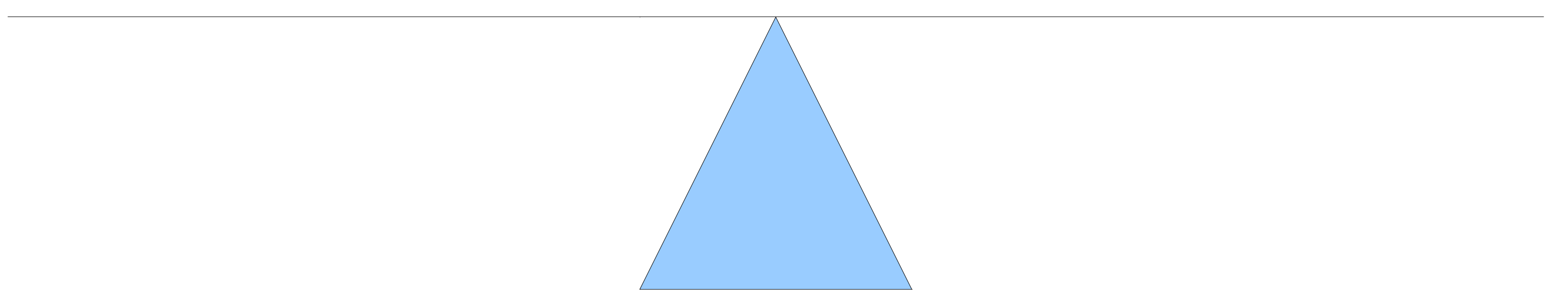
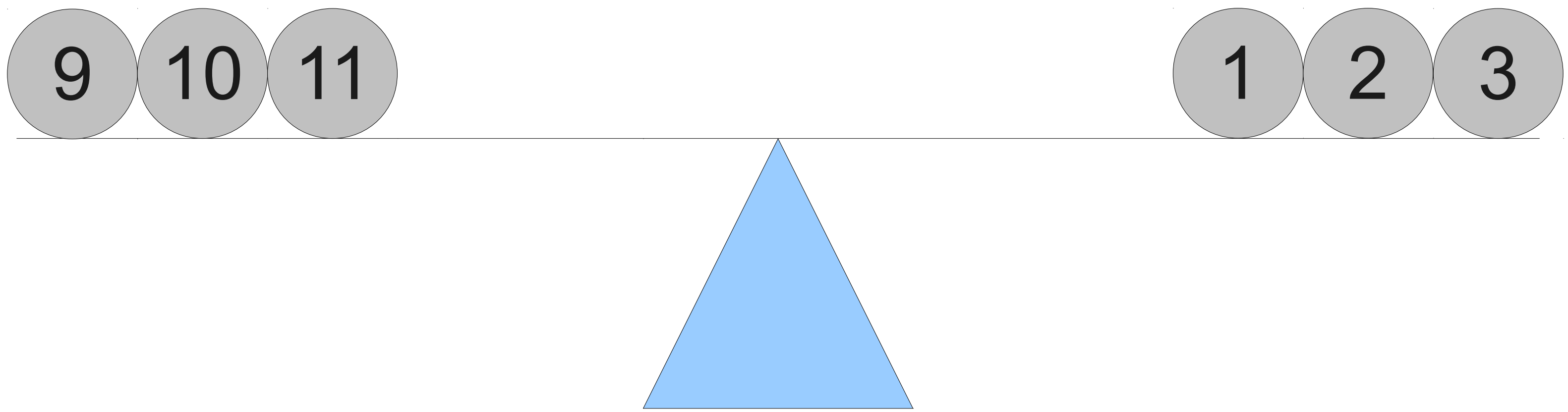
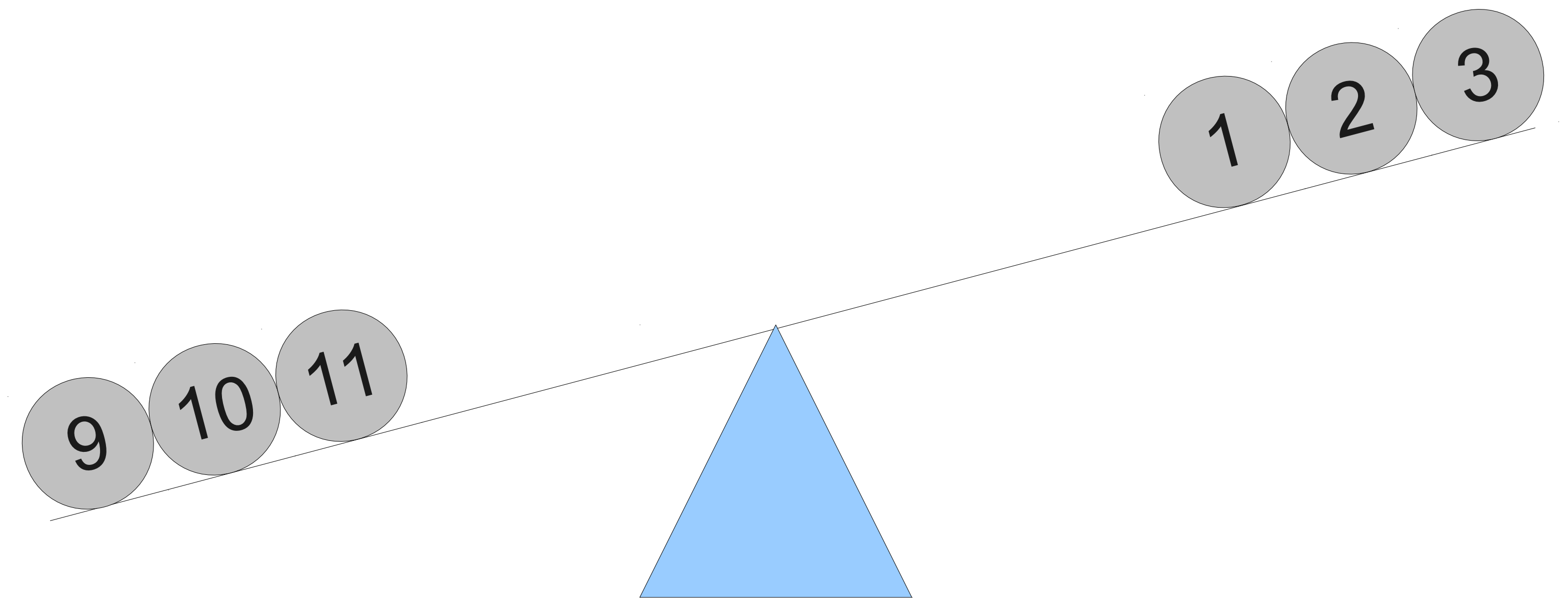
# Fun With Number Systems

