Introduction to Formal Proofs

Announcements

- Problem Set 1 out, due Friday, October 7 at 2:15PM (right before class).
  - **Start early!**
  - Later problems are harder than earlier ones; don't extrapolate from the first few problems.
  - Email us at **cs103@cs.stanford.edu** or stop by OH with questions.
- Office hours schedule is posted on the course website; Patrick is holding OH after class today (4:30PM – 6:30PM) in Gates B24A.

Office Hours

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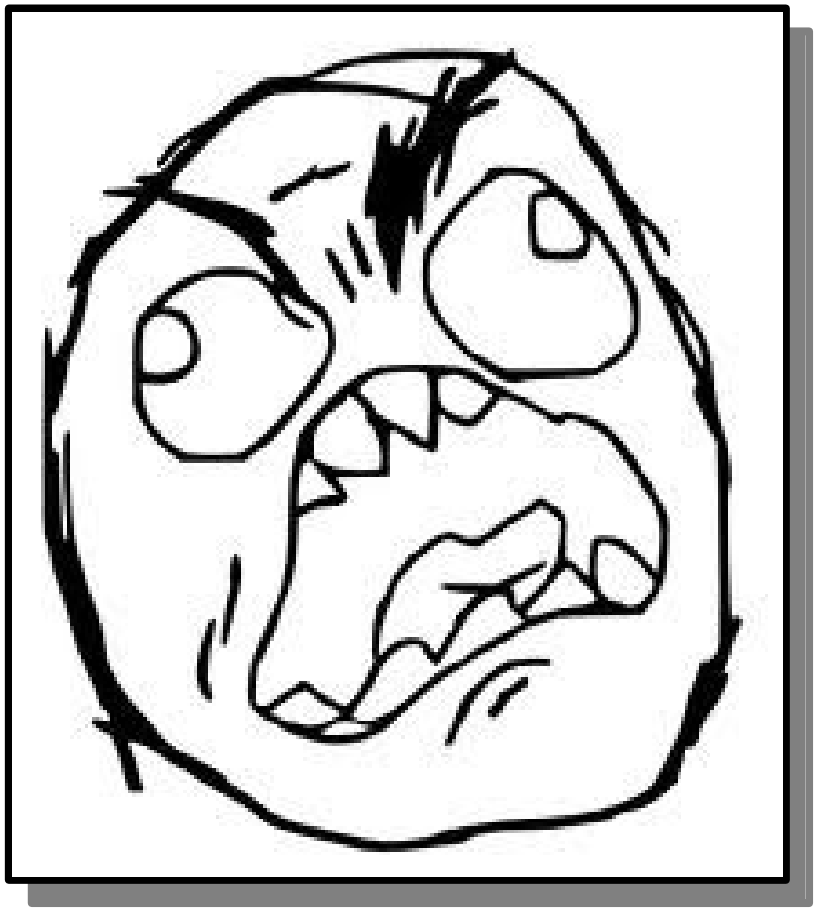

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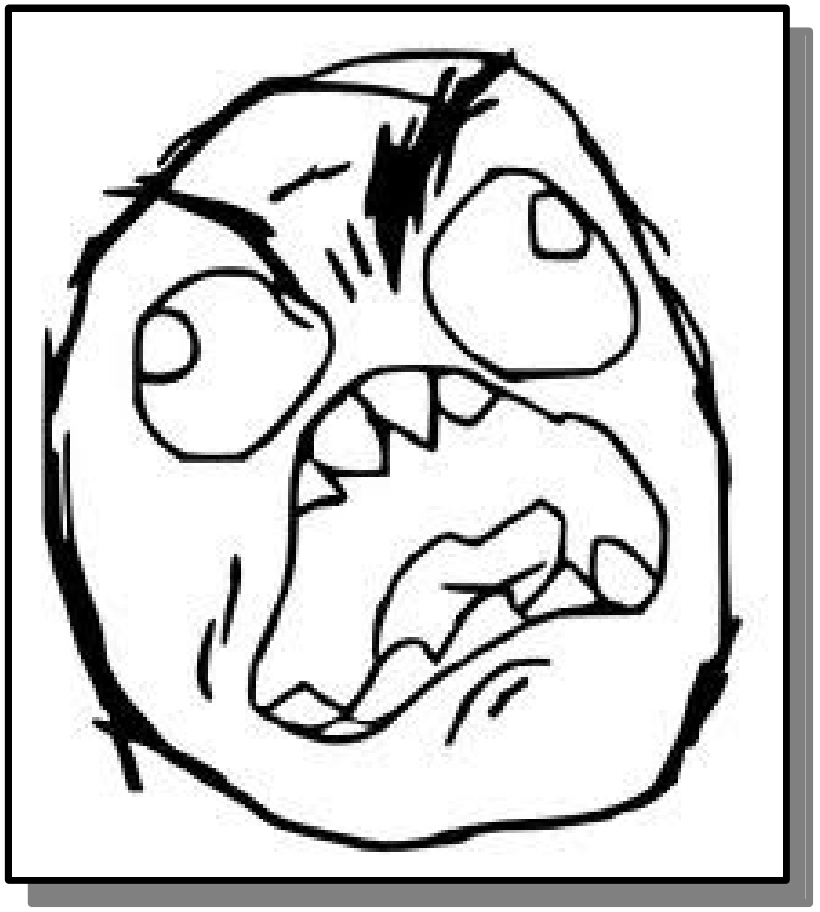

