CS103 Midterm Exam

This midterm exam is open-book, open-note, open-computer, but closed-network. This means that if you want to have your laptop with you when you take the exam, that's perfectly fine, but you **must not** use a network connection. You should only use your computer to look at notes you've downloaded in advance. Although you may use laptops, you **must** hand-write all of your solutions on this physical copy of the exam. No electronic submissions will be considered without prior consent of the course staff.

SUNetID: ______________________________

Last Name: ______________________________

First Name: ______________________________

I accept both the letter and the spirit of the honor code. I have not received any assistance on this test, nor will I give any.

(signed) ____

You have three hours to complete this midterm. There are 180 total points, and this midterm is worth 15% of your total grade in this course. The first three problems are shorter and simpler than the last three, so be sure to allocate your time appropriately. You may find it useful to read through all the questions to get a sense of what this midterm contains.

Good luck!

Question		**Points**	**Grader**
(1) First-Order Logic	(20)	/20	
(2) Finding Flaws in Proofs	(20)	/20	
(3) Finite Automata	(20)	/20	
(4) Utopian Tournament Graphs	(40)	/40	
(5) The Well-Ordering Principle	(40)	/40	
(6) Pigeonhole Party!	(40)	/40	
	(180)	**/180**	

Problem 1: Translating into Logic **(20 points total)**

In each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. The statement you write can use any first-order construct (equality, connectives, quantifiers, etc.), but you must only use the predicates and functions provided.

As an example, if you were given the predicate *Integer(x)*, which returns whether x is an integer, and the function *Plus(x, y)*, which returns x + y, you could write the statement "there is some even integer" as

$$\exists n. \exists k. (Integer(n) \land Integer(k) \land Plus(k, k) = n)$$

since this asserts that some number n is equal to $2k$ for integer k. However, you could not write

$$\exists n. (Integer(n) \land Even(n))$$

because there is no *Even* predicate.

(i) Never Gonna Give You Up **(5 Points)**

Given the predicate

Knows(x, y), which says that x and y know each other

and the constant symbols *me*, *you*, *love*, and *rules*, write a statement in first-order logic that says "We're no strangers to love. You know the rules, and so do I." You can assume that if x does not know y, then x and y are strangers.

(ii) Gotta Catch 'em All! **(5 Points)**

Given the predicates

> *WantsToBeBetterThan(x, y)*, which says that *x* wants to be better than *y*,
> *WasBetterThan(x, y)*, which says that *x* was, in the past, better than *y*.

and the constant symbol *me*, write a statement in first-order logic that says "I want to be the very best, like no one ever was." (That is, I want to be better than everyone else, and in the past no one was better than everyone else.)

iii) Good Advice **(10 Points)**

Given the predicates

> *Fools(x, y, t)*, which says that *x* fools *y* at time *t*,
> *Person(x)*, which says whether *x* is a person, and
> *Time(t)*, which says whether *t* is a time,

along with the constant *you*, write a statement in first-order logic that says "you can fool some people all the time or all the people some of the time, but not all the people all the time." To clarify, the statement "all the people some of the time" doesn't necessarily mean that there is some instant in time at which you can fool everyone; it just says that for any person, you can fool them at some point in time.

Problem 2: Finding Flaws in Proofs **(20 points)**

Consider the following modification of the *RS(x, y)* function from the third problem set:

$$RS^?(x, y) = \begin{cases} 1 & \text{if } y = 0 \\ RS^?(x, \frac{y}{2})^2 & \text{if } y > 0 \text{ and } y \text{ is even} \\ \frac{1}{x} RS^?(x, \frac{y+1}{2})^2 & \text{otherwise} \end{cases}$$

This function is *completely wrong* and, in most cases, does not correctly compute x^y. Below is a purported proof that this function does compute the correct value:

Theorem: $RS^?(x, y) = x^y$ for all $x \in \mathbb{R}$, $y \in \mathbb{N}$, where $x \neq 0$.

