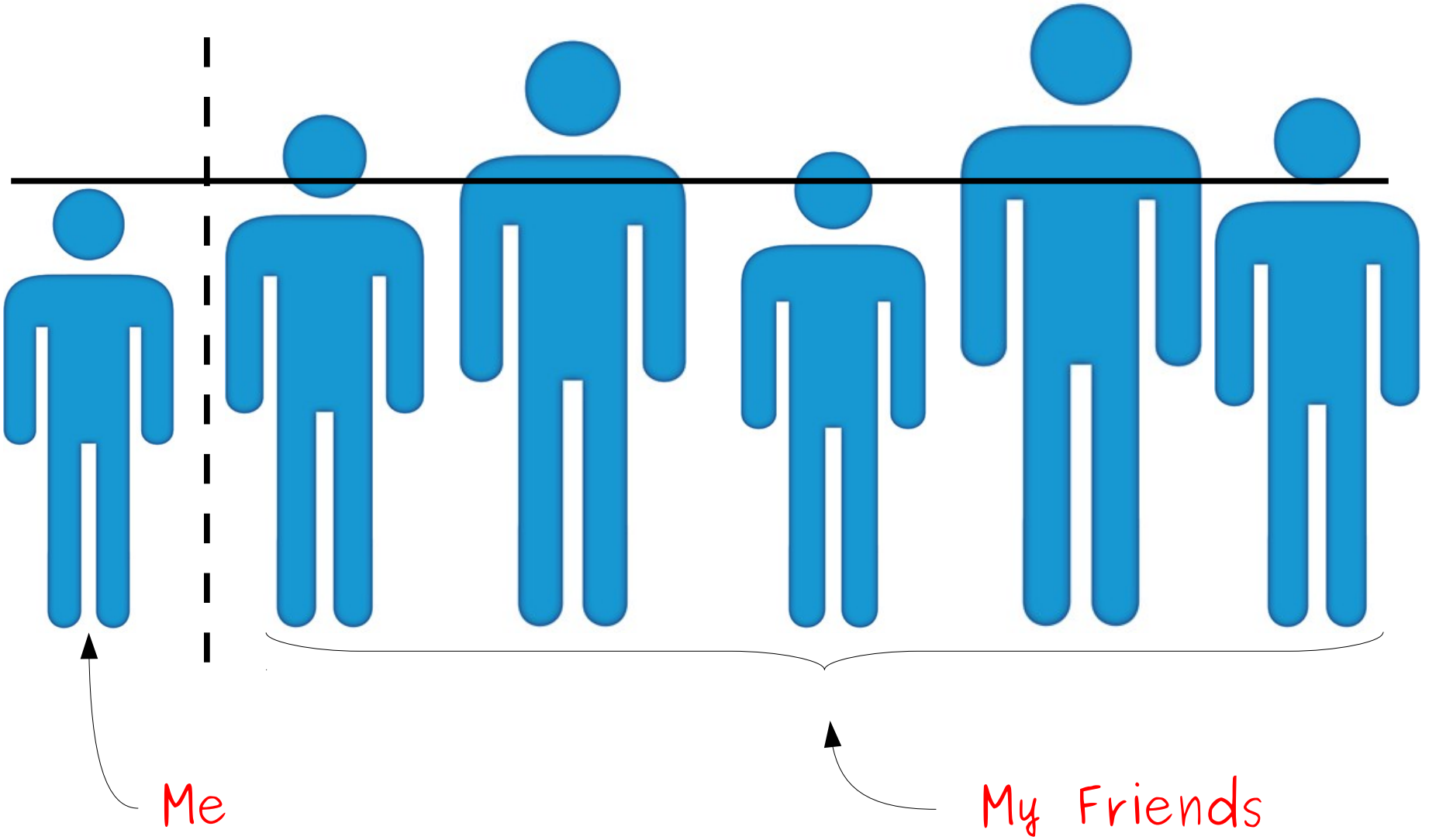

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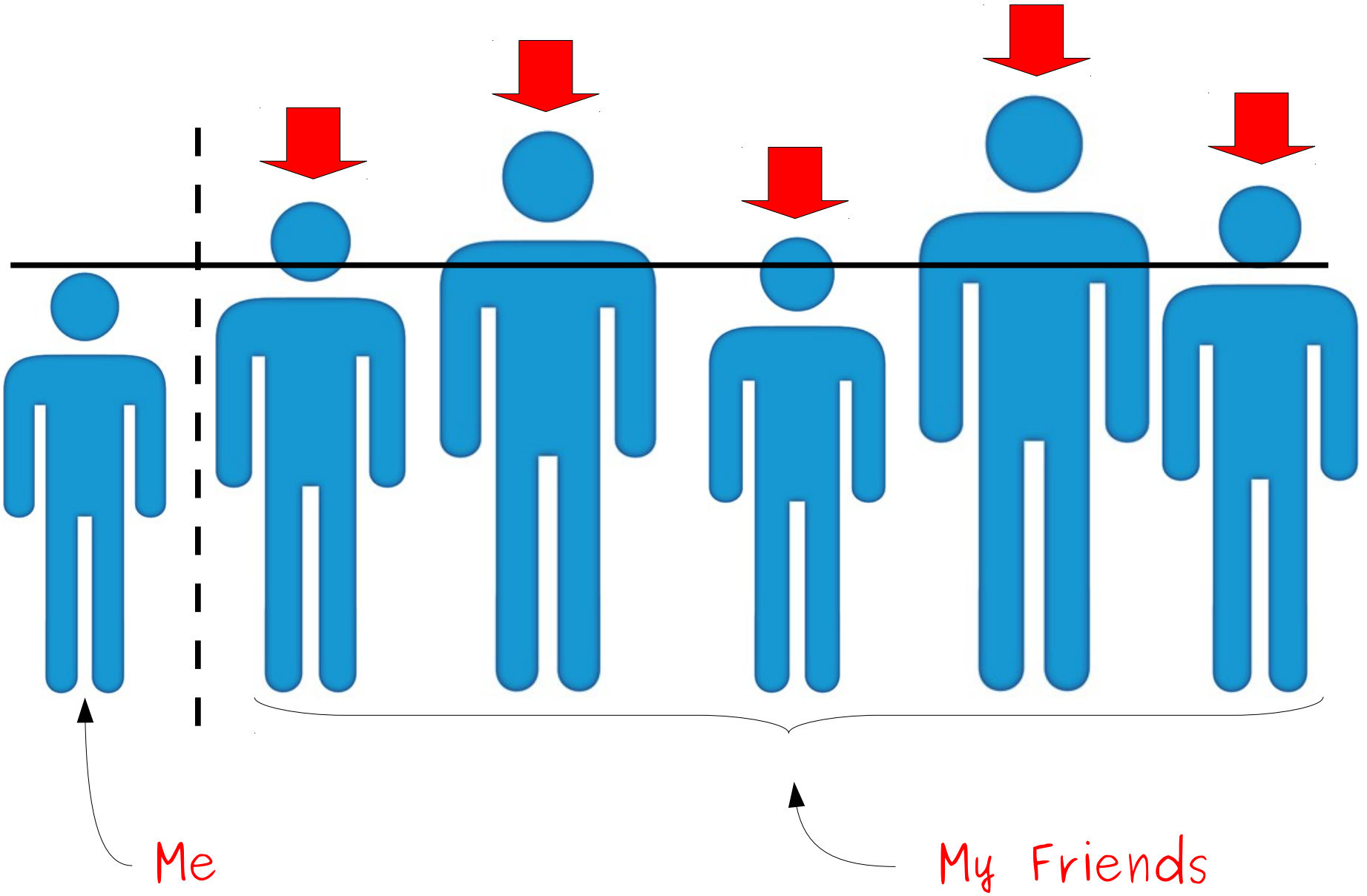

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What is the contradiction of the following statement?

"There is an integer that is smaller than all other integers."

Answer:

"**Every integer** is at least as large as some other integer."

Rational and Irrational Numbers

- A **rational number** is a number *r* that can be written as

$$r = \frac{p}{q}$$

  where
  - *p* and *q* are integers,
  - $q \neq 0$, and
  - *p* and *q* have no common divisors other than ±1.
- A number that is not rational is called **irrational**.

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A Famous and Beautiful Proof

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Proof: By contradiction; **????**

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Theorem: $\sqrt{2}$ is irrational.
Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

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Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, $p = \sqrt{2}q$, so $p^2 = 2q^2$.

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Since $p / q = \sqrt{2}$ and $q \neq 0$, $p = \sqrt{2}q$, so $p^2 = 2q^2$.

This means that p^2 is even, so by our earlier result, p is even. Thus there is an integer k such that $p = 2k$.

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Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

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Theorem: $\sqrt{2}$ is irrational.
Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

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Proof by Contrapositive

The Contrapositive

- The contrapositive of “If P, then Q” is “If **not** Q, then **not** P.”
- Example:
  - “If I'd raptor-proofed my house, then I would have survived the dinosaur attack.”
  - Contrapositive: “If I didn't survive the dinosaur attack, then I didn't raptor-proof my house.”
- Another example:
  - “If I had been a good test subject, then I would have received cake.”
  - Contrapositive: “If I didn't receive cake, then I wasn't a good test subject.”

Notation

- Recall that we can write “If P, then Q” as $P \rightarrow Q$.
- Notation: We write “not P” as **¬P**.
- Examples:
  - “If P is false, then Q is true:” $\neg P \rightarrow Q$
  - “Q is false whenever P is false:” $\neg P \rightarrow \neg Q$
- The contrapositive of $P \rightarrow Q$ is **$\neg Q \rightarrow \neg P$**.

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

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Theorem: If ¬Q → ¬P, then P → Q.
Proof: By contradiction. Assume that ¬Q → ¬P, but that P → Q is false. Since P → Q is false, it must be true that P is true and **Q is false**.

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Theorem: If $\neg Q \rightarrow \neg P$, then $P \rightarrow Q$.
Proof: By contradiction. Assume that $\neg Q \rightarrow \neg P$, but that $P \rightarrow Q$ is false. Since $P \rightarrow Q$ is false, it must be true that P is true and $\neg Q$ is true.