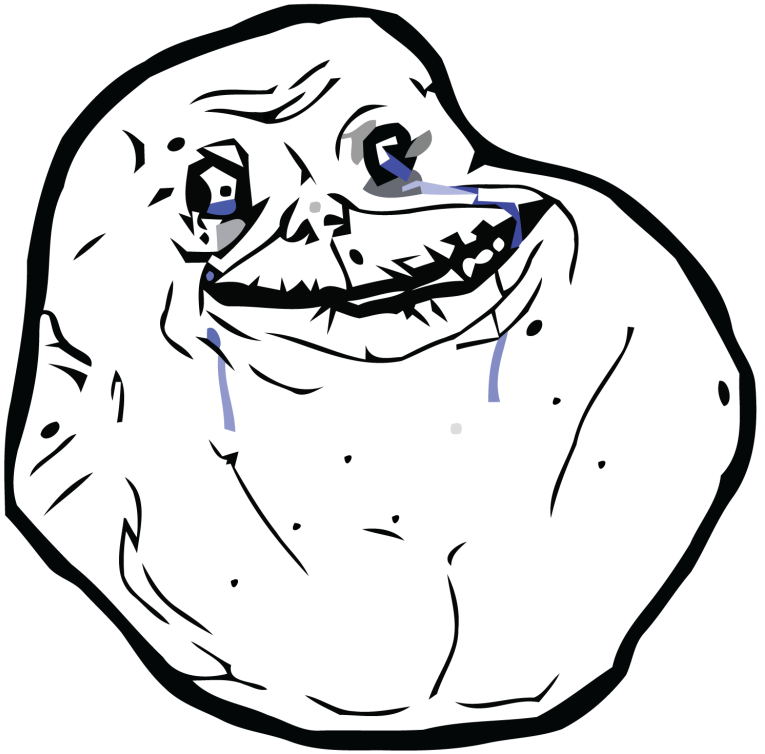

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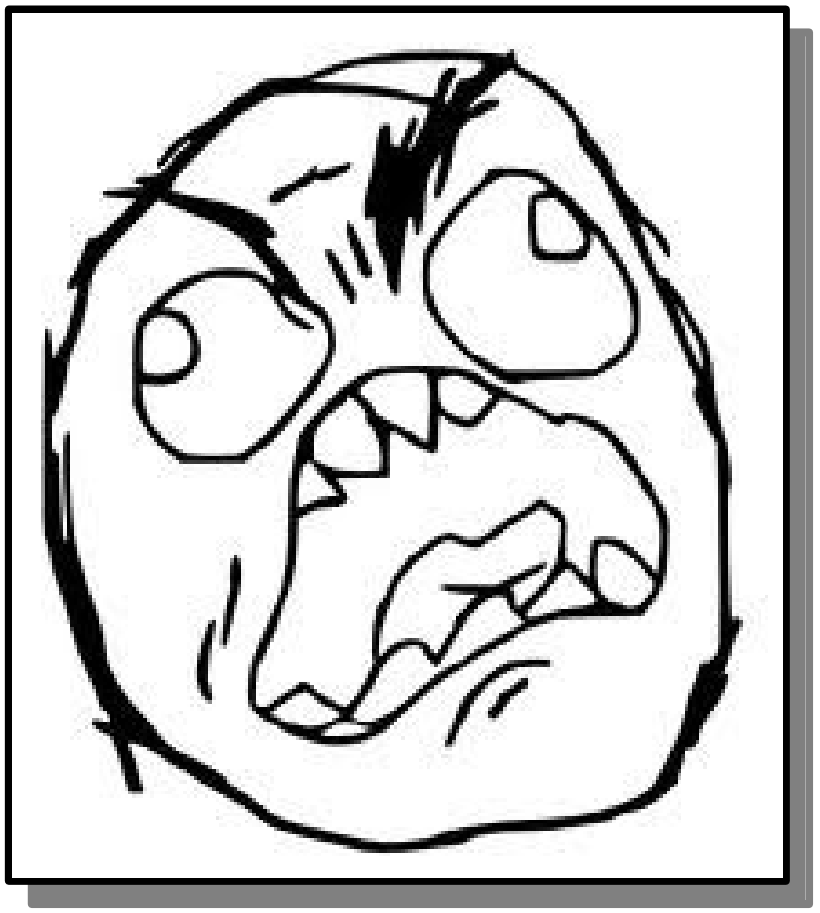