Or: How Knowing Number Systems Can Help You Interview Better

# Finding the Odd Ball

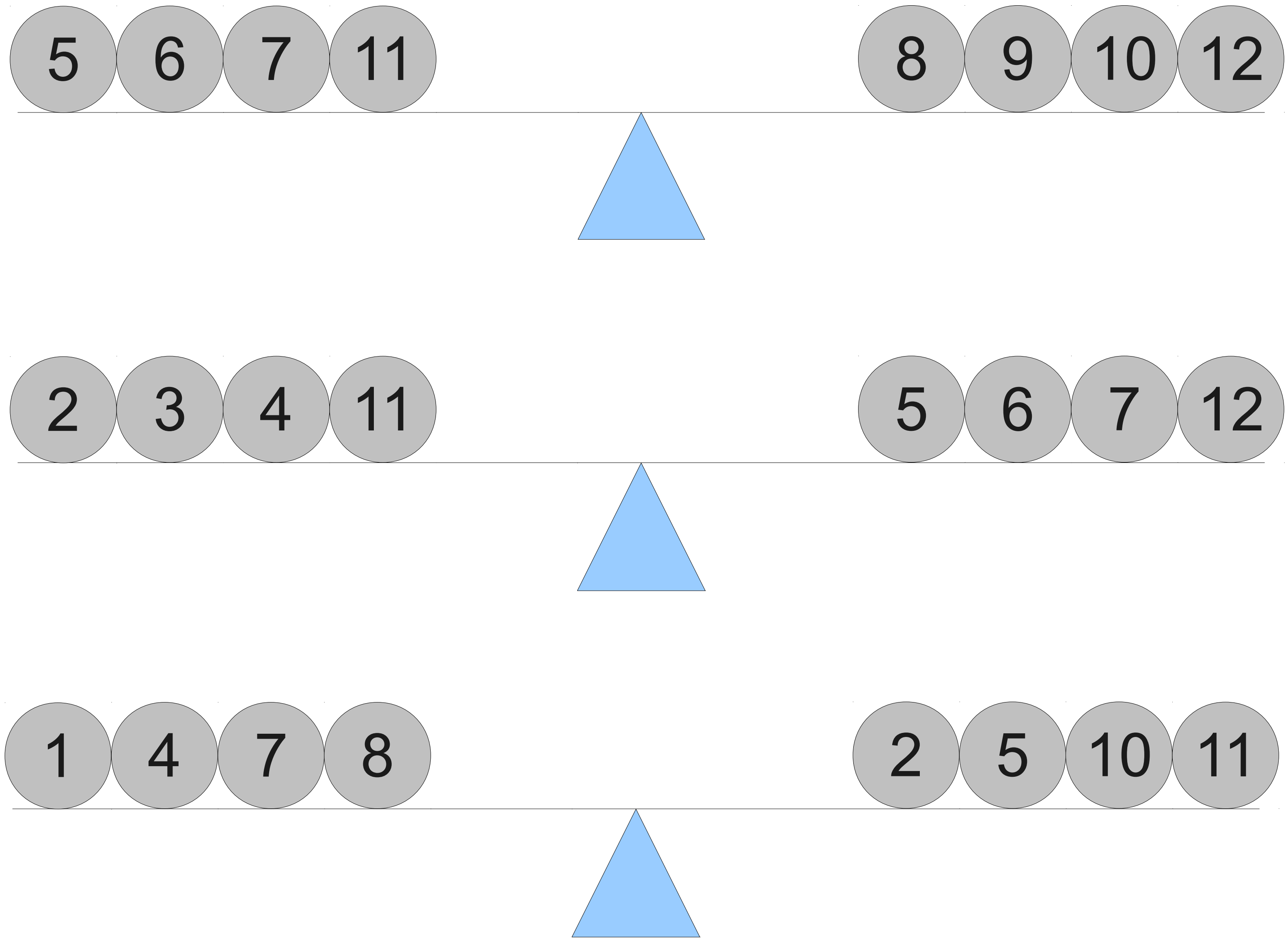



9
10
11
1
2
3


9
10
11
1
2
3

Goal: Find the odd ball and whether it's heavier or lighter in three weighings.

# The Solution



# Balanced Ternary

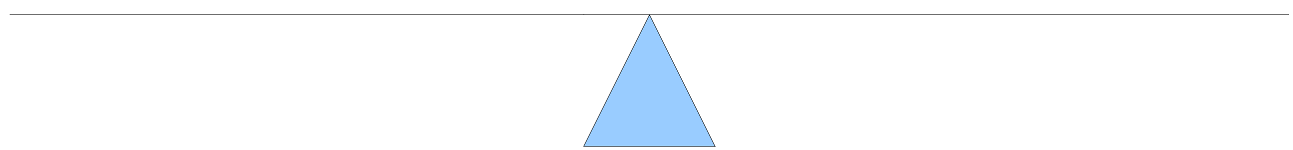
- Number system for encoding three-way comparisons.
- Each digit corresponds to a power of three.
- Digits are -1, 0, +1.
  - For notational simplicity, will use -, 0, +.
- Example: +0–0
  - $1 \times 3^3 + 0 \times 3^2 - 1 \times 3^1 + 0 \times 3^0 = 24$
- Example: ––++
  - $-1 \times 3^3 - 1 \times 3^2 + 1 \times 3^1 + 1 \times 3^0 = -32$