Theorem: If $\neg Q \rightarrow \neg P$, then $P \rightarrow Q$.
Proof: By contradiction. Assume that $\neg Q \rightarrow \neg P$, but that $P \rightarrow Q$ is false. Since $P \rightarrow Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \rightarrow \neg P$, $\neg P$ is true.

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Proof: By contradiction. Assume that $\neg Q \rightarrow \neg P$, but that $P \rightarrow Q$ is false. Since $P \rightarrow Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \rightarrow \neg P$, $\neg P$ is true. But this means that we have shown P and $\neg P$, which is impossible. We have reached a contradiction, so if $\neg Q \rightarrow \neg P$, then $P \rightarrow Q$.

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An Important Proof Strategy

To show that $P \to Q$, you may instead show that $\neg Q \to \neg P$.

This is called a **proof by contrapositive**.

Theorem: If $x + y = 16$, then either $x \geq 8$ or $y \geq 8$.

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Proof: By contrapositive. **????**

If

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then

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If

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then

either $x \geq 8$ or $y \geq 8$

If

If

$x + y = 16$

then

either $x \geq 8$ or $y \geq 8$

If

both $x < 8$ and $y < 8$

If

$x + y = 16$

then

either $x \geq 8$ or $y \geq 8$

If

both $x < 8$ and $y < 8$

Pro tip: The opposite of "either A or B" is "not A and not B"

If

$x + y = 16$

then

either $x \geq 8$ or $y \geq 8$

If

both $x < 8$ and $y < 8$

If

$x + y = 16$

then

either $x \geq 8$ or $y \geq 8$

If

both $x < 8$ and $y < 8$

then

If

$x + y = 16$

then

either $x \geq 8$ or $y \geq 8$

If

both $x < 8$ and $y < 8$

then

$x + y \neq 16$

Theorem: If $x + y = 16$, then either $x \geq 8$ or $y \geq 8$.

Proof: By contrapositive. **????**

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Proof: By contrapositive. We prove that if both $x < 8$ and $y < 8$, then $x + y \neq 16$.

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Proof: By contrapositive. We prove that if both $x < 8$ and $y < 8$, then $x + y \neq 16$. Since both $x < 8$ and $y < 8$, then $x + y < 16$, so $x + y \neq 16$.

Theorem: If $x + y = 16$, then either $x \geq 8$ or $y \geq 8$.

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then

n is even

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then

n is even

If

n is odd

then

n^2 is odd

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive. We prove that if n is odd, n^2 is odd. Since n is odd, $n = 2k + 1$ for some integer k. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so n^2 is odd. ■

An Incorrect Proof

Theorem: For any sets A and B, if $x \in A$, then $x \in A \cap B$.

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Theorem: For any sets A and B, if $x \in A$, then $x \in A \cap B$.

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An Incorrect Proof

Theorem: For …hen $x \in A \cap B$.

Proof: By cont… **$\notin A$, then**

$x \notin A \cap$ …: $x \in A \cap B$ iff

$x \in A$ a… $A \cap B$ either. ■

Common Pitfalls

To prove $P \rightarrow Q$ by contrapositive, show that

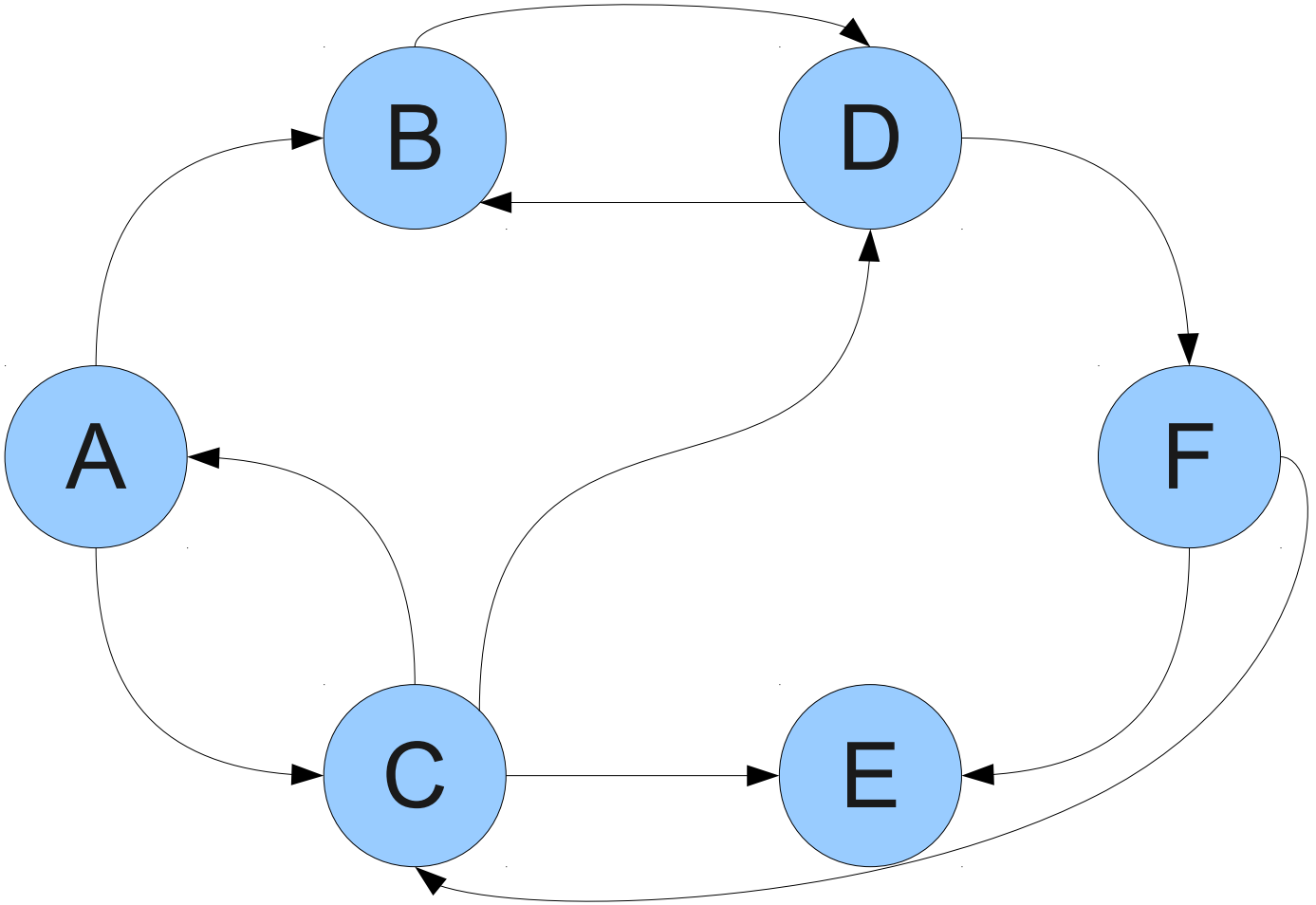
$$\neg Q \rightarrow \neg P$$

Do not show that

$$\neg P \rightarrow \neg Q$$

(This is equivalent to showing that $Q \rightarrow P$!)

Shortest Paths in a Graph



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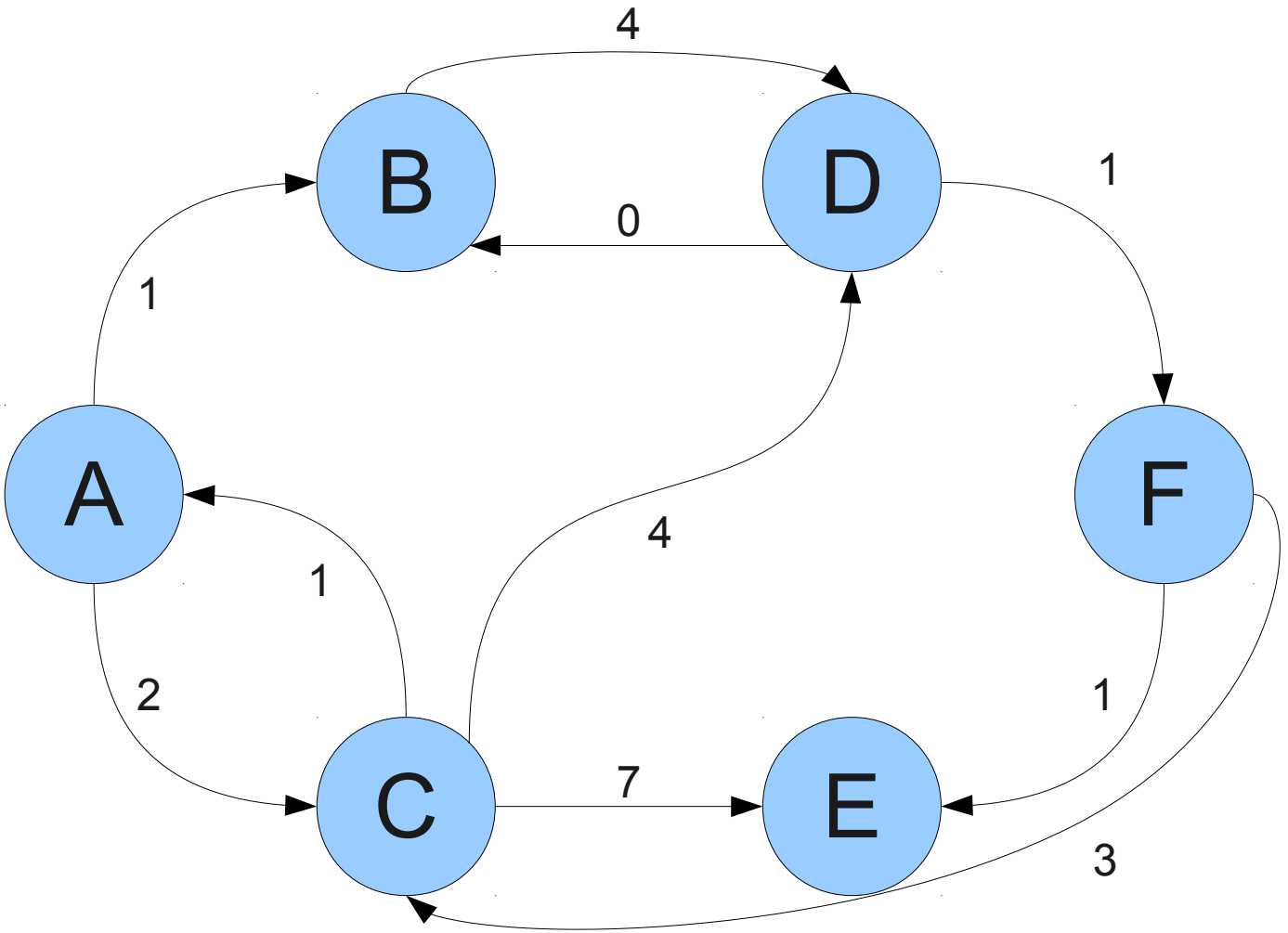