Proof: By strong induction. Let P(y) be "for all $x \in \mathbb{R}$, if $x \neq 0$, then $RS^?(x, y) = x^y$." We prove that P(y) is true for all $y \in \mathbb{N}$. As our base case, we prove P(0), that for any nonzero $x \in \mathbb{R}$, $RS^?(x, 0) = x^0$. Since $RS^?(x, 0) = 1 = x^0$ by definition, this is true.

For the inductive step, assume that for some y, for all natural numbers y' such that $0 \leq y' \leq y$, P(y') is true, so for any nonzero $x \in \mathbb{R}$, $RS^?(x, y') = x^{y'}$. We prove that P(y + 1) is true, that for all nonzero $x \in \mathbb{R}$, $RS^?(x, y + 1) = x^{y+1}$. We consider two cases:

Case 1: $y + 1$ is even. Then $RS^?(x, y+1) = RS^?(x, \frac{y+1}{2})^2$. By the inductive hypothesis, $RS^?(x, \frac{y+1}{2}) = x^{\frac{y+1}{2}}$, so $RS^?(x, y+1) = RS^?(x, \frac{y+1}{2})^2 = (x^{\frac{y+1}{2}})^2 = x^{y+1}$ as required.

Case 2: $y + 1$ is odd. Then $RS^?(x, y+1) = \frac{1}{x} RS^?(x, \frac{y+2}{2})^2$. By the inductive hypothesis, $RS^?(x, \frac{y+2}{2}) = x^{\frac{y+2}{2}}$, so $RS^?(x, y+1) = \frac{1}{x} RS^?(x, \frac{y+2}{2})^2 = \frac{1}{x}(x^{\frac{y+2}{2}})^2 = \frac{1}{x} x^{y+2} = x^{y+1}$ as required.

Thus in either case $RS^?(x, y + 1) = x^{y+1}$, so P(y + 1) is true, completing the proof by induction. ■

Feel free to tear out this page as a reference.

(i) Does Not Compute **(5 Points)**

The $RS^?$ function does not correctly compute x^y for most choices of x and y. Give an example of a choice of x and y where $x \neq 0$ and $RS^?(x, y)$ does not correctly compute x^y.

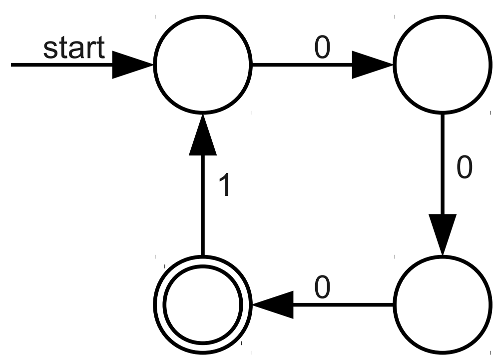
(ii) Your Argument is Invalid **(15 Points)**

The above proof is incorrect. What is wrong with its logic? It is **not enough** to simply state that the proof is incorrect or to give a counterexample; instead, cite the specific part of the proof that is incorrect and explain what logical error is being made.

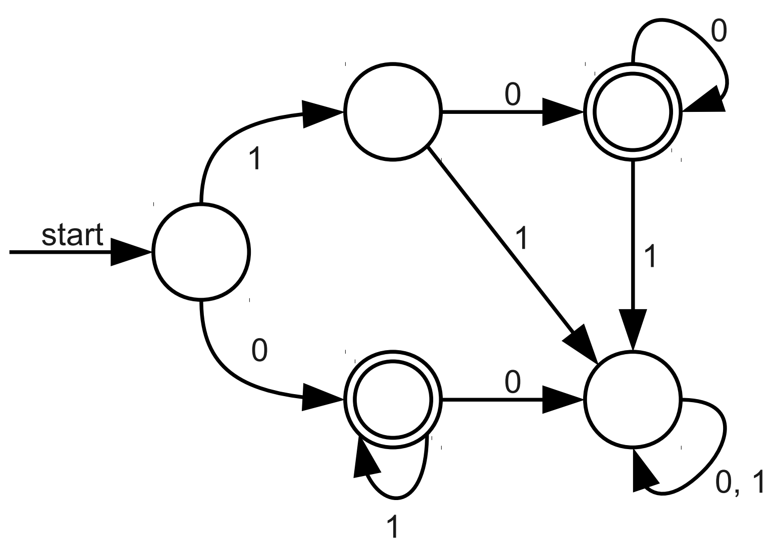
Problem 3: Finite Automata **(20 points total)**