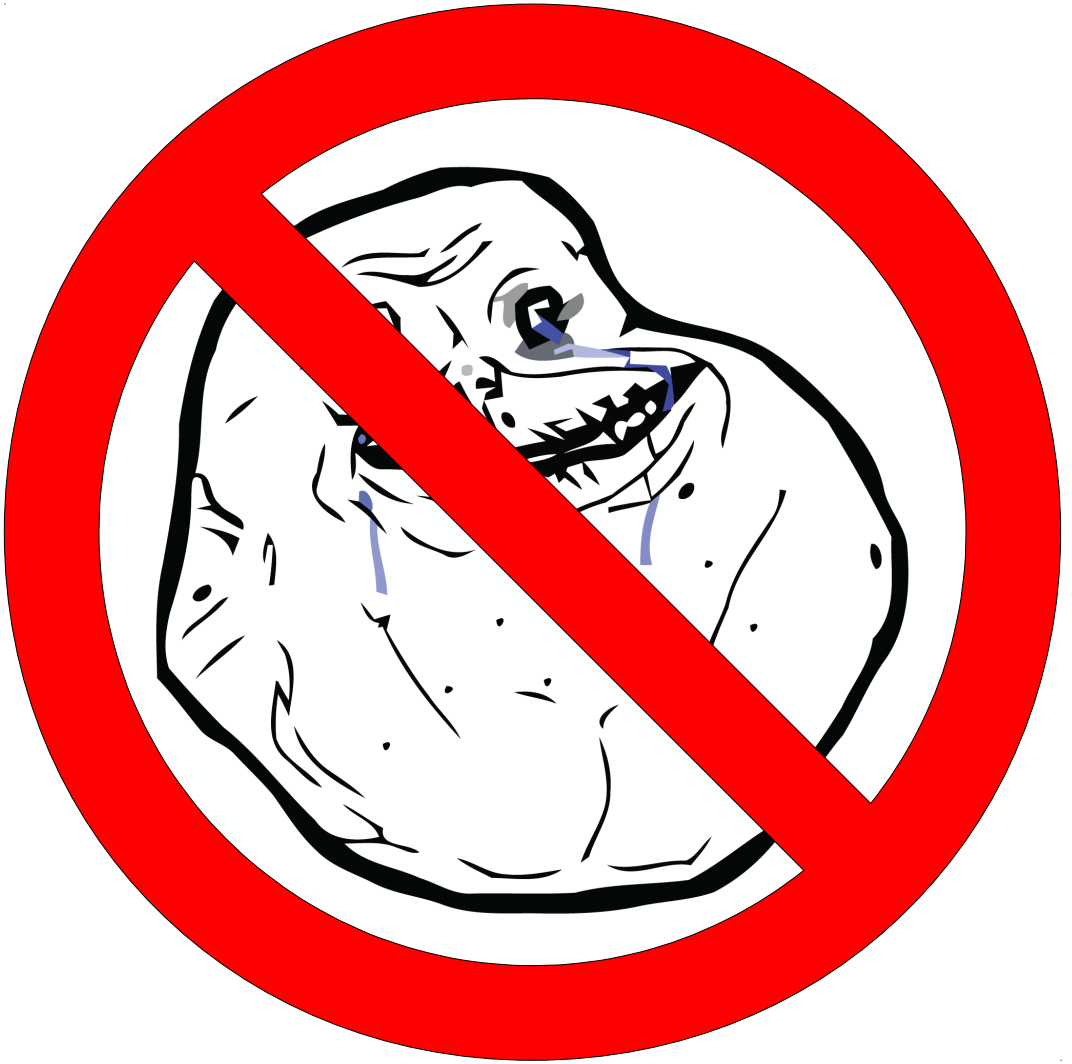

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Friday Four Square

- Good snacks!
- Good company!
- Good game!
- Good fun!
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Mathematical Proofs

Structure of a Proof

- Begin with a set of initial assumptions (**hypotheses**).
- Through creativity, use logical reasoning to derive the final result (the **conclusion**) from the hypotheses.
- Assuming that all intermediary steps are sound logical reasoning, the conclusion **follows** from the hypotheses.

Proofs are Wonderful

- End up with **incontrovertible evidence** that your claim is correct.
- Develop a **better understanding** of the underlying structure of the problem.
  - Exploit interesting properties of the hypotheses to derive a final result.
  - See the connection between your proof and other similar proofs to arrive at a more general result.

Proofs are Hard

- A **single incorrect step** can render an entire proof invalid.
- Can't “**debug**” a proof as you would a program.
- Proofs are much harder to **synthesize** than to **follow**.