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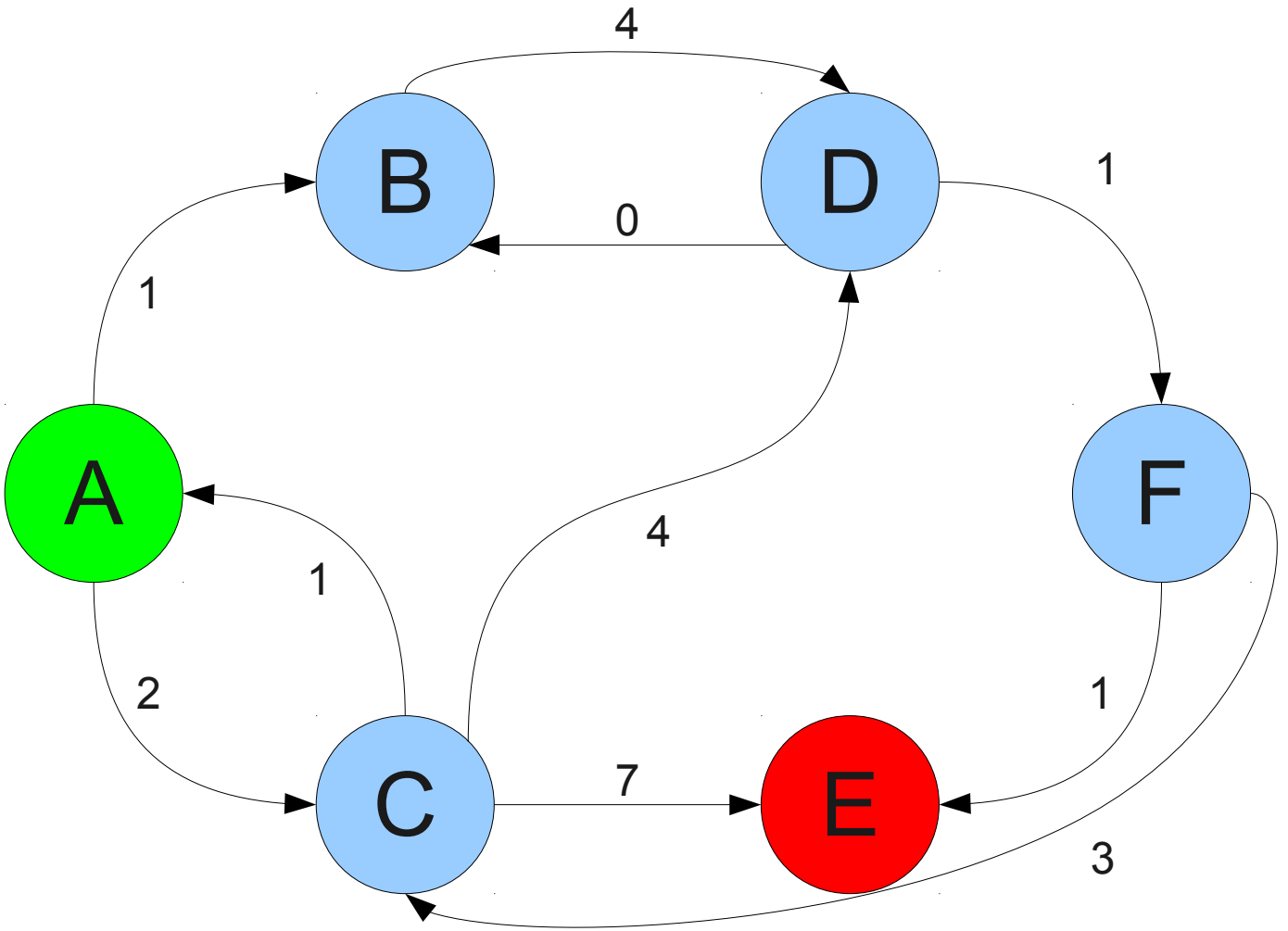

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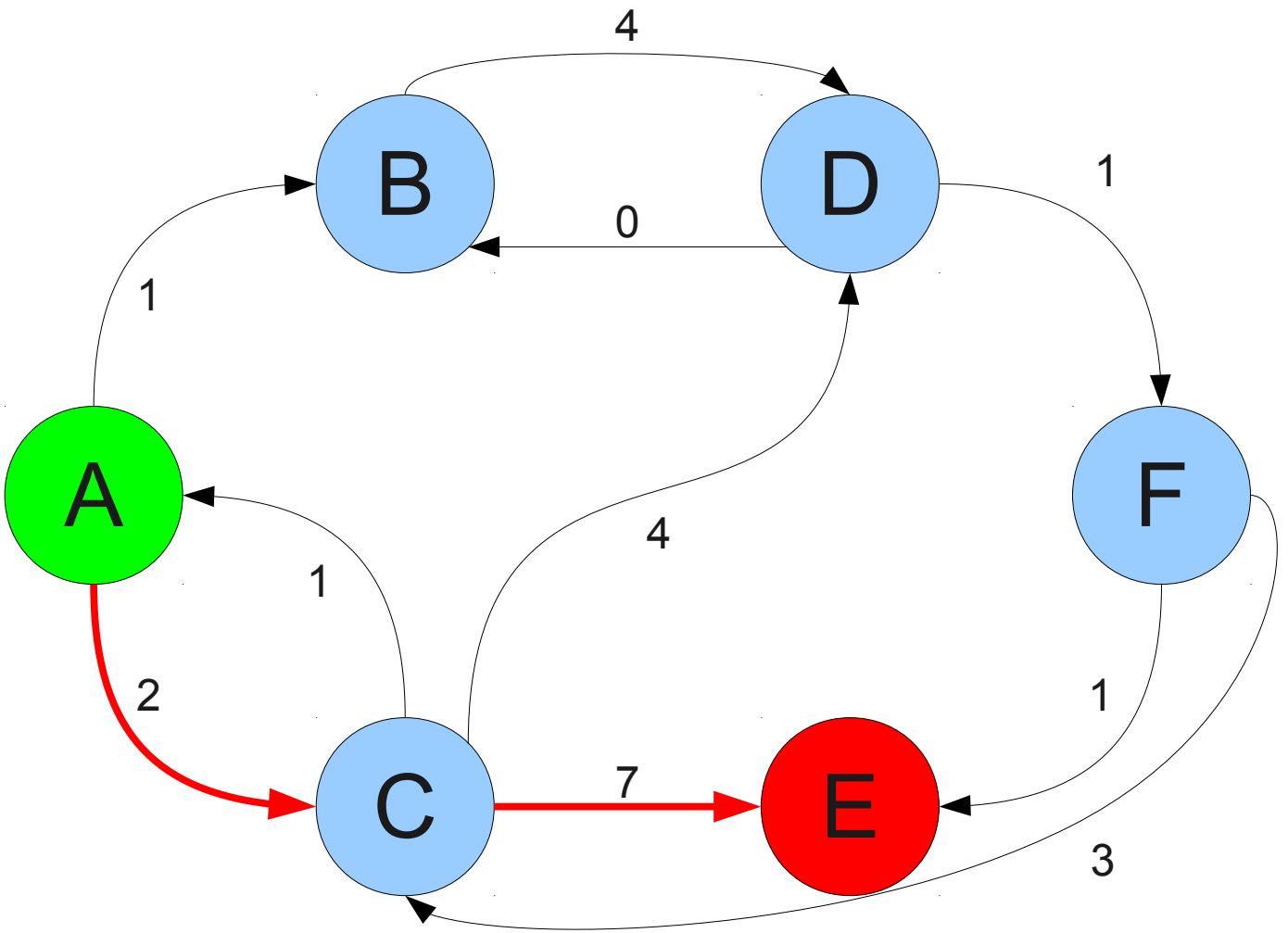

