Problem Set 1

This first problem set is designed to help you gain a familiarity with sets, relations, graphs, and proof techniques. There are a variety of problems here. Some are designed to get you playing around with the material, while others will push you to apply the techniques to more advanced domains.

Start this problem set early. It contains ten problems (plus one survey question), several of which require a fair amount of thought. I would suggest reading through this problem set at least once as soon as you get it to get a sense of what it covers.

As much as you possibly can, please try to work on this problem set individually. If you work too much in a group, you'll miss the chance to strengthen your mathematical muscles.* That said, if you do work with others, please be sure to cite who you are working with and on what problems. For more details, see the section on the honor code in the course information handout.

In any question that asks for a proof, you **must** provide a rigorous mathematical proof. You cannot draw a picture or argue by intuition. You should, at the very least, state what type of proof you are using, and (if proceeding by contradiction or contrapositive) state exactly what it is that you are trying to show. If we specify that a proof must be done a certain way, you must use that particular proof technique; otherwise you may prove the result however you wish.

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

This problem set has 125 possible points and eleven questions. It is weighted at 7% of your total grade. The earlier questions serve as a warm-up for the later problems, so do be aware that the difficulty of the problems does increase over the course of this problem set.

Good luck, and have fun!

Due October 7th at 2:15 PM

* Trust me, they exist. ☺

Problem One: Elementary Set Theory (4 points)

For the purposes of this problem, suppose that we are dealing with the following sets:

$A = \{ 1, 2, 3, 4 \}$
$B = \{ 2, 2, 2, 1, 4, 3 \}$
$C = \{ 1, 3 \}$
$D = \{ 2, 3, 4 \}$
$E = \{ x \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$

For each of the following, is the claim true or false? Explain why. You do not need to prove your assertions.

i. $A \neq B$
ii. $C \cup D \subseteq A$
iii. $A \cap C = C$
iv. $C \subseteq \mathbb{N} - E$

Problem Two: Finding Flaws in Proofs (12 points)

The following proofs all contain errors that allow them to prove results that are patently false. Before you go on to do proofs of your own, take the time to work through these incorrect proofs. For each proof, identify at least one flaw in the proof, explain what the problem is, then give a counterexample to the alleged theorem. In each case, **make sure you understand what logical error is being made**. The mistakes made here are extremely common. If you're having trouble with these questions, feel free to email us or stop by office hours.

Theorem: For any sets A and B, $A \cup B \subseteq A \cap B$
Proof: We need to show that for any arbitrary $x \in A \cup B$, $x \in A \cap B$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \in A \cup B$. ■

Theorem: If a binary relation R over the set A is not reflexive, then it is irreflexive.
Proof: Since R is not reflexive, there must be at least one $x \in A$ such that xRx does not hold. Since the choice of x was arbitrary, we must therefore have that for any $x \in A$, xRx does not hold. Thus R is irreflexive. ■

Theorem: For any set A, $\emptyset \subseteq A$ is false.
Proof: In order for $\emptyset \subseteq A$ to hold, we need to have that for every $x \in \emptyset$, $x \in A$. However, $x \notin \emptyset$ for every x, and so this is clearly impossible. Thus $\emptyset \subseteq A$ must be false. ■

Theorem: Every positive integer can be written as $2x + 3y$ for some positive integers x and y.
Proof: By contradiction; assume that no positive integer can be written as $2x + 3y$ for some positive integers x and y. But this is clearly false; for example, $50 = 2 \cdot 10 + 3 \cdot 10$. We have reached a contradiction, so our assumption must have been wrong, so any positive integer can be written as $2x + 3y$ for some positive integers x and y. ■

Problem Three: Multiples of Three (16 points)

A number is a *multiple of three* if it can be written as 3k for some integer k. A number is *congruent to one modulo three* if it can be written as 3k + 1 for some integer k, and a number is *congruent to two modulo three* if it can be written as 3k + 2 for some integer k. For each integer n, exactly one of the following is true:

- n is a multiple of three.
- n is congruent to one modulo three.
- n is congruent to two modulo three.

Suppose that we want to prove this result:

n is a multiple of three iff n^2 is a multiple of three.

To do this, we will prove the following two statements:

If n is a multiple of three, n^2 is a multiple of three.
If n^2 is a multiple of three, n is a multiple of three.

i. Prove the first of these statements with a direct proof.
ii. Prove the second of these statements using the contrapositive. Make sure that you explicitly state the contrapositive of the statement before you attempt to prove it.
iii. Prove, by contradiction, that $\sqrt{3}$ is irrational. Make sure that you explicitly state what assumption you are making before you derive a contradiction from it. Recall from lecture that a rational number is one that can be written as p/q for integers p and q where $q \neq 0$ and p and q have no common divisor other than ±1.

Problem Four: Modular Arithmetic (12 Points)

For any integer k, consider the relation $\equiv_k$, which is defined as follows:

$$a \equiv_k b \text{ if there exists an integer } q \text{ such that } a - b = kq$$

For example, $7 \equiv_3 4$, because $7 - 4 = 3 \cdot 1$, and $13 \equiv_4 5$ because $13 - 5 = 8 = 4 \cdot 2$. If $x \equiv_k y$, we say that *x is congruent to y modulo k*, hence the notation from the previous problem.

i. What three properties must a relation have to be an equivalence relation?
ii. Prove that, for any integer k, $\equiv_k$ is an equivalence relation by proving that it satisfies all three of the properties that you just listed.