Looking Forward

- **Basic Proof Techniques** (Today/Monday)
  - What do proofs look like?
  - What strategies can be used to prove a result?
- **Proof by Induction** (Monday/Wednesday)
  - How do you prove properties of complex structures?
- **Application: Dijkstra's Algorithm** (Wednesday)
  - How do proofs help you better understand something?
- **Formal Logic** (Friday/Monday)
  - What are the formal laws governing proofs?

Direct Proofs

Direct Proofs

- A **direct proof** is the simplest type of proof.
- Starting with an initial set of hypotheses, apply simple logical steps to prove the conclusion.

Two Quick Definitions

- An integer is **even** if it can be written as 2k for some integer k.
- An integer is **odd** if it can be written as 2k + 1 for some integer k.
- Every integer is either even or odd.
- No integer is both even and odd.

A Simple Direct Proof

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This symbol means "end of proof"

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Proofs with Implications

- To prove a statement of the form “if P is true, Q is true:”
  - Assume that P is true.
  - Using P and other hypotheses, show that Q is true as well.
- In the previous proof, we **assumed** that n was even, then used that to prove that n^2 is even.

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If you want to prove that P implies Q, assume P and prove Q.

Don't assume Q and then prove P!

A More Elaborate Direct Proof

- Suppose that we have two equivalence relations $(A, =_A)$ and $(B, =_B)$.
- Define a new relation $(A \times B, =_{A \times B})$ as
  - $(a_1, b_1) =_{A \times B} (a_2, b_2)$ if $a_1 =_A a_2$ and $b_1 =_B b_2$.
- Examples:
  - Are two shapes the same color and size?
  - Did two Olympic teams earn the same number of gold and silver medals?
- **Prove**: $(A \times B, =_{A \times B})$ is an equivalence relation.

Recall

- An equivalence relation is a relation that is
  - **Reflexive**: For any a, aRa
  - **Symmetric**: If aRb, bRa.
  - **Transitive**: If aRb and bRc, aRc.
- Using the definition, let's prove that $=_{A \times B}$ is an equivalence relation.

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To see that it is reflexive, note that for any $(a, b) \in A \times B$, $a =_A a$ and $b =_B b$ because $=_A$ and $=_B$ are reflexive. Thus $(a, b) =_{A \times B} (a, b)$.

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To see that it is symmetric, note that if $(a_1, b_1) =_{A \times B} (a_2, b_2)$, then $a_1 =_A a_2$ and $b_1 =_B b_2$. Since $=_A$ and $=_B$ are symmetric, $a_2 =_A a_1$ and $b_2 =_B b_1$, so $(a_2, b_2) =_{A \times B} (a_1, b_1)$.

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To see that is is transitive, suppose that $(a_1, b_1) =_{AxB} (a_2, b_2)$ and that $(a_2, b_2) =_{A \times B} (a_3, b_3)$. Thus $a_1 =_A a_2$ and $a_2 =_A a_3$, and because $=_A$ is transitive, $a_1 =_A a_3$. Similarly, $b_1 =_B b_2$ and $b_2 =_B b_3$, and because $=_B$ is transitive, $b_1 =_B b_3$. Consequently, $(a_1, b_1) =_{AxB} (a_3, b_3)$ as required.

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Proofs with Quantifiers

- To prove a statement of the form “for any x, P(x)”:
  - Assign a name to some object representing an arbitrary choice of x.
  - This does **not** mean “choose an arbitrary x;” rather, the symbol x stands in for any choice of x that we could ever make.
  - Prove that P(x) is true under the assumption that x could be anything and the choice is out of your control.

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Proof: Consider an arbitrary natural number, say, 16. 16 is even, and it has no odd divisors. Since our choice was arbitrary, for any arbitrary n, if n is even, n has no odd divisors. ■

