Practice CS103 Midterm Exam

This midterm exam is open-book, open-note, open-computer, but closed-network. This means that if you want to have your laptop with you when you take the exam, that's perfectly fine, but you must **not** use a network connection. You should only use your computer to look at notes you've downloaded in advance.

This practice midterm contains questions that are similar to the ones that may appear on the actual midterm. You will have three hours to complete the midterm, and I **strongly** suggest taking this midterm under similar time conditions. There are 180 total points, and the midterm is worth 15% of your total grade in this course.

Normally, I would leave extra space between problems so that you would have room to write out your answers, but to save paper I have tried to minimize the amount of blank space in this handout. You do not need to bring extra scratch paper to the midterm, but I would suggest doing so in case you want to try out various solutions to the problem.

Good luck!

Question	Points
(1) First-Order Logic	/20
(2) Finding Flaws in Proofs	/20
(3) Irrational Numbers Revisited	/40
(4) Graph Coloring	/40
(5) Euclid's Algorithm	/40
(6) Finite Automata	/20
	/180

Problem 1: Translating into Logic **(20 points total)**

In each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. The statement you write can use any first-order construct (equality, connectives, quantifiers, etc.), but you must only use the predicates and functions provided.

As an example, if you were given just the predicate *Integer(x)*, which returns if x is an integer, and the function *Plus(x, y)*, which returns x + y, you could write the statement "there is some even integer" as

$$\exists n.\ \exists k.\ (\mathit{Integer}(k) \wedge \mathit{Plus}(k, k) = n)$$

since this asserts that some number n is equal to $2k$ for integer k. (We could have included the fact that n should be an integer as well, but this is implicit from the fact that $2k$ is an integer). However, you could not write

$$\exists n.\ (\mathit{Integer}(n) \wedge \mathit{Even}(n))$$

because there is no *Even* predicate. The point of this question is to get you to think how to express certain concepts in first-order logic given a limited set of predicates, so feel free to write any formula you'd like as long as you don't invent your own predicates or functions.

(i) Medical Advice **(10 Points)**

Given the predicates

Apple(x), which says that *x* is an apple,
Day(x), which says that *x* is a day,
Eats(x, y, z), which says that *x* eats *y* on day *z*, and
Near(x, y), which says that *x* is near *y*,

and the constant symbols *me* and *doctor*, write a statement in first-order logic that says "An apple a day keeps the doctor away." That is, if you eat an apple every day, then the doctor will stay away from you.

(ii) Newtonian Physics **(10 Points)**

Given the predicate

Action(x), which says that *x* is an action,

and the function

Opposite(x), which returns the opposite of *x*,

write a statement in first-order logic that says "for every action there is an equal and opposite reaction." That is, every action is accompanied by another action that is its opposite.

Problem 2: Finding Flaws in Proofs **(20 points)**

The following proofs each contain errors that let them prove results that are patently false. For each of these proofs, state what logical error the proof makes and explain why it is wrong.

(i) A Hairy Result **(5 Points)**

Theorem: For every resident of San Jose, there is some other resident of San Jose with the same number of hairs on their head as the first person.

Proof: The maximum number of hairs ever found on a human head is less than 400,000, and the population of San Jose is (as of 2011) around 900,000. By the generalized pigeonhole principle, since there are 900,000 people to distribute into 400,000 buckets, every bucket must contain at least two people. Thus for any resident of San Jose, there is some other resident of San Jose with the same number of hairs on their head. ■

(ii) Real-ity Check **(5 Points)**

Theorem: $|\mathbb{R}| \leq |\mathbb{N}|$

Proof: We give an injective function $f : \mathbb{N} \to \mathbb{R}$, which proves that $|\mathbb{R}| \leq |\mathbb{N}|$. Let the function $f : \mathbb{N} \to \mathbb{R}$ be defined as $f(x) = x$. This function is injective, since if $f(x_0) = f(x_1)$, then $x_0 = x_1$. ■

(iii) Horses Revisited **(10 Points)**

As you saw in the second problem set, the proof that all horses are the same color broke down because the inductive step did not apply to the case where there are two horses. Here is an attempt to revise that proof so that the particular error is gone.

Theorem: Any nonempty set of horses that does not have exactly two horses in it contains horses that are all the same color.

Proof: By induction. Let H(n) be "Either n = 2, or for any set of horses of size n, all horses in that set are the same color." We prove that H(n) holds for all natural numbers greater than zero by induction on n.

For the base case, we prove H(1), that every horse in any nonempty set of horses of size 1 has the same color. But in a set of horses with just one element, all horses in the set are trivially the same color and H(1) holds.