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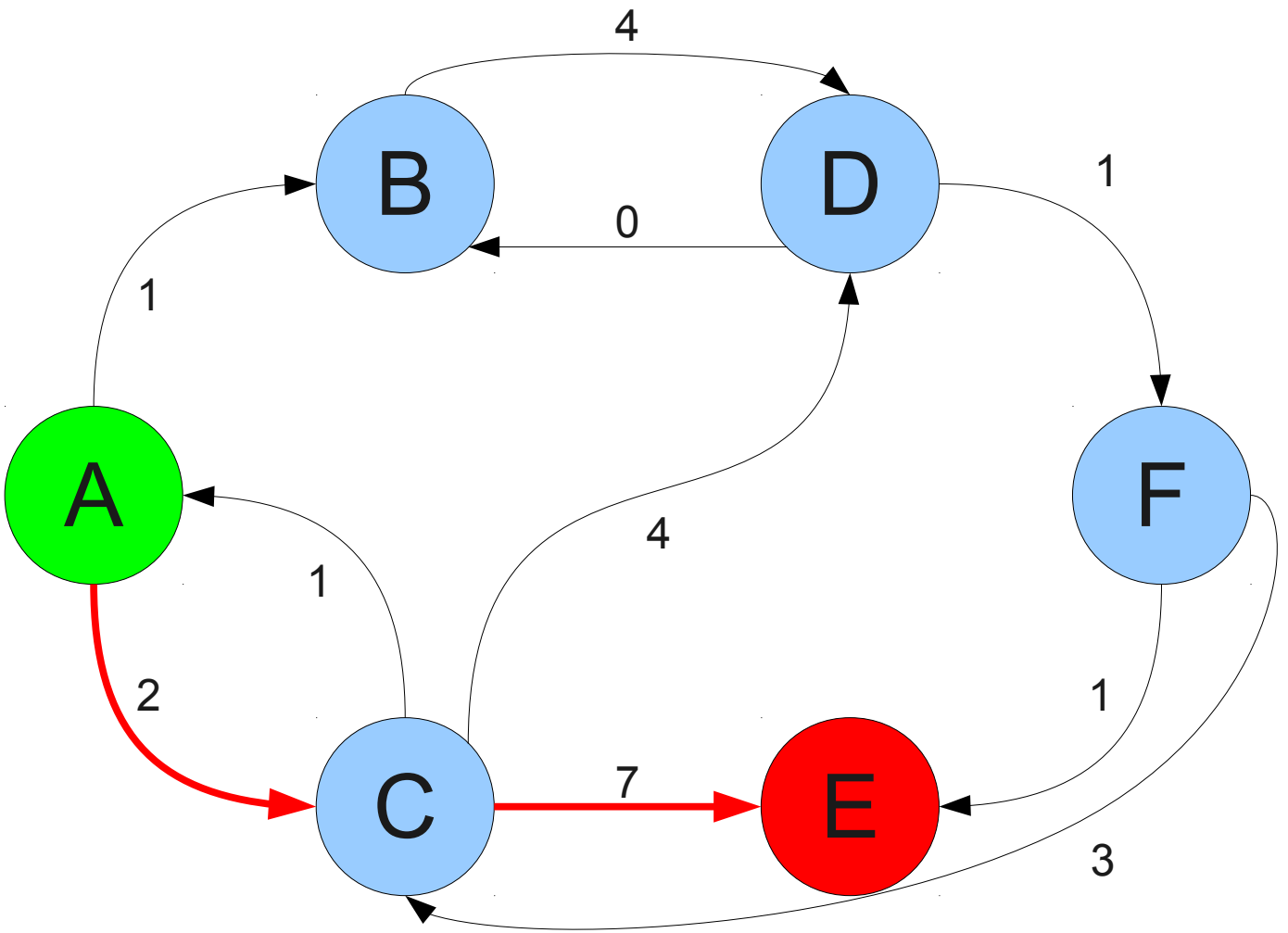

Cost: **9**

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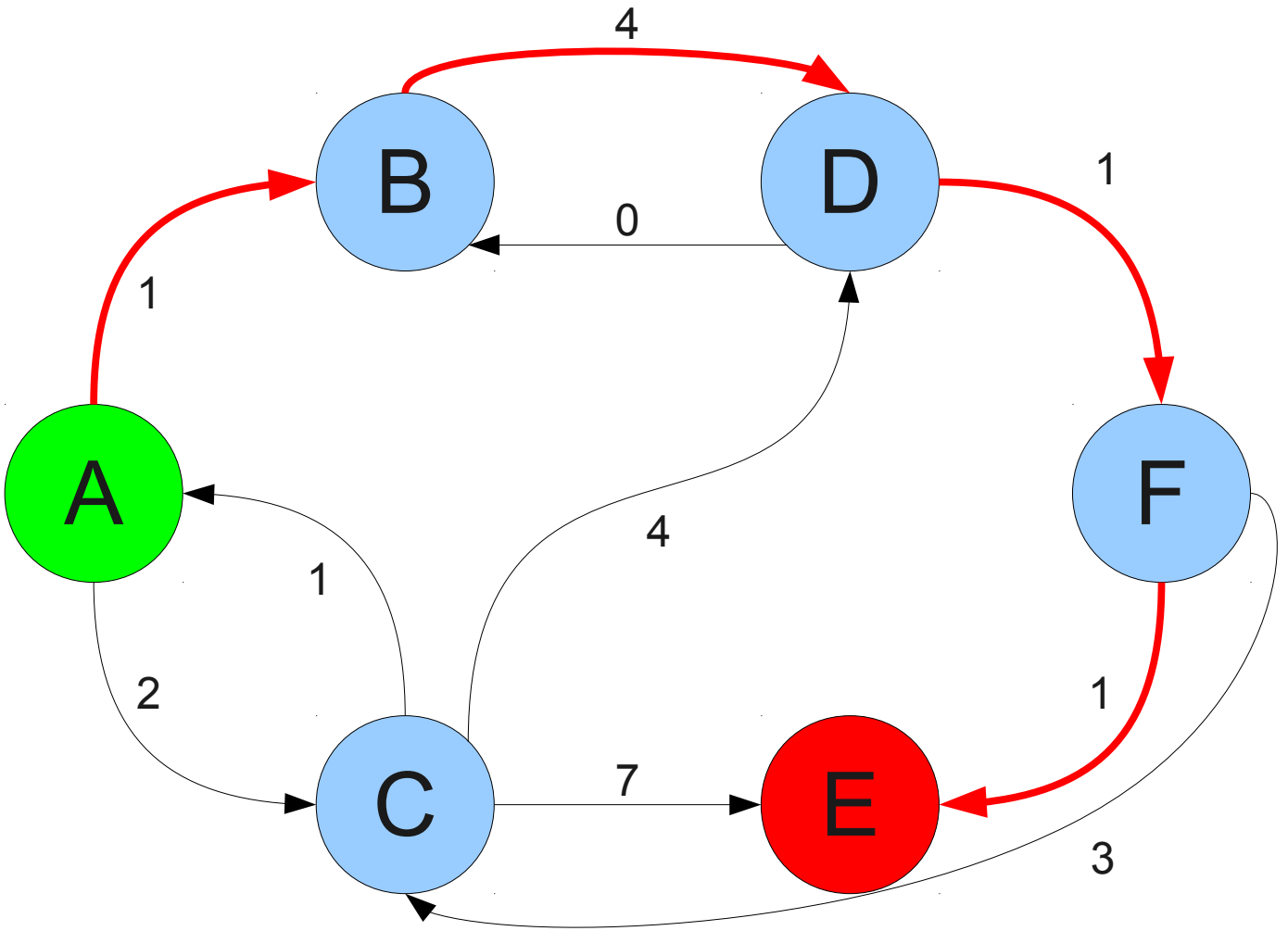