Problem Five: Relations (12 points)

Below are four descriptions of binary relations. For each of the descriptions, provide an example of a set and a binary relation over that set that satisfies each of the properties, then prove why your relation has the indicated properties.

i. A relation that is neither symmetric nor antisymmetric.
ii. A relation that is neither reflexive nor irreflexive.
iii. A relation that is both symmetric and antisymmetric.
iv. A relation that is both reflexive and irreflexive.

Problem Six: Lexicographical Orderings (12 points)

Suppose that $(A, <_A)$ and $(B, <_B)$ are ordered sets such that $<_A$ is a strict order and $<_B$ is a strict order. Consider the set $A \times B$ and the relationship $<_{lex}$ on $A \times B$ defined as follows: Given pairs (a_1, b_1) and (a_2, b_2):

- If $a_1 <_A a_2$, then $(a_1, b_1) <_{lex} (a_2, b_2)$.
- Otherwise, if $a_1 = a_2$ and $b_1 <_B b_2$, then $(a_1, b_1) <_{lex} (a_2, b_2)$

Intuitively, you can think of this as follows. Compare the first elements of the two pairs. If the first element of the first pair is less than the first element of the second pair, then the first pair is less than the second pair. Otherwise, if the first elements are equal, then look at the second elements of each pair. If the second element of the first pair is less than the second element of the second pair, then the first pair is less than the second pair. This relation is called a **lexicographical ordering** and arises frequently in computer science.

Prove that $<_{lex}$ is a strict order. Remember that your proof needs to show that $<_{lex}$ is a strict order regardless of the choice of $(A, <_A)$ and $(B, <_B)$. *(Hint: Use the fact that* $(A, <_A)$ *and* $(B, <_B)$ *are strict orders to prove that* $<_{lex}$ *meets all the criteria necessary of a strict order).*

Problem Seven: Properties of Sets (12 points)

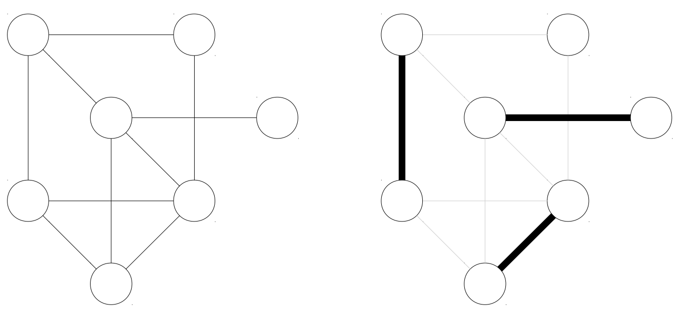
Below are three claims about sets. For each statement, if it is always true, prove it. If it is always false, prove that it is always false. If it is sometimes true and sometimes false, provide an example for which it is true and an example for which it is false and explain why your examples are correct.

To prove that two sets are equal, remember that you need to show that any element of the first set must also be an element of the second set and vice versa. Recall that this is equivalent to showing that the two sets are subsets of one another. It is **not** sufficient to use Venn diagrams or any other informal reasoning here. You need to formally prove each result.

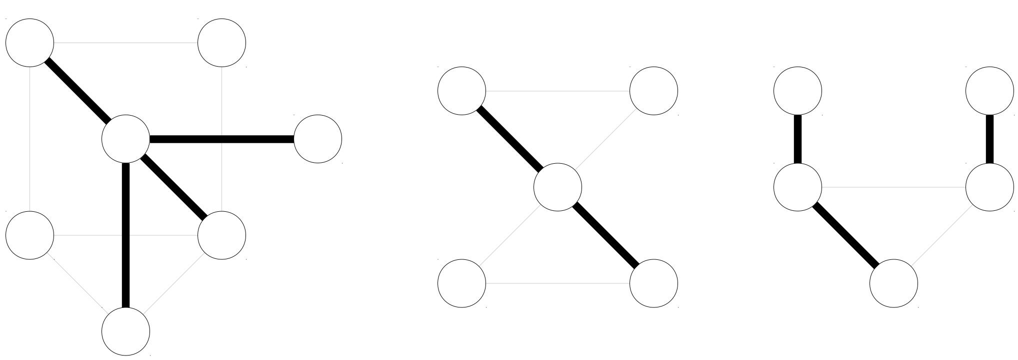
i. If $x \in A$, then $x \in \mathcal{P}(A)$
ii. $A \cap (B - A) \neq \emptyset$
iii. If $A \subsetneq B$, then $B - A \neq \emptyset$

Problem Eight: Matchings (12 Points)