For the inductive step, assume that H(n) holds (any set of n horses are all the same color) and consider H(n + 1). If n + 1 = 2, then H(n + 1) is trivially true and we are done. Otherwise we need to show that any set of n + 1 horses are all the same color. We can see this as follows. Consider any set of n + 1 horses, then number the horses 1, 2, 3, …, n + 1. The set of the first n horses, by the inductive hypothesis, must all be the same color as one another (call it "color one"); likewise the set of the last n horses must be the same color as one another (call it "color two"). But these two sets overlap with one another, so some horse must be both color one and color two, which means that color one and color two are the same color. Consequently, all horses in the set are the same color, completing the proof of H(n + 1). ■

Problem 3: Irrational Numbers Revisited **(40 points total)**

In lecture, we said that a real number r is *rational* if it can be written as r = p / q for integers p and q such that q ≠ 0 and p and q have no common divisors other than ±1. This allowed us to prove that $\sqrt{2}$ and $\sqrt{3}$ were irrational by showing that any p / q that would be equal to these numbers would have to have a common divisor.

In some sense, this feels like cheating. The definition of "rational number" that you are probably more familiar with excludes the restriction that p and q have no common divisors, and says that a number is rational iff it can be written as p / q for integers p and q such that q ≠ 0. In this problem, you'll explore an alternative proof that $\sqrt{2}$ is irrational that uses this weaker definition of rationality.

For the purposes of this problem, suppose that we restrict ourselves to the positive real numbers. Denote the set of *positive* real numbers by $\mathbb{R}^+$ and the set of *positive* natural numbers by $\mathbb{N}^+$. We say that a positive real number $r \in \mathbb{R}^+$ is rational if there exist $p, q \in \mathbb{N}^+$ such that $p / q = r$. We will show using this new definition that $\sqrt{2}$ is irrational.

(i) Relating the Numerator and Denominator **(10 Points)**

Prove that if $p, q \in \mathbb{N}^+$ and $p / q = \sqrt{2}$, then $q < p < 2q$.

You've just shown that $q < p < 2q$. Now, let's go back to our original equation:

$$p / q = \sqrt{2}$$

We can rearrange this as follows:

$$p / q = \sqrt{2}$$
$$p = \sqrt{2}\, q$$
$$p^2 = 2q^2$$

Now, for a cute trick: let's suppose that we subtract pq from both sides of this equation:

$$p^2 = 2q^2$$
$$p^2 - pq = 2q^2 - pq$$
$$p(p - q) = q(2q - p)$$
$$p / q = (2q - p) / (p - q) = \sqrt{2}$$

Using your result from (i), we can see that that $p - q > 0$ and that $2q - p > 0$. This shows that the fraction $(2q - p) / (p - q)$ is well-defined and a ratio of positive natural numbers (since both p and q are positive natural numbers).

With the above result, we have shown that if we can find some positive natural numbers p and q such that $p / q = \sqrt{2}$, then we can also find two new natural numbers $p' = 2q - p$ and $q' = p - q$ such that $(2q - p) / (p - q) = p' / q' = p / q = \sqrt{2}$.

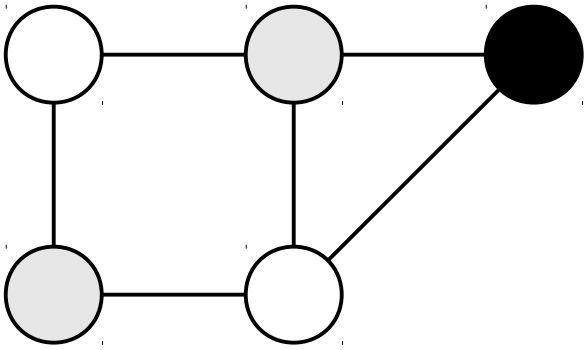
At this point we can make two important observations. From your result in (ii), we have that $2q - p > 0$, so the numerator of this new fraction is a positive natural number. Additionally, since in (i) you proved that $p < 2q$, we have that $p - q < q$. In other words, the new denominator is a positive natural number that is smaller than the old denominator. This means that if $\sqrt{2}$ is rational, we can always find a rational representation of $\sqrt{2}$ with a smaller positive denominator than before. This seems problematic, though, because by the well-ordering principle we know that there is no infinite decreasing sequence of natural numbers.

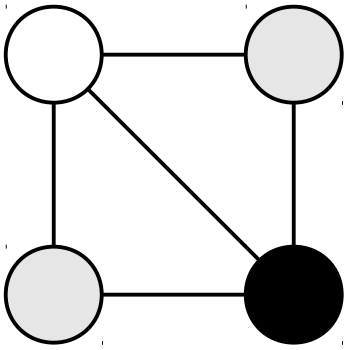
(ii) The Well-Ordering Principle **(30 Points)**

Using the above intuition as a starting point, prove that $\sqrt{2}$ is irrational using the well-ordering principle. The beauty of this proof is that it does not make any assumptions about the common divisors of p and q. You may use any of the results described above in your proof without needing to formally prove them.