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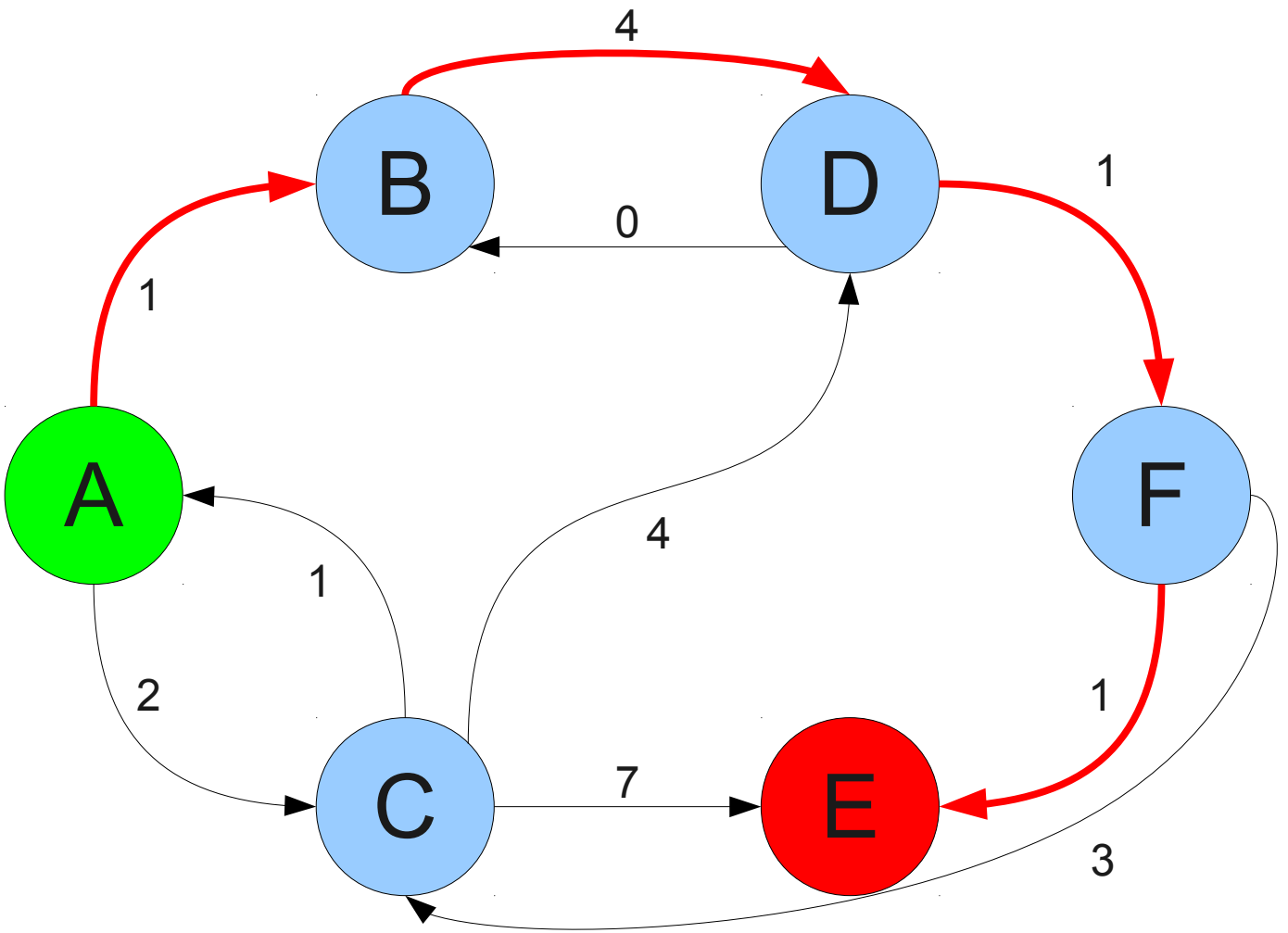
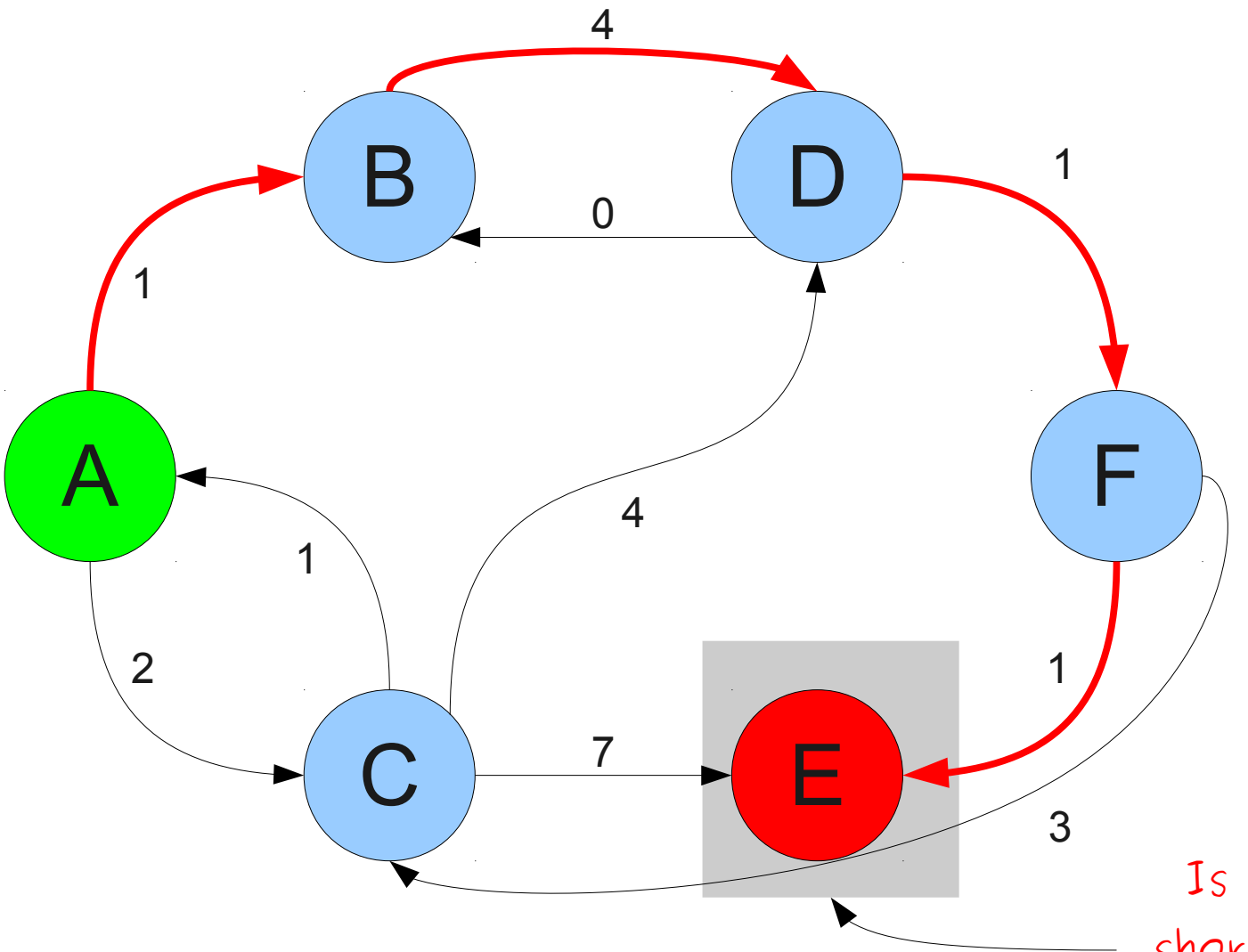


Cost: **7**

Shortest Paths in a Graph



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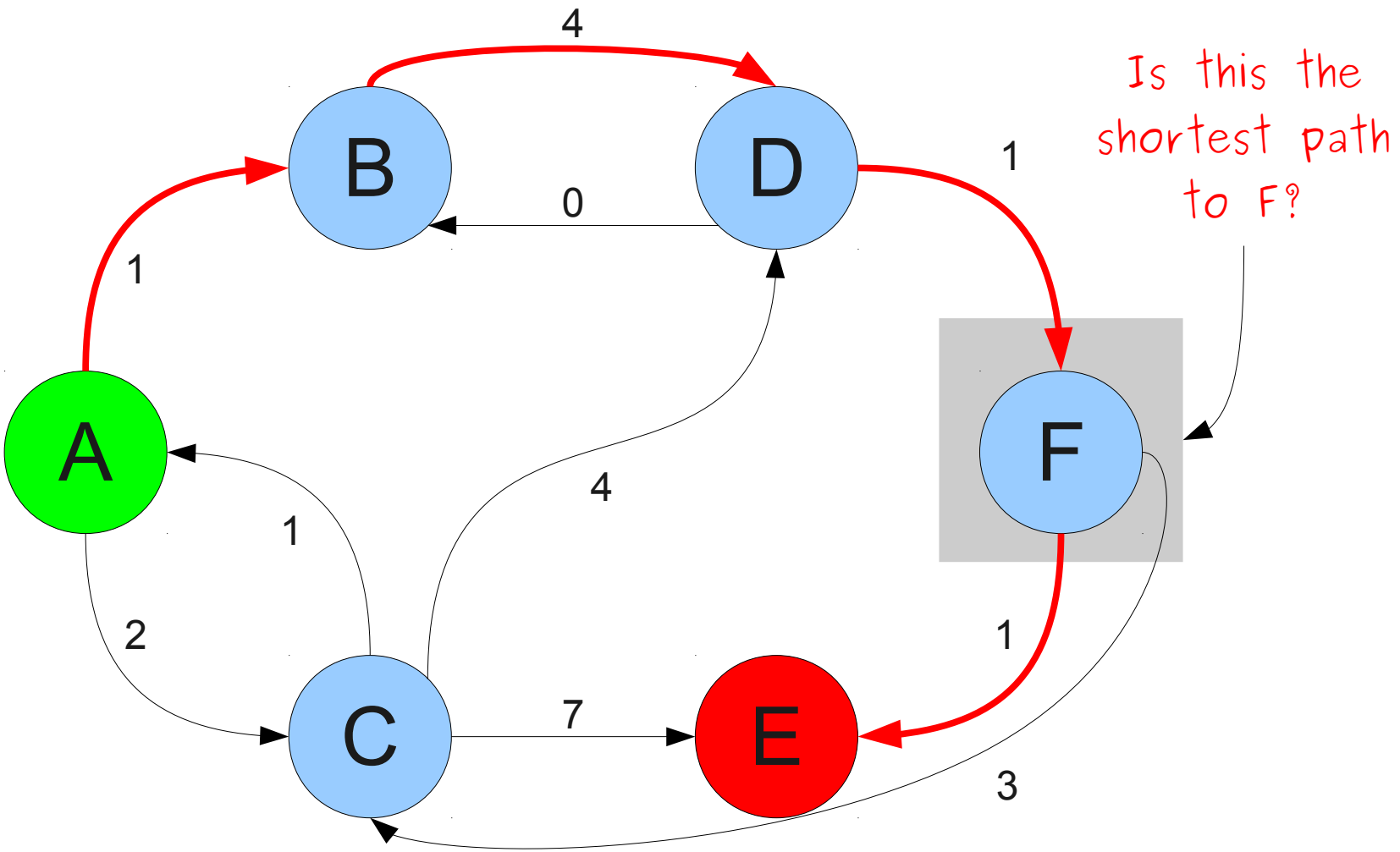

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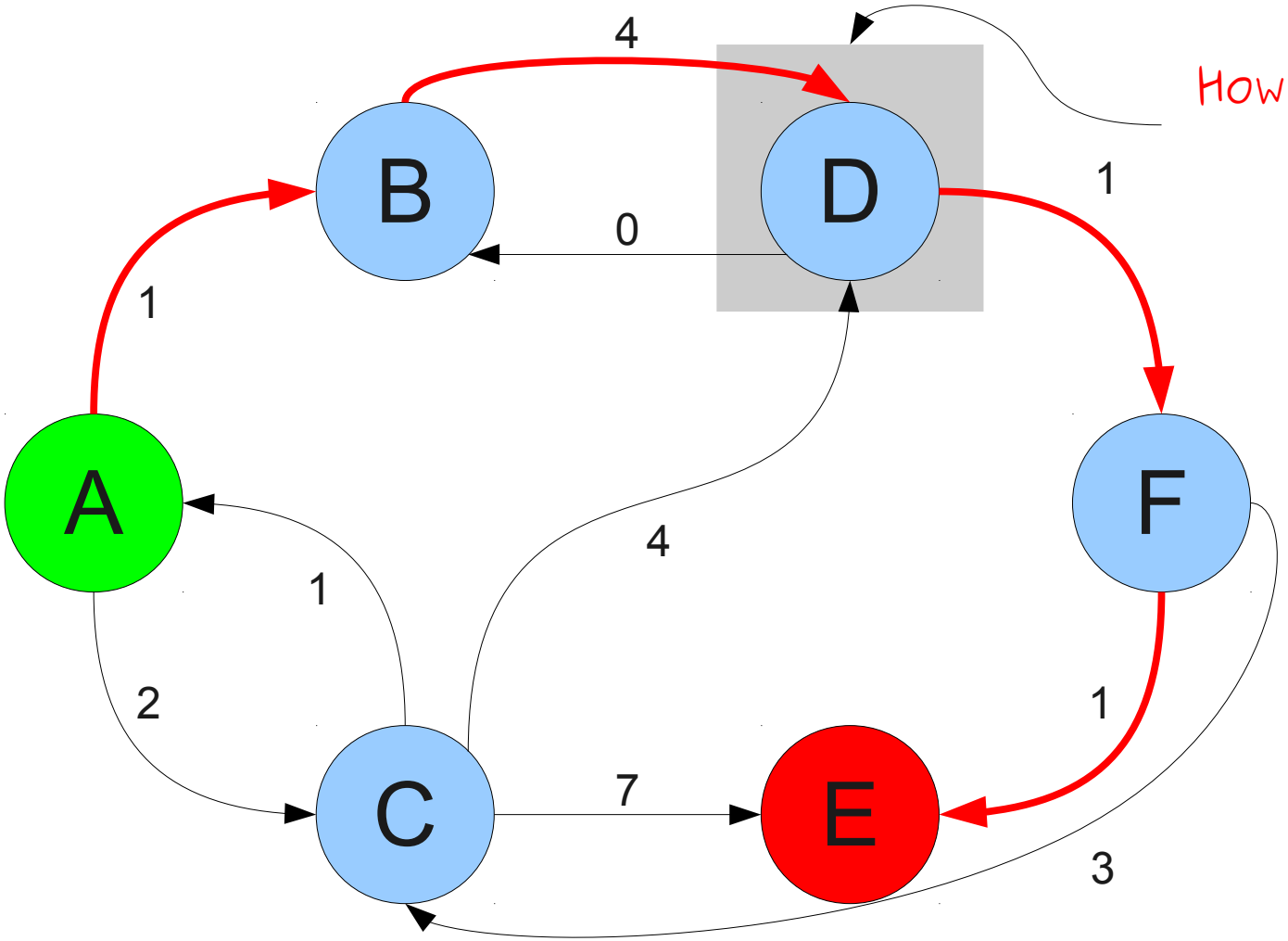