# -12 to +12 in Balanced Ternary

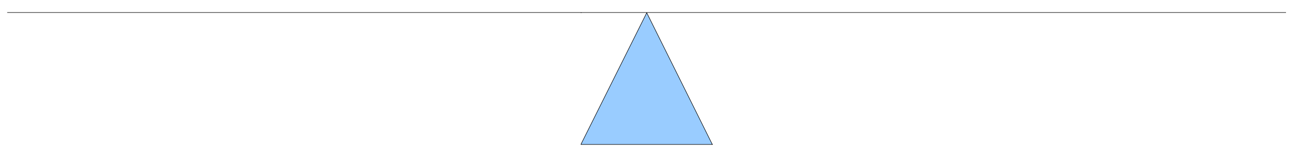
| | | | |
|---|---|---|---|
| | | 0 | 000 |
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| −12 | −−0 | 12 | ++0 |

| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |

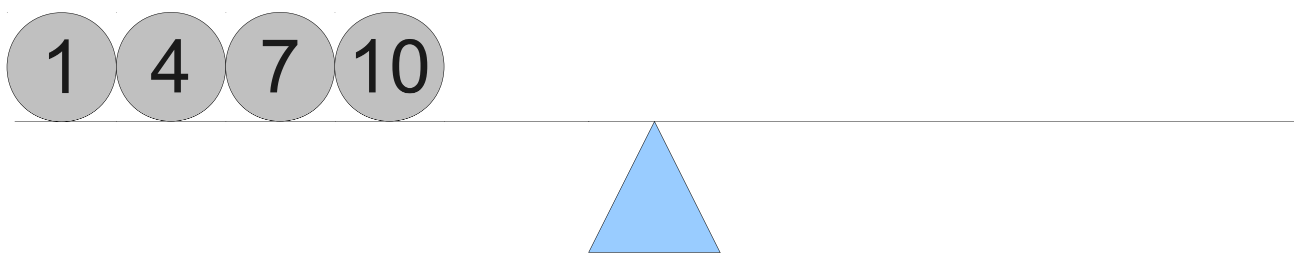
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



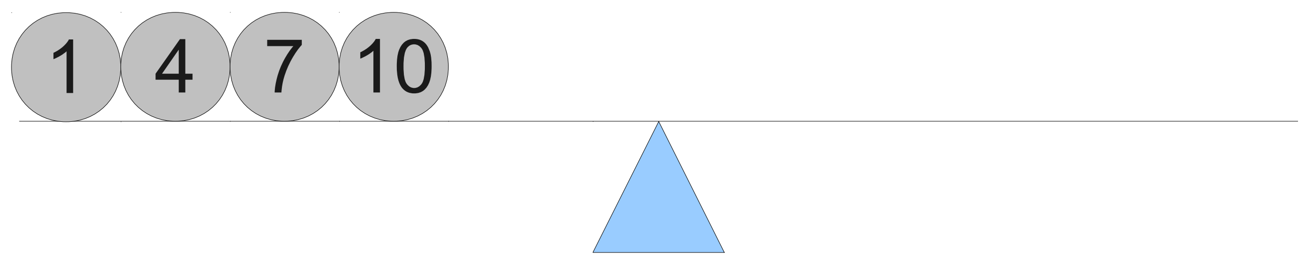
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



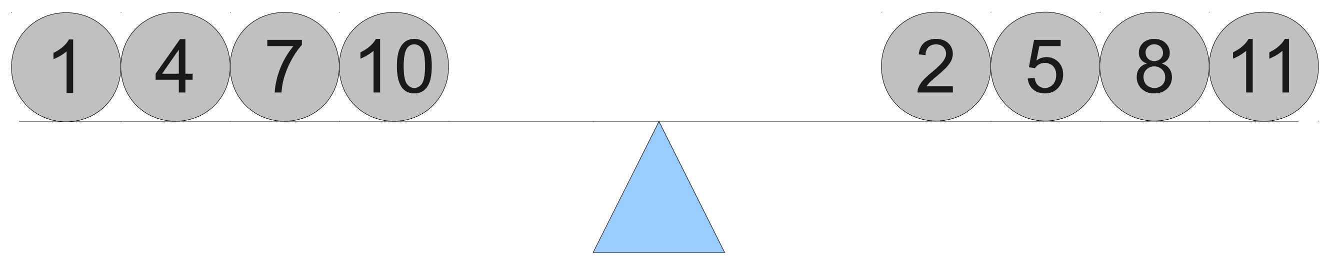
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |



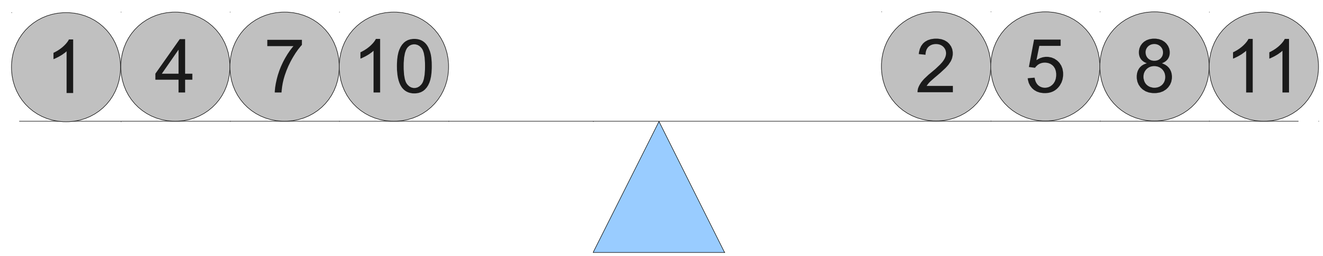
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