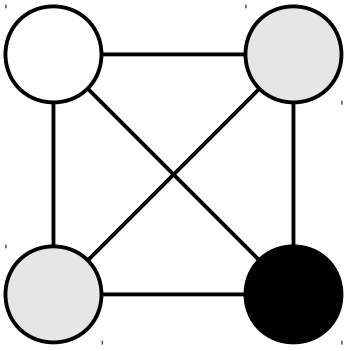
Problem 4: Graph Coloring **(40 points)**

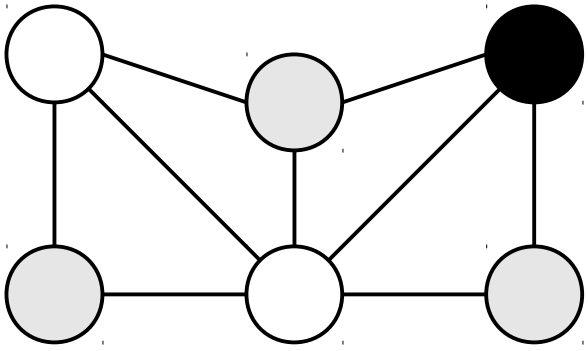
Given an undirected graph G, a **coloring** of G is a function that associates each node in the graph with a color such that no two nodes joined by an edge have the same color. For example, the following are legal colorings:





While the following are not legal colorings:





Any graph with no self-loops (edges from a vertex to itself) with n nodes can be colored with n colors by assigning each node a unique color, though some graphs can be colored with fewer colors than this. For example, some graphs (the **bipartite graphs**) can be colored with as few as two colors. The minimum number of colors with which it is possible to color a graph is called the graph's *chromatic number*.

Prove that if G is an undirected graph with no self-loops where each node has degree at most k (for some natural number k), then G can be colored with k + 1 colors.

Problem 5: Euclid's Algorithm **(40 points total)**

For any pair of integers x and y, a number d is a *common divisor* of x and y if d divides x and d divides y. That is, there are integers m and n such that $x = md$ and $y = nd$. If either $x \neq 0$ or $y \neq 0$, then the *greatest common divisor* of x and y is the largest number d that is a common divisor of x and y.

One of the oldest known algorithms is *Euclid's algorithm*, which is used to find the greatest common divisor of two integers. Euclid's algorithm is sometimes employed in RSA cryptography, which needs to search for numbers whose greater common divisor is 1. In this problem, you will explore Euclid's algorithm and will formally prove its correctness.

(i) Same Difference **(5 Points)**

Prove that if d is a common divisor of x and y, then d is a divisor of ax + by for any integers a and b.

(ii) The Division Algorithm **(5 Points)**

Recall from lecture that the *division algorithm* says that for any integers x and y, with $y \neq 0$, that x can be written as $x = qy + r$ for integers q and r such that $0 \leq r < y$.

Prove that d is a common divisor of x and y iff it is a common divisor of y and r.

The result you've just proven shows that the set of divisors of x and y is the same as the set of divisors of y and r (if $y \neq 0$). This means that the greatest common divisor of x and y must be the same as the greatest common divisor of y and r. Euclid's algorithm, which dates back almost 2300 years, is based on this principle. The algorithm is defined as follows:

$$gcd(x, y) = \begin{cases} x & \text{if } y = 0 \\ gcd(y, r) & \text{otherwise,} \end{cases}$$ where r is found by the division algorithm

In the remainder of this problem, you will prove that Euclid's algorithm is correct for all natural numbers x and y (except the special case where x = 0 and y = 0, where the greatest common divisor is not defined).

(iii) Proving the Base Case **(10 Points)**

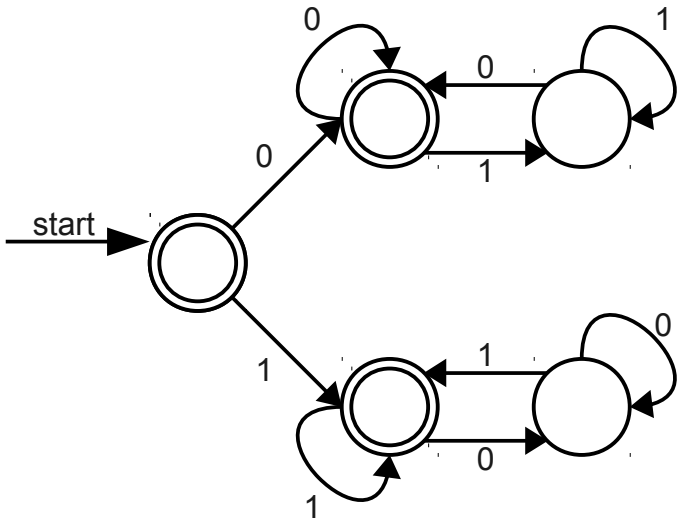
Prove that the greatest common divisor of x and 0 is x for any nonzero natural number x.

(iv) Verifying the Algorithm **(20 Points)**

Prove that *gcd(x, y)* returns the greatest common divisor of x and y, assuming that x and y are natural numbers and either $x \neq 0$ or $y \neq 0$. As a hint, try using strong induction on y.

Problem 6: Finite Automata **(20 points total)**

Below is a DFA over the alphabet {0, 1}. For each of the five strings below, does the DFA accept or reject the string?



i. 01010
ii. 000011
iii. 10001
iv. 1100
v. ε