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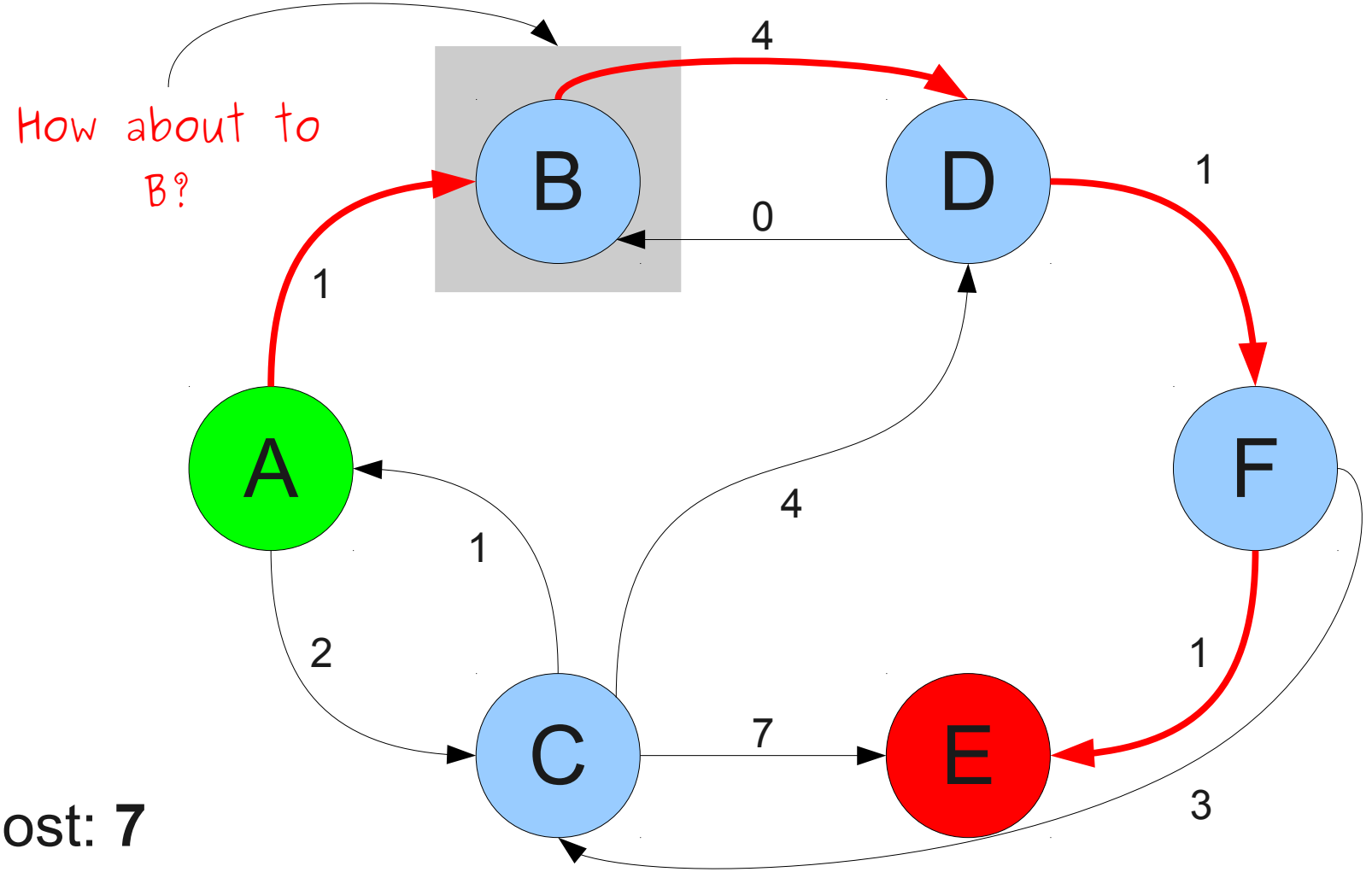

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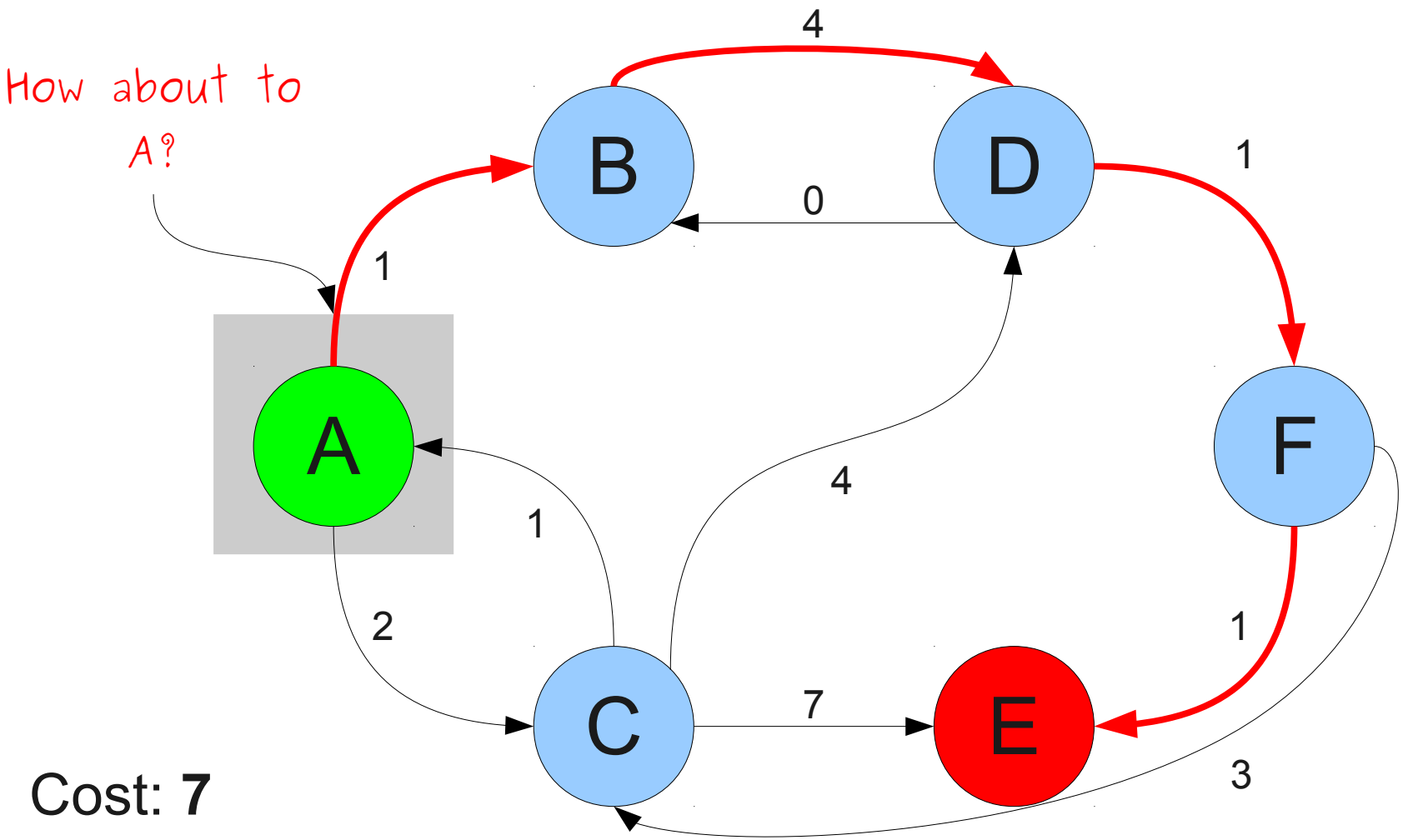

Question

Suppose we have a shortest path from u to v.

Is that path also the shortest path to each other node in the path?

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Suppose **$((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n.**

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Pro tip: Assign numbers to elements of a sequence.

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Suppose $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n.

For any $k \leq n$, is the path a shortest path from v_0 to v_k?

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Question

Suppose $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n.

For any $k \leq n$, is **$((v_0, v_1), \ldots, (v_{k-1}, v_k))$** a shortest path from v_0 to v_k?

Question

Suppose $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n.