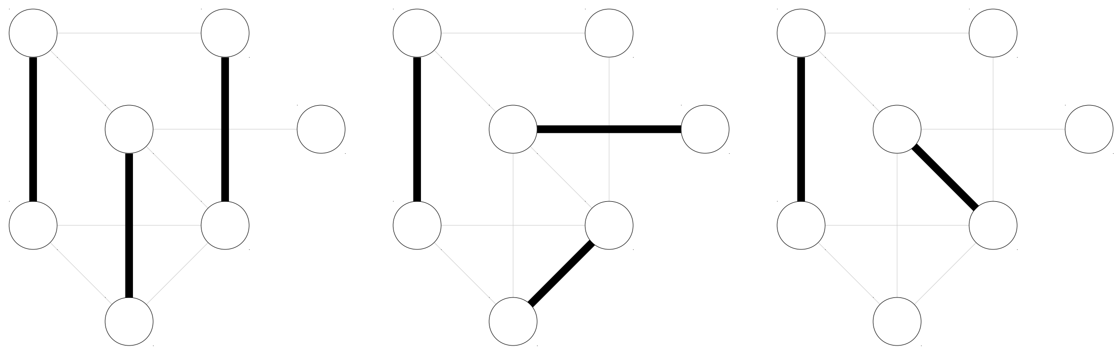
A **matching** in an undirected graph is a set of edges in the graph such that no two edges share an endpoint. Thus the following is a matching:



But these are not:



A graph may have multiple matchings in it, some of which may have more edges than others. For example, the following are all valid matchings in the given graph:



The **size** of a matching is the number of edges in it. A **maximum matching** is a matching in a graph such that no other matching has a larger size.

i. Find a graph with at least five nodes whose maximum matching has size n / 2, where n is the number of nodes in the graph.
ii. Find a graph with at least five edges whose maximum matching has size one.
iii. Prove that if a graph has a maximum matching of size zero, then it must not contain any edges.
iv. Are maximum matchings necessarily unique? That is, does every graph have exactly one maximum matching, or can there be many? If maximum matchings are unique, prove it. If not, give a counterexample.

Problem Nine: DAGs (12 Points)

Recall from lecture that a directed acyclic graph (DAG) is a directed graph that contains no cycles. A **finite DAG** is a DAG that has finitely many nodes.

In lecture, we discussed the topological sort algorithm, which lists the nodes in a DAG such that no node is listed before all nodes that it has edges into. As part of the algorithm, we assumed that at each step, it was possible to find some node in the DAG that has no outgoing edges. To be mathematically rigorous, we need to prove that this is actually correct.

i. Prove that in any finite DAG, there must be at least one node with no outgoing edges.
ii. Prove or disprove: If a graph satisfies property (i), it is a finite DAG.
iii. *(Extra credit)* Give an example of an *infinite* DAG where every node has at least one outgoing edge. Prove that the graph is a DAG, then explain why your proof from (i) breaks down when the graph is infinite.

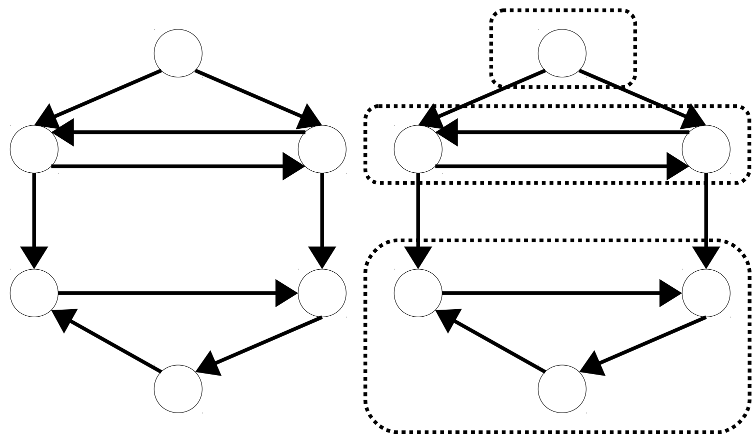
For future reference, the term "graph" almost always refers to finite graphs when no clarification is given.

Problem Ten: Strongly Connected Components (16 points)

Given a directed graph G = (V, E), define the relation u → v to mean "v is reachable from u." A **strongly connected component** (SCC) in a directed graph is a set of nodes U in the graph with the following properties:

- For any u ∈ U and v ∈ U, u → v and v → u.
- For any u ∈ U and v ∈ V – U, either u ↛ v or v ↛ u.

The first property says that every node in U is reachable from every other node. The second property says that the set U is as large as possible, meaning that any node in the graph that is reachable from the nodes in U and can reach the nodes in U is also contained in U. For example, here is a sample graph labeled with its strongly connected components:



Prove that each node in a graph belongs to exactly one strongly connected component. *(Hint: Try showing that each node is in at* least *one strongly connected component and that each node is in at* most *one strongly connected component.)*

Problem Eleven: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. How hard did you find this problem set? How long did it take you to finish?
ii. Does that seem unreasonably difficult or time-consuming for a five-unit class?
iii. Did you attend Monday's problem session? If so, did you find it useful?
iv. How is the pace of this course so far? Too slow? Too fast? Just right?
v. Is there anything in particular we could do better? Is there anything in particular that you think we're doing well?

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled “Stanford Venture Fund Laboratories.” There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to **cs103@cs.stanford.edu**.