Below are four finite automata, some of which are DFAs and some of which are not. For each automaton, state whether or not it is a DFA. If it is not, explain why that automaton is not a DFA. You do not need to provide an explanation if the automaton is a DFA. You may assume that the language is $\Sigma = \{0, 1\}$.

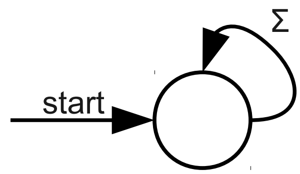
(i) The Preantepenultimate Automaton **(5 Points)**



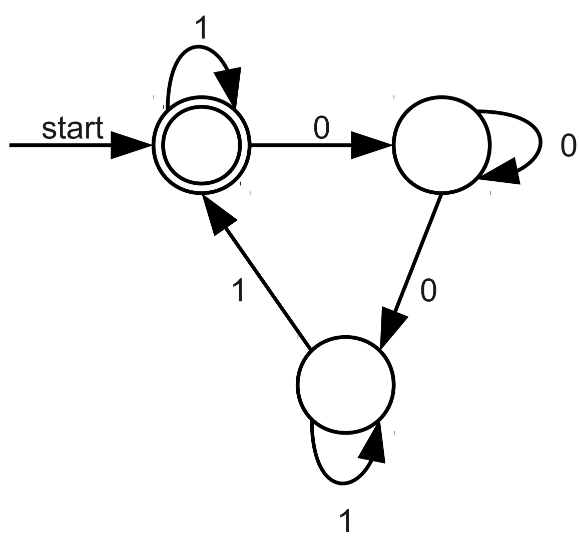
(ii) The Antepenultimate Automaton **(5 Points)**



(iii) The Penultimate Automaton **(5 Points)**



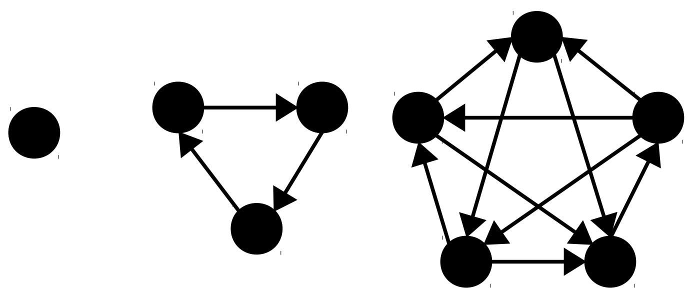
(iv) The Ultimate Automaton **(5 Points)**



Problem 4: Utopian Tournament Graphs **(40 points total)**

Recall from the second problem set that a *tournament graph* is a graph representing the outcome of a tournament with n > 0 players, in which each player plays each other player exactly once. Each game has a winner and a loser, and there are no draws. A tournament graph is a graph of the outcome of the tournament, where each node corresponds to a player and each edge (u, v) means that player u won her game against player v. In the second problem set, you proved that in any tournament graph, there is at least one tournament winner (a player who, for each other player, either won her game against that player, or won a game against someone who in turn beat that player).

It is possible to construct tournament graphs with more than one tournament winner, and in fact it's possible to construct tournament graphs where *everyone* is a winner. For example, here are tournament graphs with 1, 3, and 5 nodes where each player wins:



Prove that for any odd natural number *n*, there is at least one tournament graph for *n* players such that every player is a tournament winner.