For any $k \leq n$, is $((v_0, v_1), \ldots, (v_{k-1}, v_k))$ a shortest path from v_0 to v_k?

Theorem: If $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n, then for any $k \leq n$, $((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k))$ is a shortest path from v_0 to v_k.

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Proof: By contrapositive; **???**

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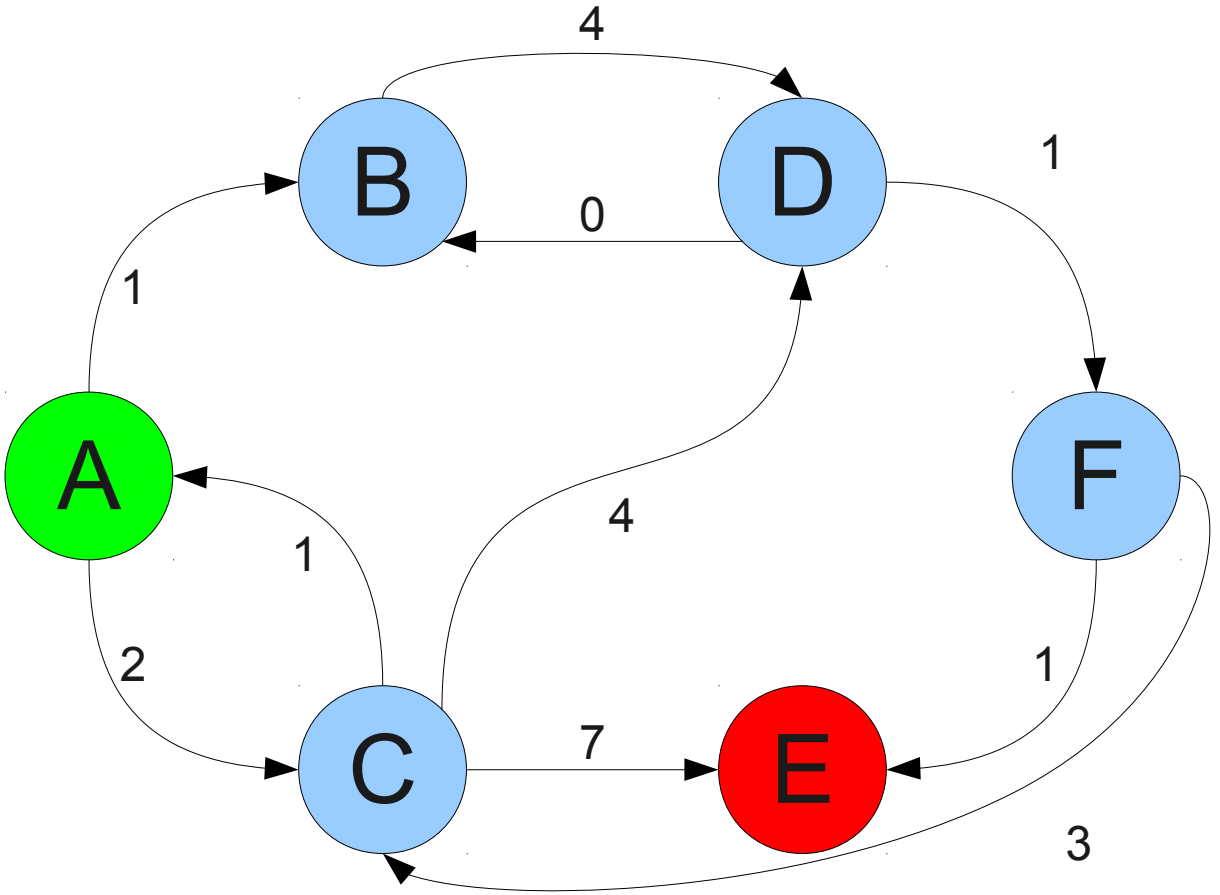
An Intuition



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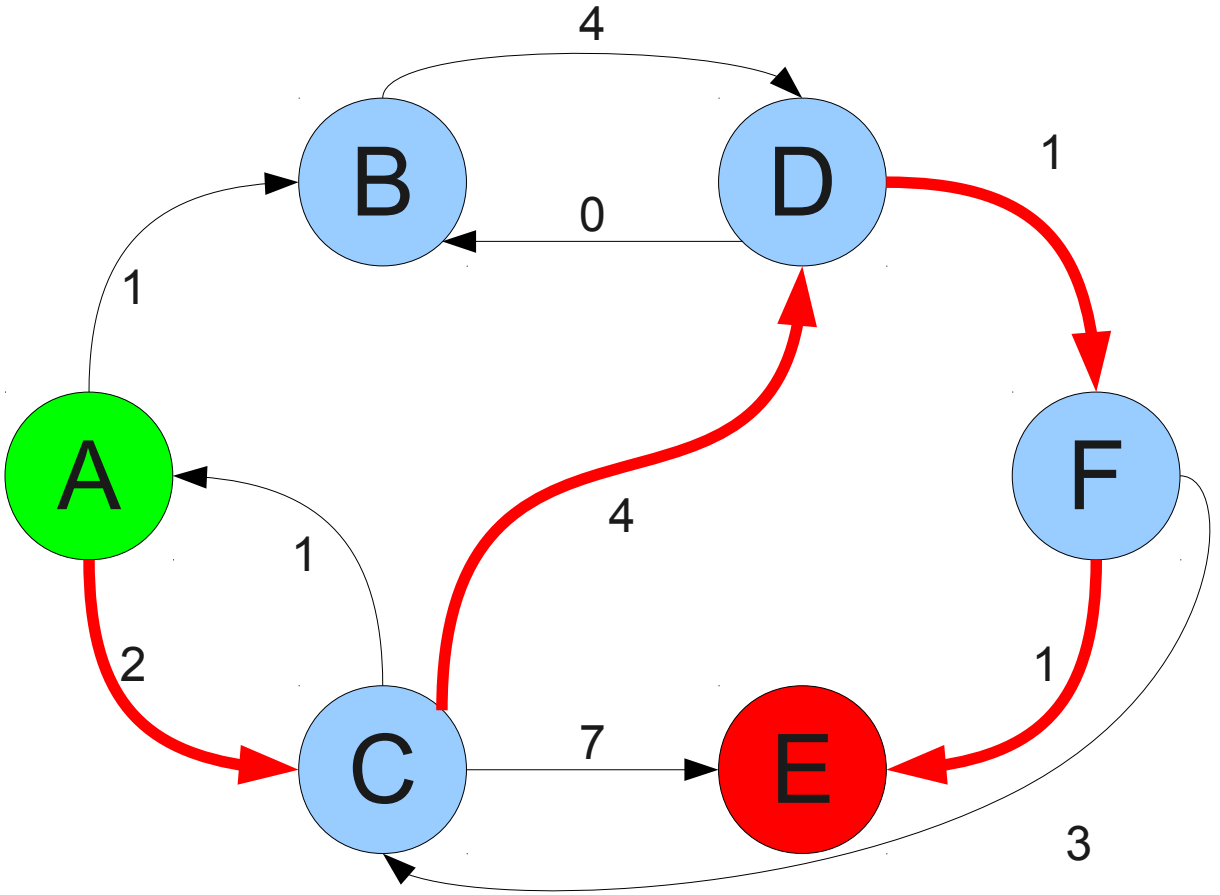

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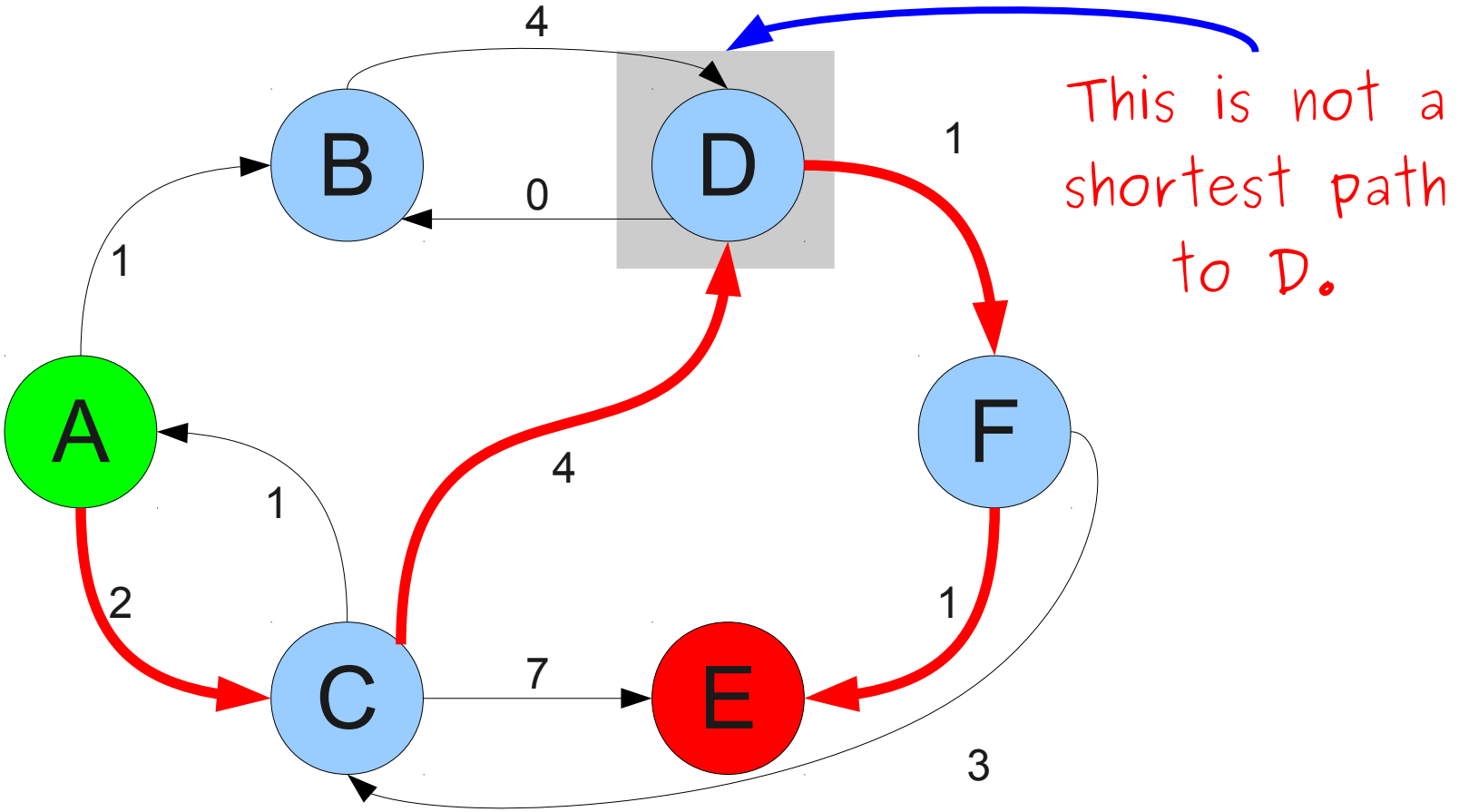