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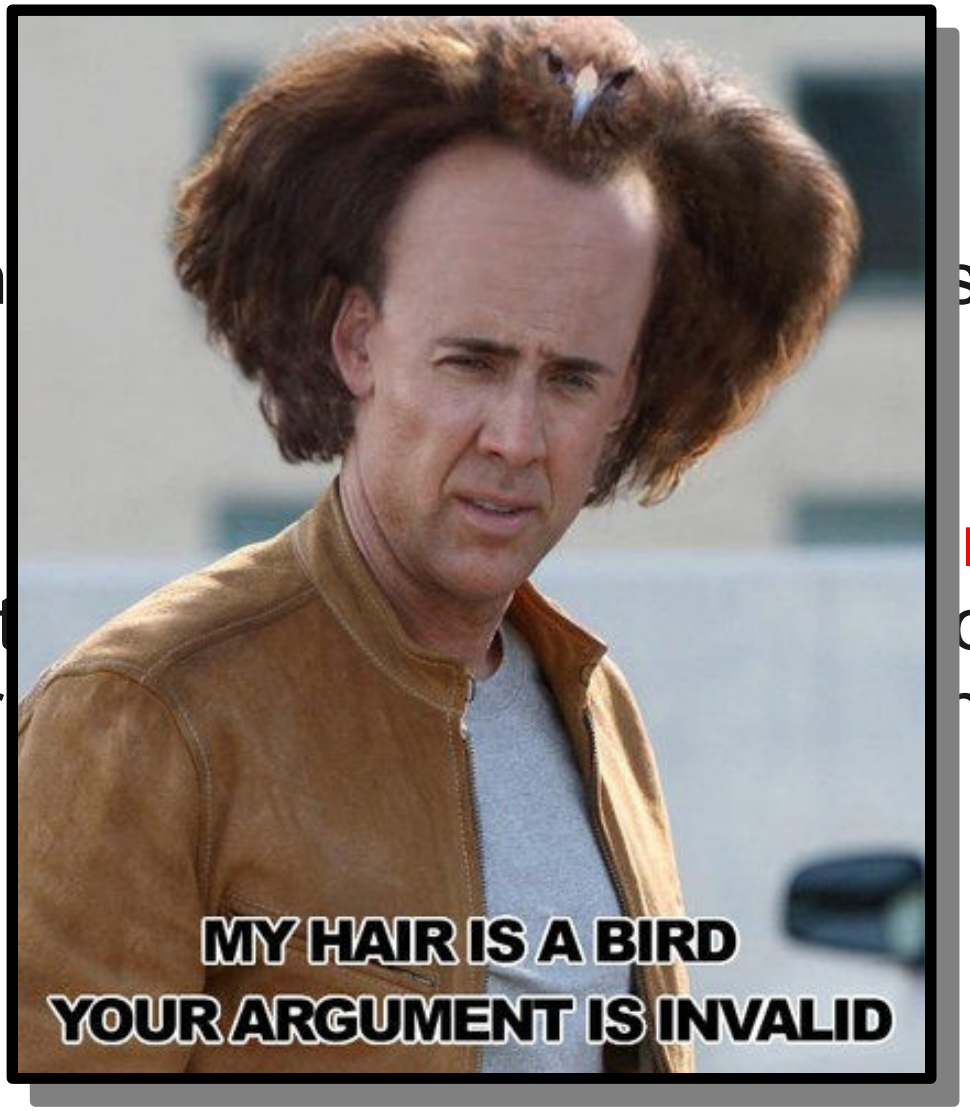

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adjective /ˈärbiˌtrerē/

1. Based on random choice or personal whim, rather than any reason or system - *“his mealtimes were entirely arbitrary”*

2. *(of power or a ruling body)* Unrestrained and autocratic in the use of authority - *“arbitrary rule by King and bishops has been made impossible”*

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Use this definition

Source: Google

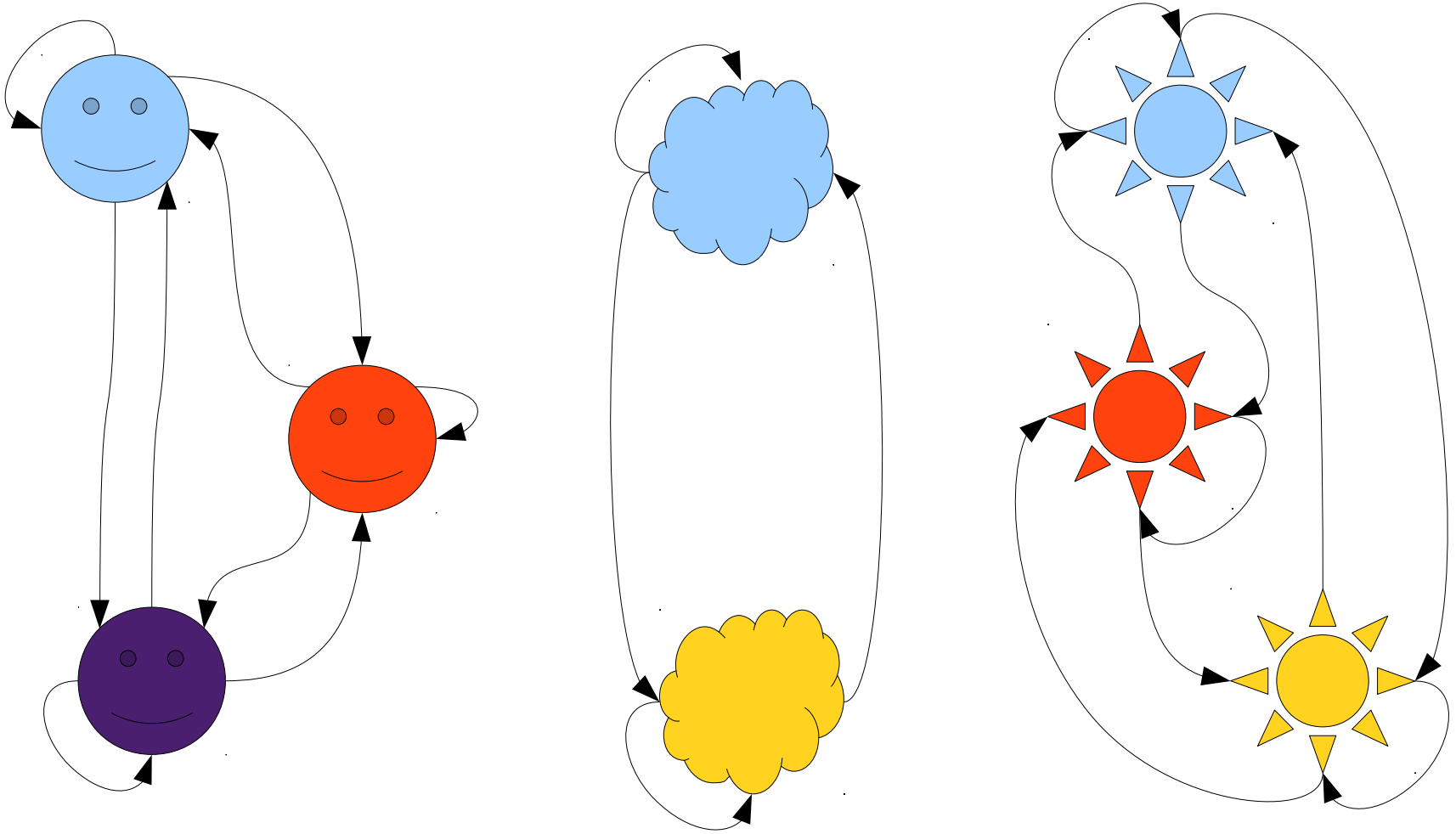
To prove something is true for all x, **do not** choose an x and base the proof off of your choice!