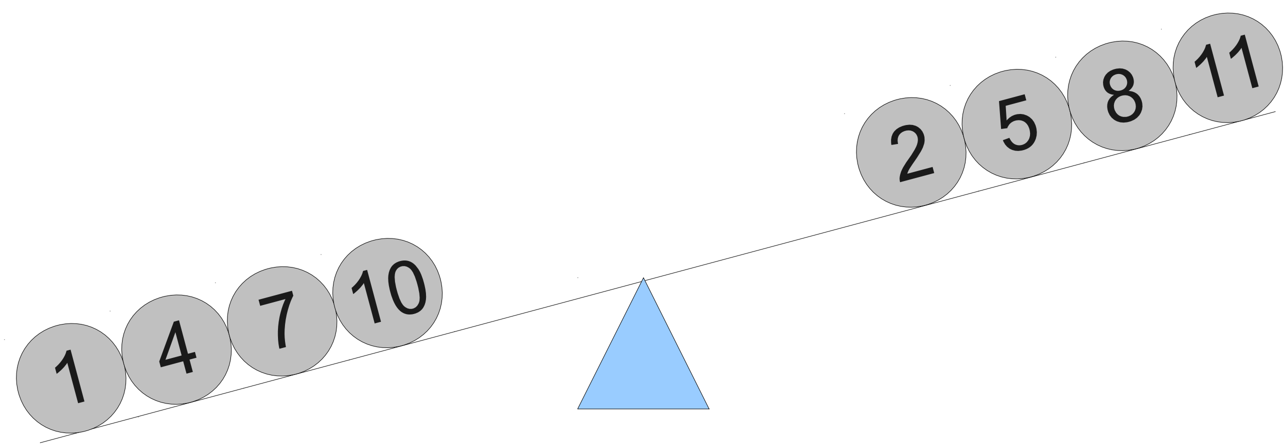
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



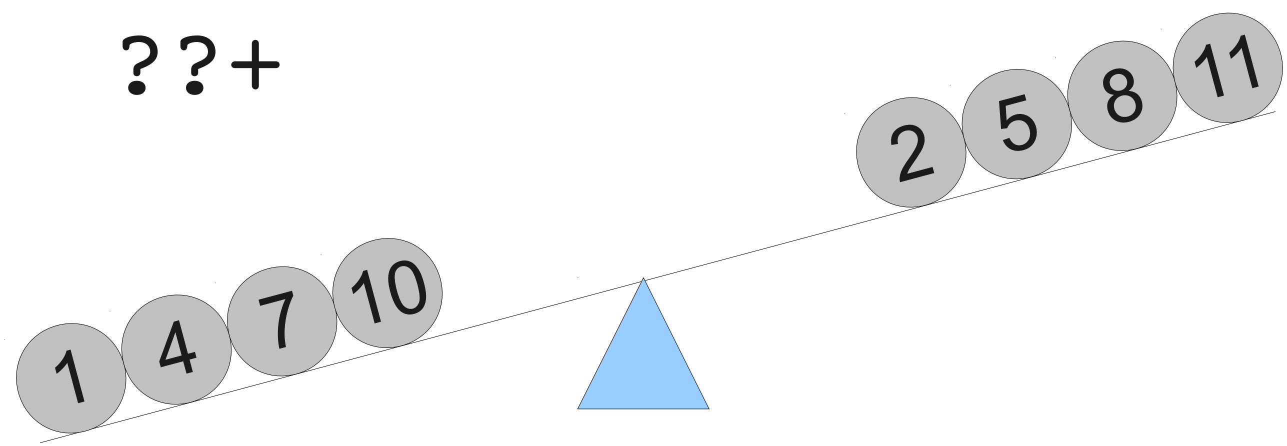
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |



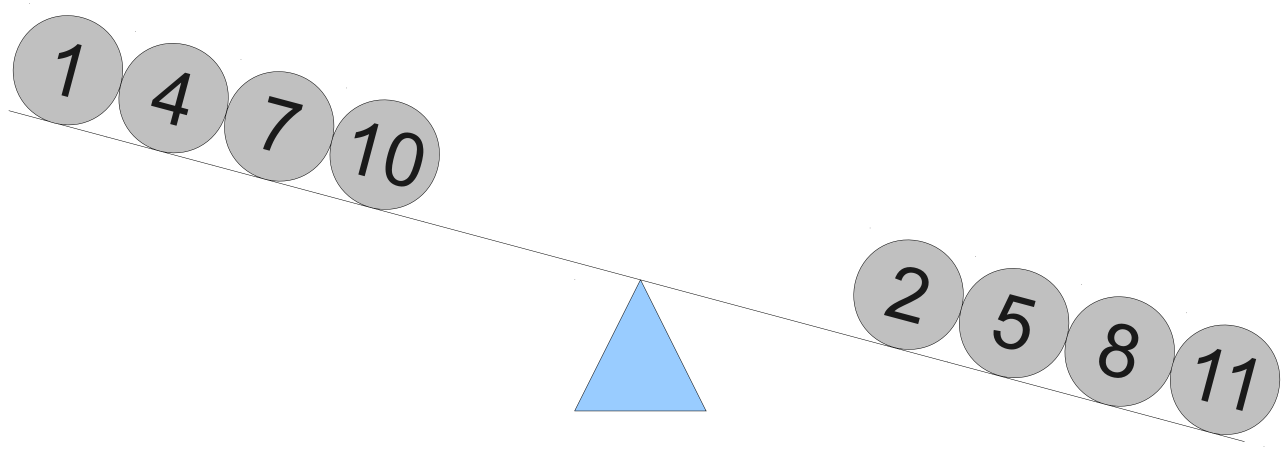
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |