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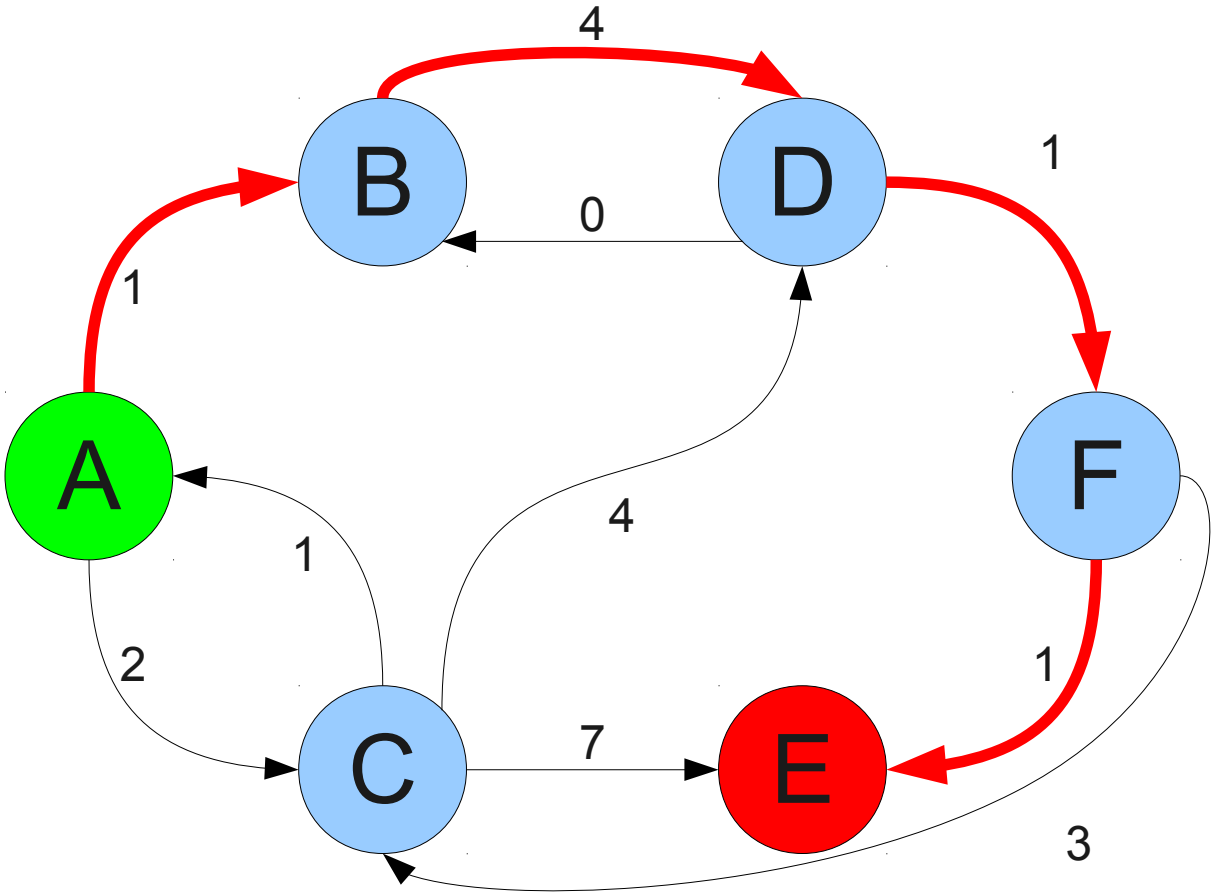
An Intuition



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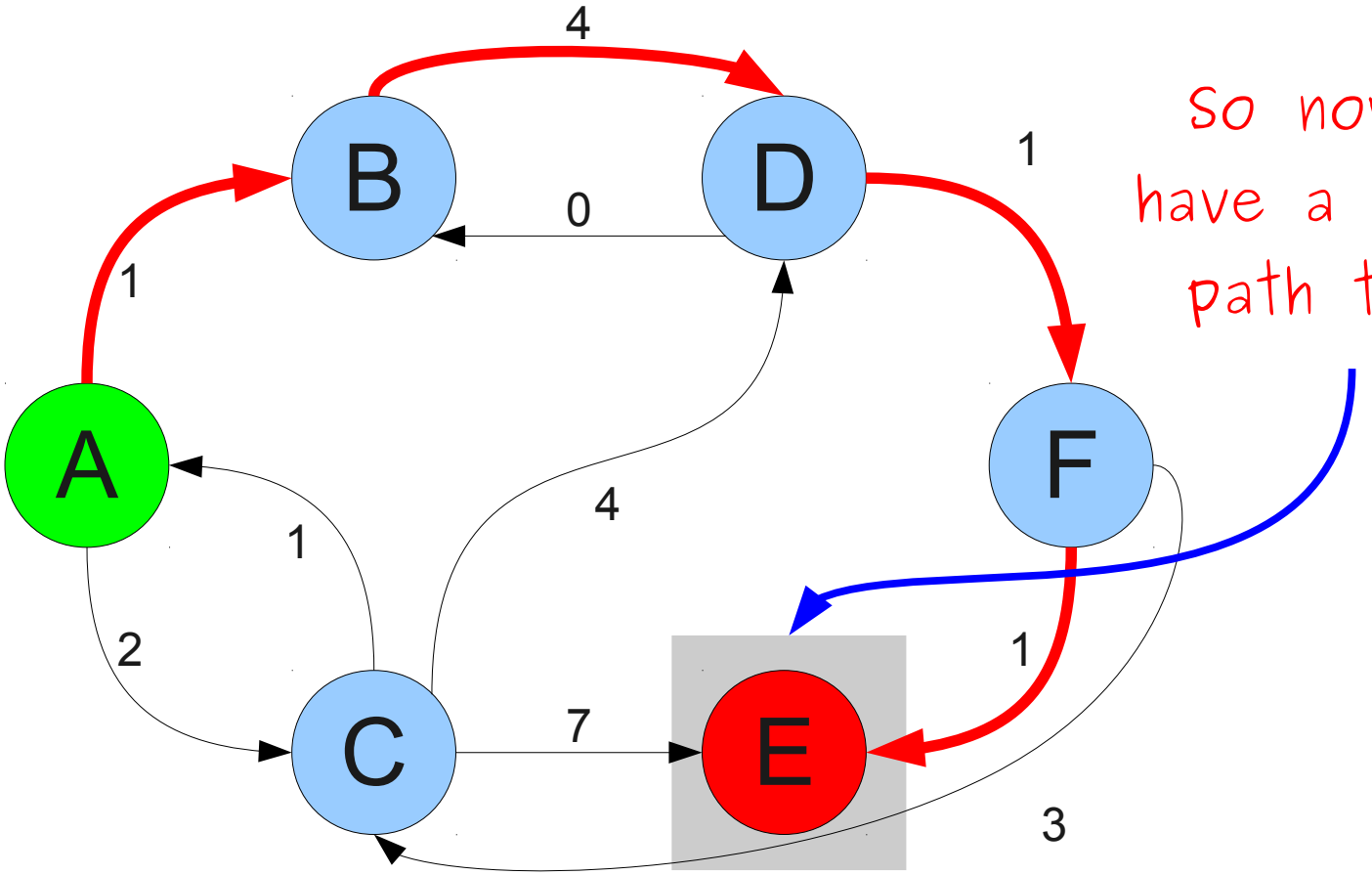

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Theorem: If $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n, then for any $k \leq n$, $((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k))$ is a shortest path from v_0 to v_k.

Proof: By contrapositive; we show that if for some $k \leq n$, if $((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k))$ is not a shortest path from v_0 to v_k, then $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is not a shortest path from v_0 to v_n.

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Since $((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k))$ is not a shortest path from v_0 to v_k, there must be some path $((v_0, u_1), (u_1, u_2), \ldots, (u_m, v_k))$ that is a shorter path.

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Then the path $((v_0, u_1), (u_1, u_2), \ldots, (u_m, v_k), (v_k, v_{k+1}), \ldots, (v_{n-1}, v_n))$ is a shorter path from v_0 to v_n than our original path, so our original path is not a shortest path.

Theorem: If $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$ is a shortest path from v_0 to v_n, then for any $k \leq n$, $((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k))$ is a shortest path from v_0 to v_k.

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Review

- A **proof** is a series of logically sound steps that proves a **conclusion** from a set of **hypotheses**.
- A **direct proof** works by showing the conclusion directly follows from the hypotheses.
  - To prove “if A, then B,” assume A and use it to show B.
  - To prove “for any x, P is true,” prove that for any choice of x, P is true.
  - To prove “there is some x for which P is true”, choose a specific x that demonstrates it.
- A **proof by contradiction** works by assuming the conclusion is false and arriving at an impossibility.
  - The contradiction of “if A, then B” is “A and not B.”
  - The contradiction of “for all x, P is true” is “for some x, P is false.”
  - The contradiction of “for some x, P is true” is “for all x, P is false.”
- A **proof by contrapositive** proves that A implies B by proving that not B implies not A.

Next Time

Proof by Induction