(more space for your answer to Problem 4, in case you need it)

Problem 5: The Well-Ordering Principle **(40 Points)**

On the third problem set, you explored the *well-ordering principle*, which states that any nonempty set of natural numbers, whether finite or infinite, contains some smallest natural number. Here, you will explore some other applications of the well-ordering principle.

Suppose that we have two ordered sets $(A, <_A)$ and $(B, <_B)$, where $<_A$ and $<_B$ are strict orders. A **homomorphism from A to B** is a function $f : A \to B$ with the property that

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 <_A a_2 \to f(a_1) <_B f(a_2))$$

That is, if one element of A is less than some other element of A, then after applying *f* to both of those elements the image of the first element is still less than the image of the second element. If *f* is a homomorphism from A to B, we say that A is **homomorphic** to B.

Let $\mathbb{Z}^-$ be the set of negative integers. That is, $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$.

(i) Integer Homomorphisms **(10 Points)**

Consider the set $\mathbb{Z}^-$ ordered by the greater-than relation >. Show that $(\mathbb{Z}^-, >)$ is homomorphic to $(\mathbb{N}, <)$ by giving an example of a function $f : \mathbb{Z}^- \to \mathbb{N}$ such that if z_0 and z_1 are negative integers with $z_0 > z_1$, then $f(z_0) < f(z_1)$. Then prove that your function *f* is a homomorphism.

Suppose that $(S, <_S)$ is a strictly, totally ordered set; that is, $<_S$ is a strict total order over S. Given a nonempty subset $T \subseteq S$, we say that some element $t \in T$ is a **least element** of T if for all other $t' \in T$, $t <_S t'$.

(ii) Homomorphisms and Well-Orderings (10 Points)

Suppose that some set $(S, <_S)$ is homomorphic to $(\mathbb{N}, <)$. Prove that any nonempty subset of S contains a least element.

On the third problem set, you proved the well-ordering principle must using strong induction. In the third problem session, we saw how strong and weak induction are equivalent to one another. In this problem, you'll prove the principle of mathematical induction using the well-ordering principle. This shows that weak induction, strong induction, and the well-ordering principle are all equivalent to one another.

(iii) Well-Ordering and Induction **(20 Points)**

Suppose that you have some property P(n) where:

- P(0)
- $\forall n \in \mathbb{N}. (P(n) \rightarrow P(n + 1))$

Prove, using the well-ordering principle, that P(n) is true for all natural numbers n. *(Hint: Suppose that P(n) is not true for all natural numbers n, and consider the set of natural numbers for which it is false.)*

(More space for 5.iii, if you need it)

Problem 6: Pigeonhole Party! **(40 points total)**

Consider a party in which any pair of people are either *acquaintances* (they have met before) or *strangers* (they have never met before). A group of people are all *mutual acquaintances* if everyone in that group knows everyone else, and a group of people are all *mutual strangers* if no one in the group knows anyone else. On the last problem set you used the pigeonhole principle to prove that at a party of six people, there are at least three mutual acquaintances or at least three mutual strangers. Now, you'll consider a substantial generalization of this result.

In this problem, you will prove that there is a function T(a, s) such that in any group of T(a, s) people, there are at least a mutual acquaintances or at least s mutual strangers. This function is defined as follows:

$$T(a,s)=\begin{cases} 1 & \text{if } a=1 \\ 1 & \text{if } s=1 \\ T(a-1,s)+T(a,s-1) & \text{otherwise} \end{cases}$$

Let $\mathbb{N}^+$ be the set of positive natural numbers. In order to prove that the above result is true, we need to show that for any choice of $a, s \in \mathbb{N}^+$, in any group of T(a, s) people, there are at least a mutual acquaintances or at least s mutual strangers. To do this, we'll use well-founded induction over $\mathbb{N}^+ \times \mathbb{N}^+$, using the $<_{lex}$ relation you explored in the problem set. The structure of the proof is as follows:

Theorem: In any group of T(a, s) people, there are at least a mutual acquaintances or at least s mutual strangers.