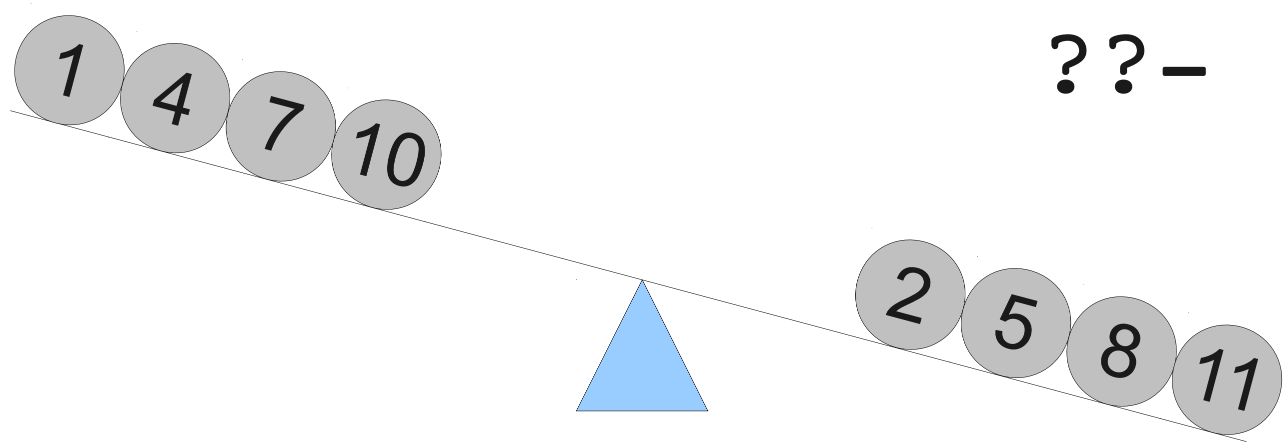
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



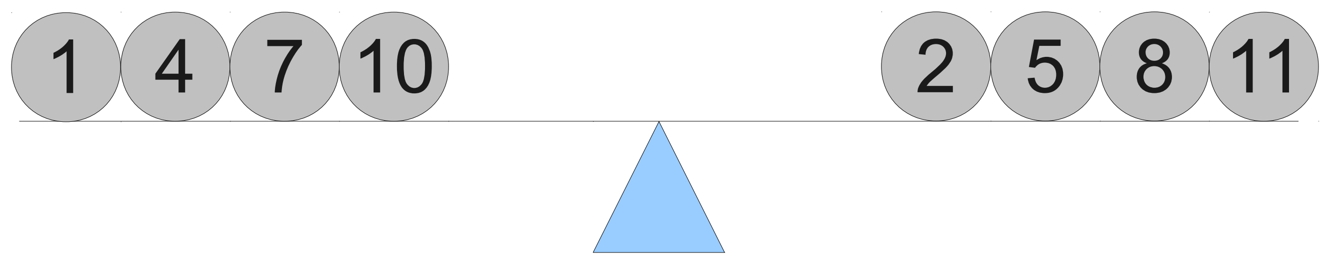
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |



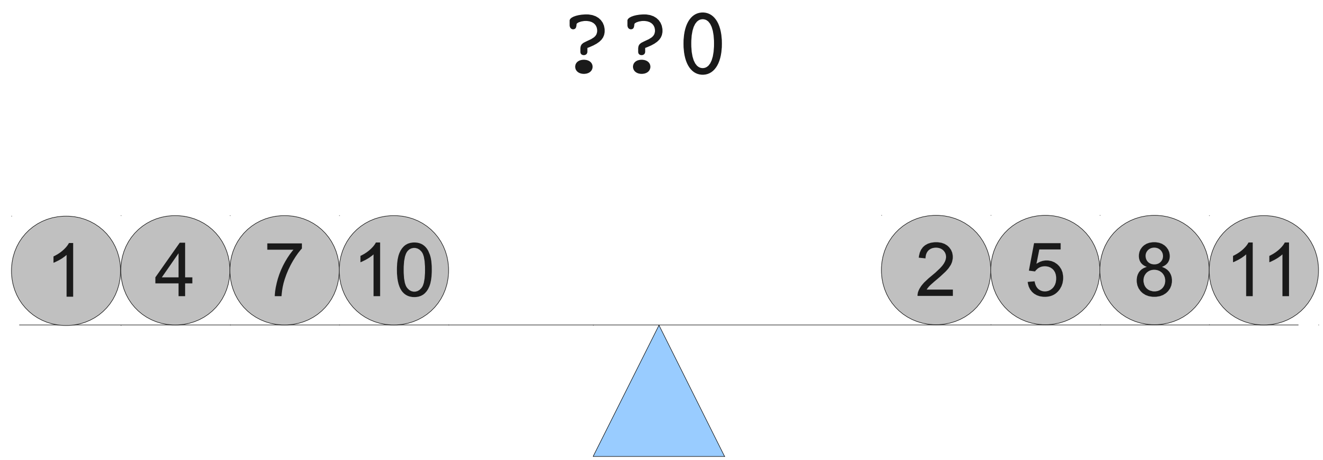
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |



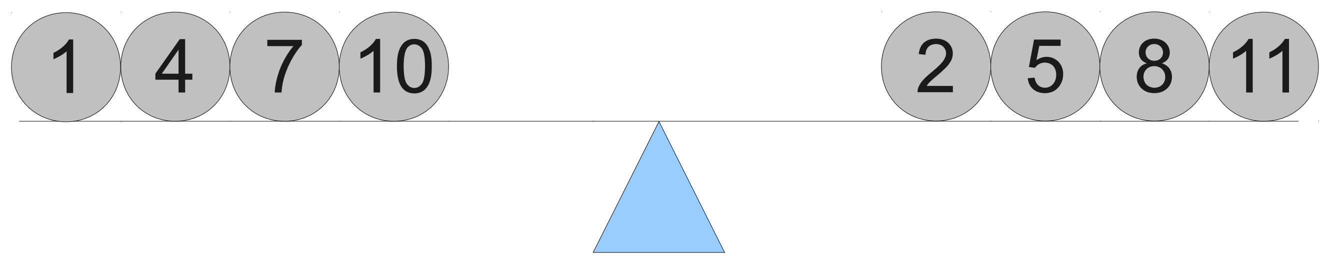
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