Instead, leave x unspecified and show that no matter what x is, the specified property must hold.

Making Arguments Readable

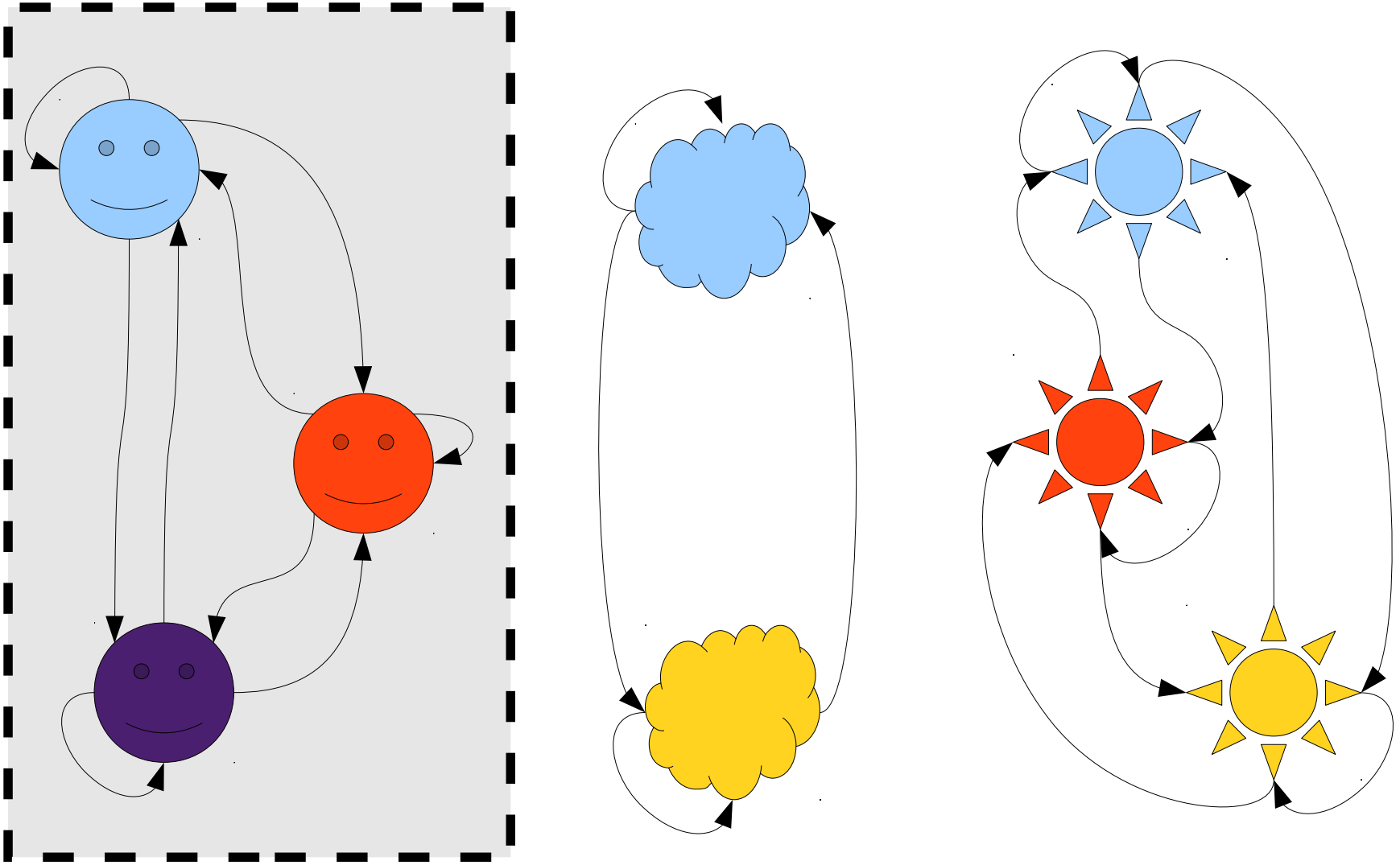
- Often, to prove some result, you must prove a weaker result first.
- From the weaker result, you then prove the final result.
- The weaker result is often called a **lemma**.
- Most lemmas are specific to a particular proof, though some lemmas are quite famous.
  - The **pumping lemma**.
  - **Schwarz's lemma** (no relation).