Proof: Let P(a, s) be "In any group of T(a, s) people, there are at least a mutual acquaintances or at least s mutual strangers." We prove P(a, s) holds for all $(a, s) \in \mathbb{N}^+ \times \mathbb{N}^+$ by well-founded induction, using the $<_{lex}$ relation over $\mathbb{N}^+ \times \mathbb{N}^+$.

Assume that for some $(a, s) \in \mathbb{N}^+ \times \mathbb{N}^+$, that for any $(a', s') \in \mathbb{N}^+ \times \mathbb{N}^+$ with $(a', s') <_{lex} (a, s)$, we know that P(a', s') holds. We need to prove P(a, s). We consider three cases:

Case 1: $a = 1$. ***You will prove P(a, s) is true in this case.***
Case 2: $s = 1$. ***You will prove P(a, s) is true in this case.***
Case 3: $a > 1$ and $s > 1$. ***You will prove P(a, s) is true in this case.***

Feel free to tear this page out as a reference.

(i) Forever Alone **(5 Points)**

Prove that Case 1 is true; namely, that if $a = 1$, then in any group of $T(a, s) = T(1, s) = 1$ people, there is a group of at least one mutual acquaintance or there is a group of at least s people that are all mutual strangers. The proof for Case 2 is analogous to this one, so we won't ask you to prove it here. *(Hint: If there's only one person in the group, what do you know about the statement "that person knows everyone else in the group?")*

This brings us to Case 3, which is the most interesting part of the proof. In order to prove this result, we will need to prove a slight variant on the pigeonhole principle.

(ii) A Modified Pigeonhole Principle **(10 Points)**

Suppose that you have two bins labeled X and Y. Prove that if you distribute at least $m + n - 1$ objects into the two bins, then either bin X will have at least m objects in it or bin Y will have at least n objects in it.

If we are in Case 3, then we need to show that in a group with T(a, s) = T(a – 1, s) + T(a, s – 1) people in it, there are at least a mutual acquaintances or at least s mutual strangers.

(iii) Applying the Result **(5 Points)**

Using your result from part (ii), show that in a group of T(a – 1, s) + T(a, s – 1) people, any person in the group has either T(a – 1, s) acquaintances (who may not be mutual acquaintances) or is a stranger to at least T(a, s – 1) people (who themselves might not be mutual strangers).

Your result from (iii) shows that if we pick any person from the group, that person has either T(a – 1, s) acquaintances or is strangers to T(a, s – 1) people. This next question asks you to finish the proof by using this fact to show that a group of T(a, s) people, where $a > 1$ and $s > 1$, must contain a group of at least a mutual acquaintances or a group of at least s mutual strangers. Remember that we have the inductive hypothesis working for us, so if we can find a group of T(a', s') people, where $(a', s') <_{lex} (a, s)$, then there must be at least a' mutual acquaintances or at least s' mutual strangers within that smaller group.

(iv) Completing the Proof **(20 Points)**

Prove that if some arbitrary person in a group of T(a, s) people has T(a – 1, s) acquaintances, then the group contains at least a mutual acquaintances or at least s mutual strangers. The case where some arbitrary person in the group is strangers to T(a, s – 1) people is analogous, so we won't ask you to prove it here.

(More space for 6.iv, if you need it)

(This section just shows off how cool of a result you've just proved; feel free to skip it)

This result is incredibly powerful – it shows that if you keep adding more and more people to a group, regardless of which people know one another, it is possible to guarantee that larger and larger groups of mutual acquaintances or mutual strangers must exist!

Here is a table of some values of T(*a*, *s*):

	s=1	**s=2**	**s=3**	**s=4**	**s=5**
a=1	1	1	1	1	1
a=2	1	2	3	4	5
a=3	1	3	6	10	15
a=4	1	4	10	20	35
a=5	1	5	15	35	70

So, for example, any group of twenty people must have at least four mutual acquaintances or four mutual strangers, and any group of ten people must have either four mutual acquaintances or three mutual strangers.