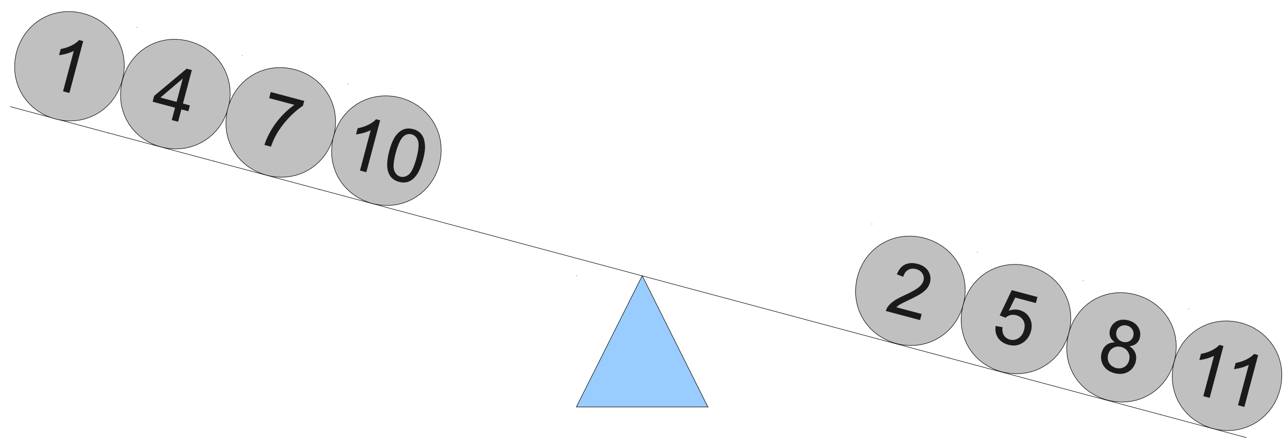
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



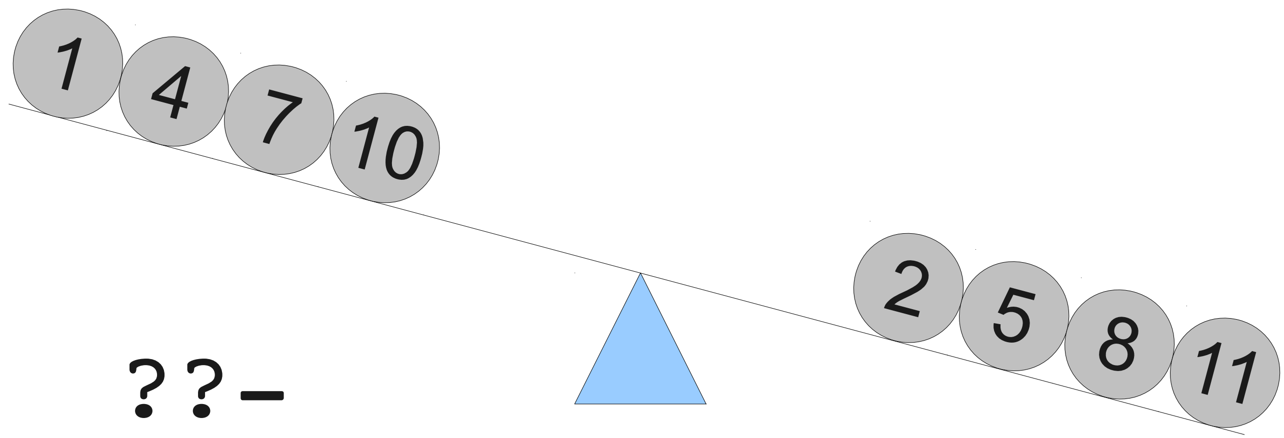
| | |
|---|---|
| -1 | 00- |
| -2 | 0-+ |
| -3 | 0-0 |
| -4 | 0-- |
| -5 | -++ |
| -6 | -+0 |
| -7 | -+- |
| -8 | -0+ |
| -9 | -00 |
| -10 | -0- |
| -11 | --+ |
| -12 | --0 |



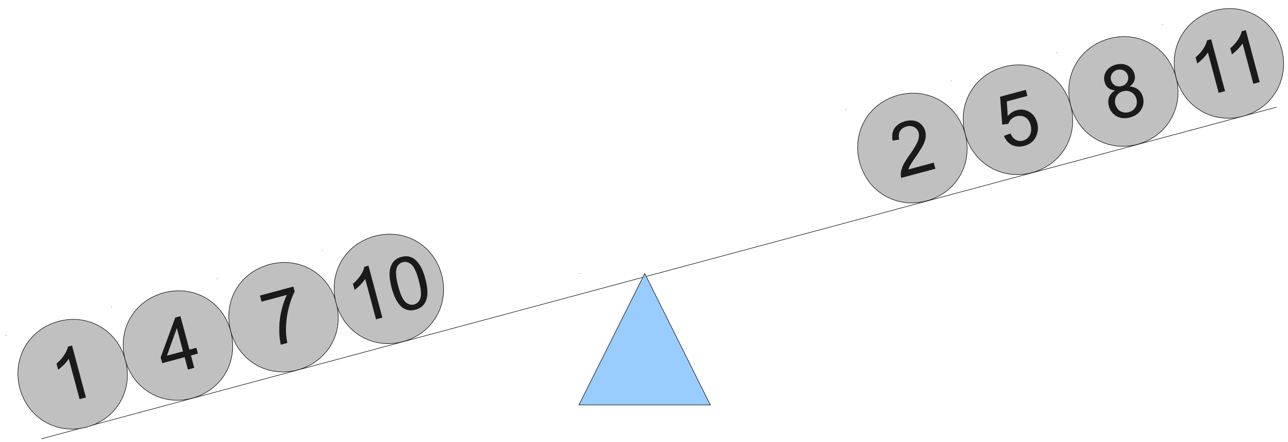
| | |
|---|---|
| −1 | 00− |
| −2 | 0−+ |
| −3 | 0−0 |
| −4 | 0−− |
| −5 | −++ |
| −6 | −+0 |
| −7 | −+− |
| −8 | −0+ |
| −9 | −00 |
| −10 | −0− |
| −11 | −−+ |
| −12 | −−0 |