Equivalence Classes



xRy ≡ x and y are the same shape.

Equivalence Classes



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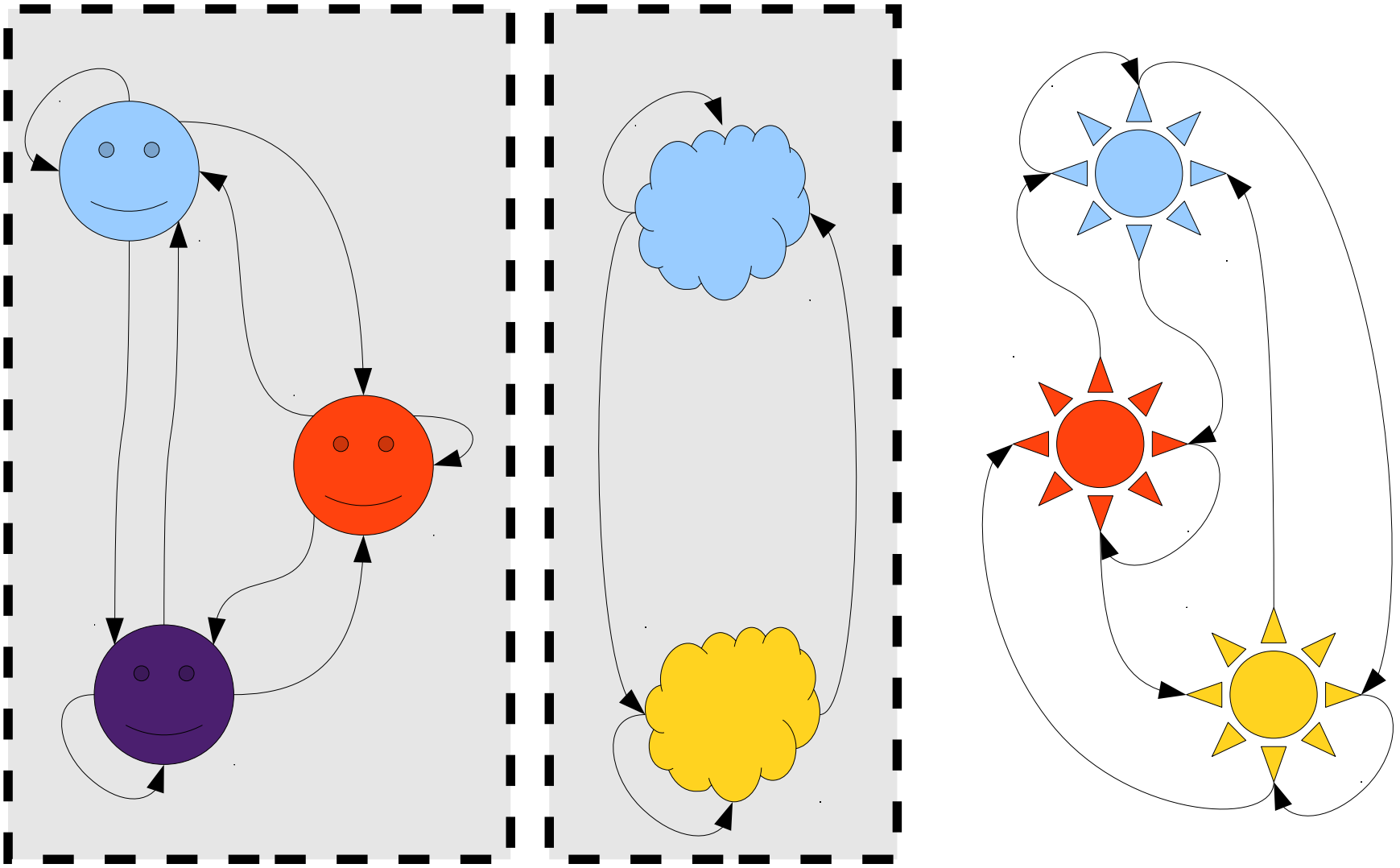