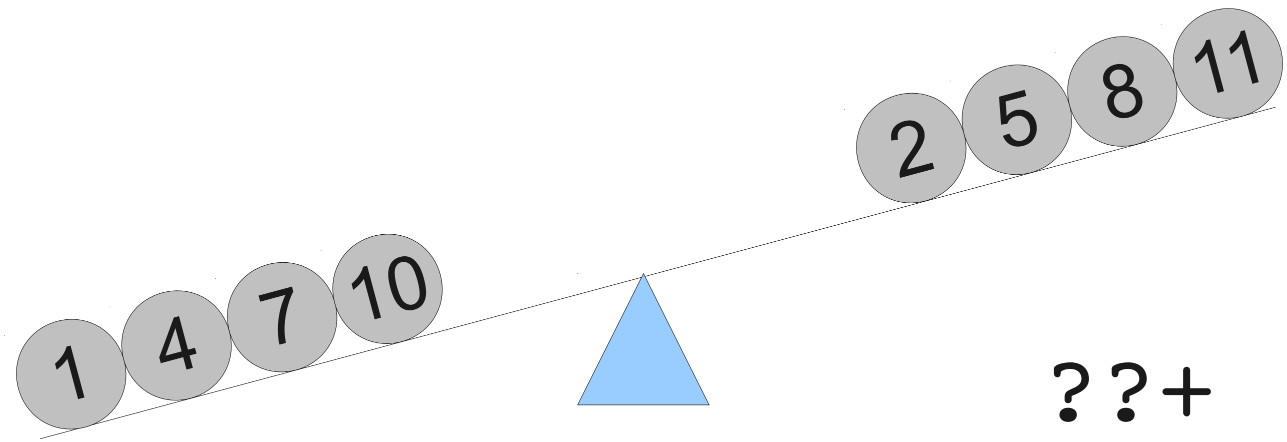
| | |
|---|---|
| -1 | 00- |
| -2 | 0-+ |
| -3 | 0-0 |
| -4 | 0-- |
| -5 | -++ |
| -6 | -+0 |
| -7 | -+- |
| -8 | -0+ |
| -9 | -00 |
| -10 | -0- |
| -11 | --+ |
| -12 | --0 |



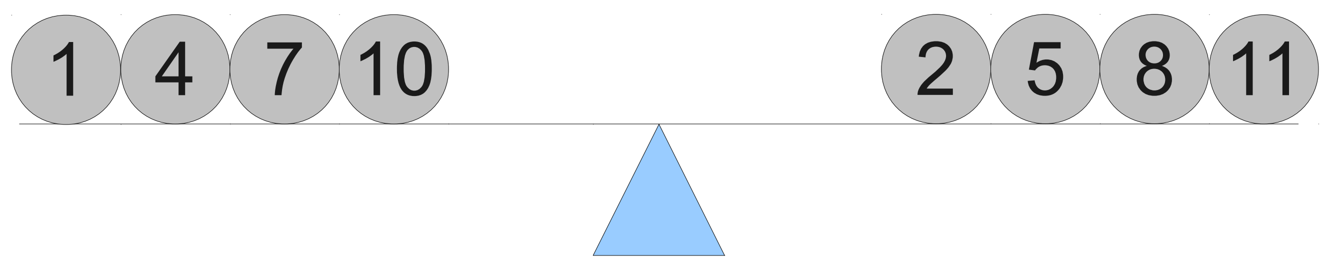
| | |
|---|---|
| −1 | 00− |
| −2 | 0−+ |
| −3 | 0−0 |
| −4 | 0−− |
| −5 | −++ |
| −6 | −+0 |
| −7 | −+− |
| −8 | −0+ |
| −9 | −00 |
| −10 | −0− |
| −11 | −−+ |
| −12 | −−0 |



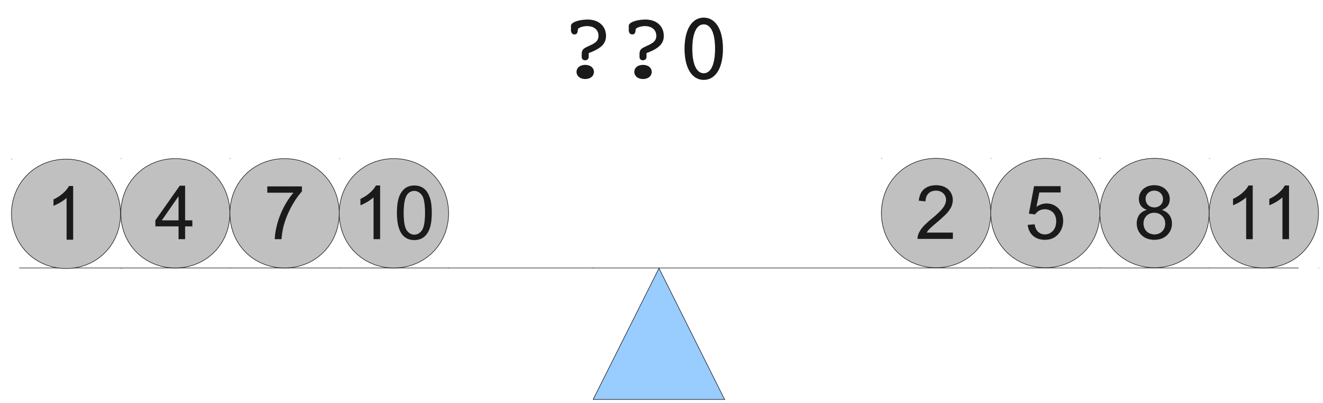
| | |
|---|---|
| −1 | 00− |
| −2 | 0−+ |
| −3 | 0−0 |
| −4 | 0−− |
| −5 | −++ |
| −6 | −+0 |
| −7 | −+− |
| −8 | −0+ |
| −9 | −00 |
| −10 | −0− |
| −11 | −−+ |
| −12 | −−0 |



| | |
|---|---|
| -1 | 00- |
| -2 | 0-+ |
| -3 | 0-0 |
| -4 | 0-- |
| -5 | -++ |
| -6 | -+0 |
| -7 | -+- |
| -8 | -0+ |
| -9 | -00 |
| -10 | -0- |
| -11 | --+ |
| -12 | --0 |



| | |
|---|---|
| -1 | 00- |
| -2 | 0-+ |
| -3 | 0-0 |
| -4 | 0-- |
| -5 | -++ |
| -6 | -+0 |
| -7 | -+- |
| -8 | -0+ |
| -9 | -00 |
| -10 | -0- |
| -11 | --+ |
| -12 | --0 |