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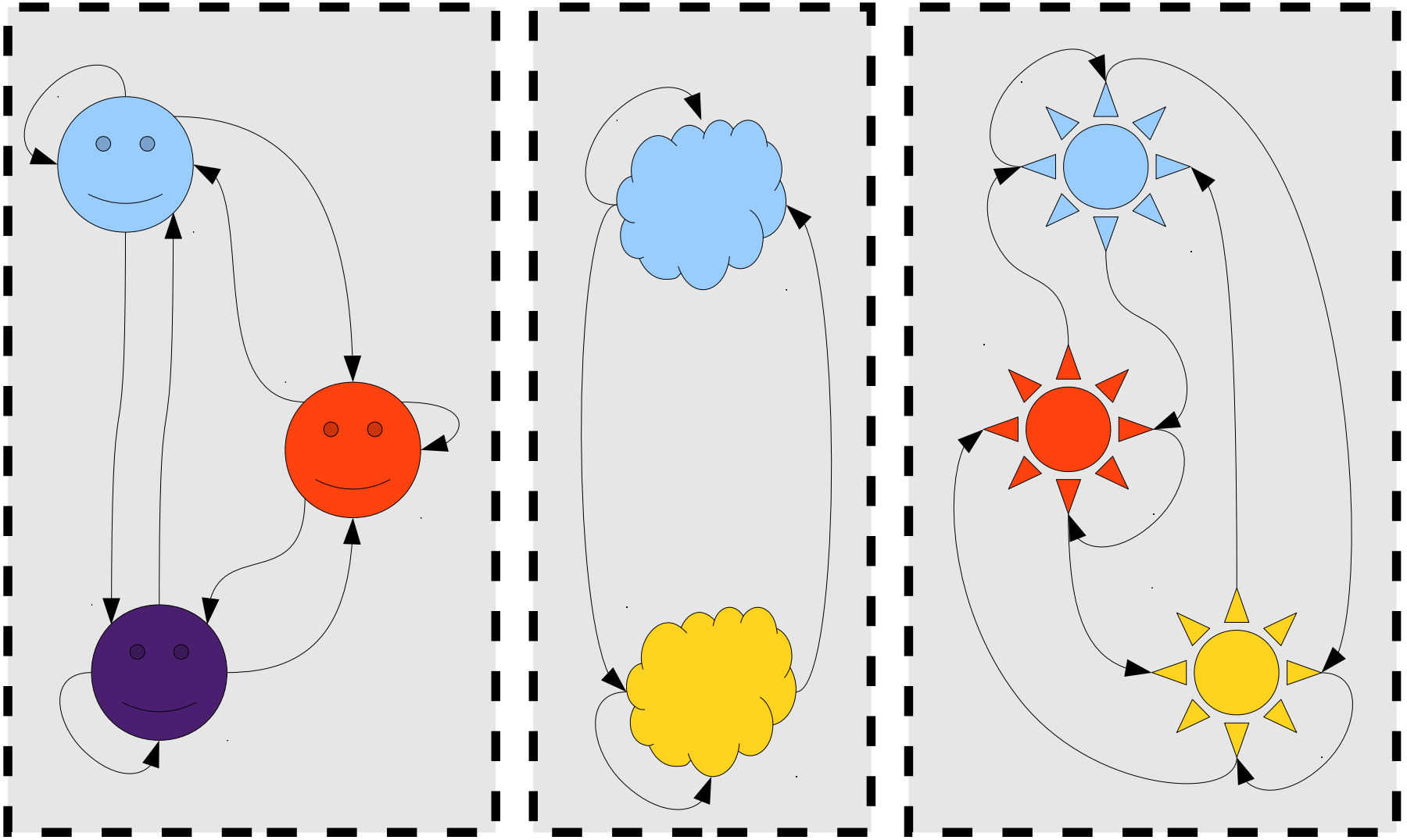

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Proofs With Lemmas

Theorem: Let R be an equivalence relation over A. Then if $x \in [a]_R$ and $x \in [b]_R$, then $[a]_R = [b]_R$.

(Recall:

$$[a]_R = \{ x \mid aRx \}$$

that is, the set of elements equal to a)

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Pro tip: This is a standard technique for proving two sets are equal. You can use this result without having to prove it yourself.

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Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A **proof by contradiction** is a proof that works as follows:
  - To prove that P is true, assume that P is not true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only option is that the assumption that P is not true is incorrect.
  - Conclude, therefore, that P is true.

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