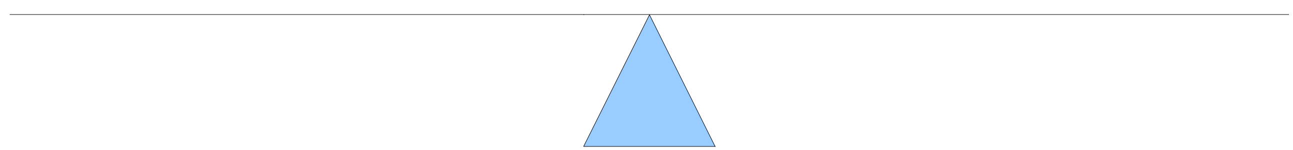


If the left side is heavier, record a +.

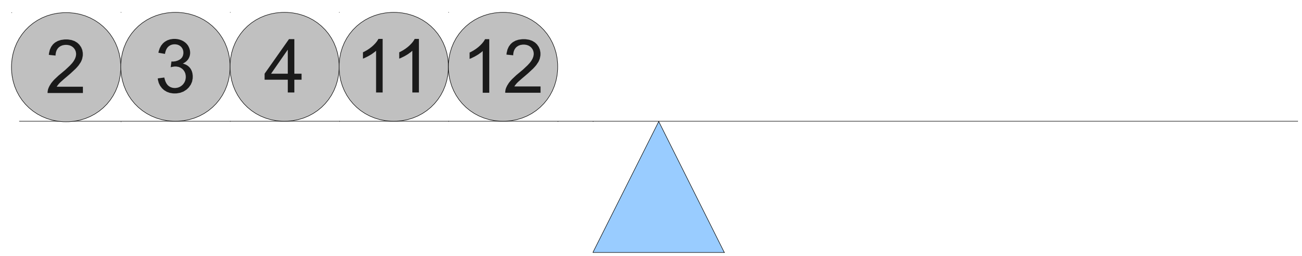
If the right side is heavier, record a -.

If the scale balances, record a 0.

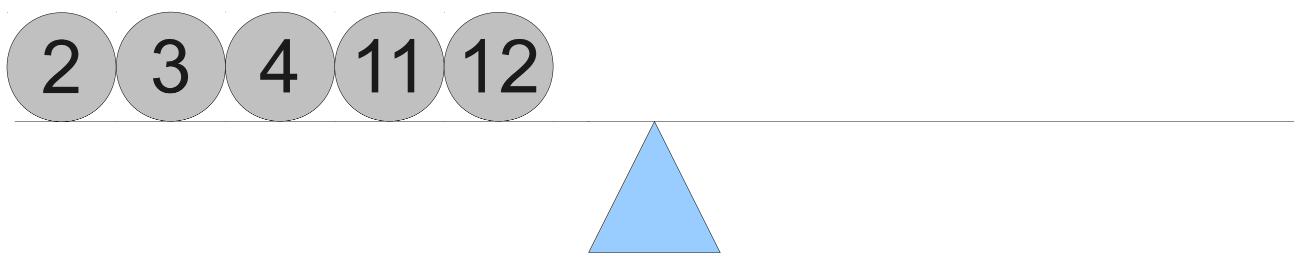
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



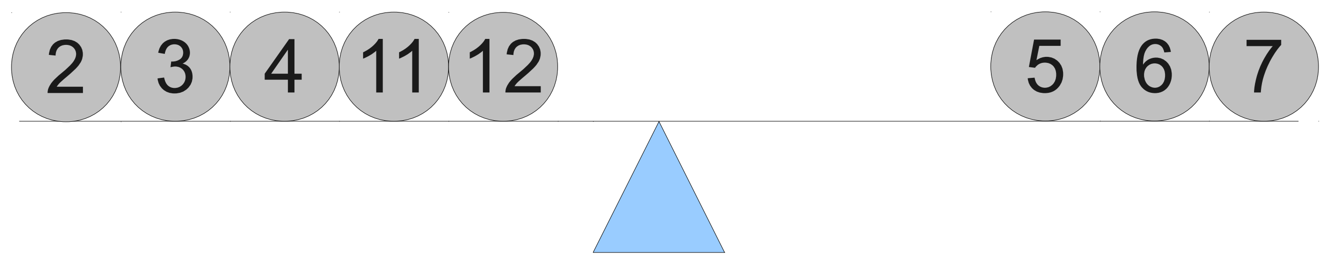
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



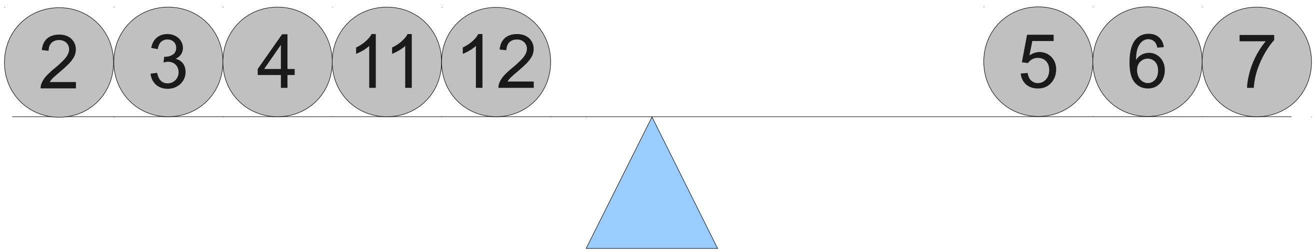
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



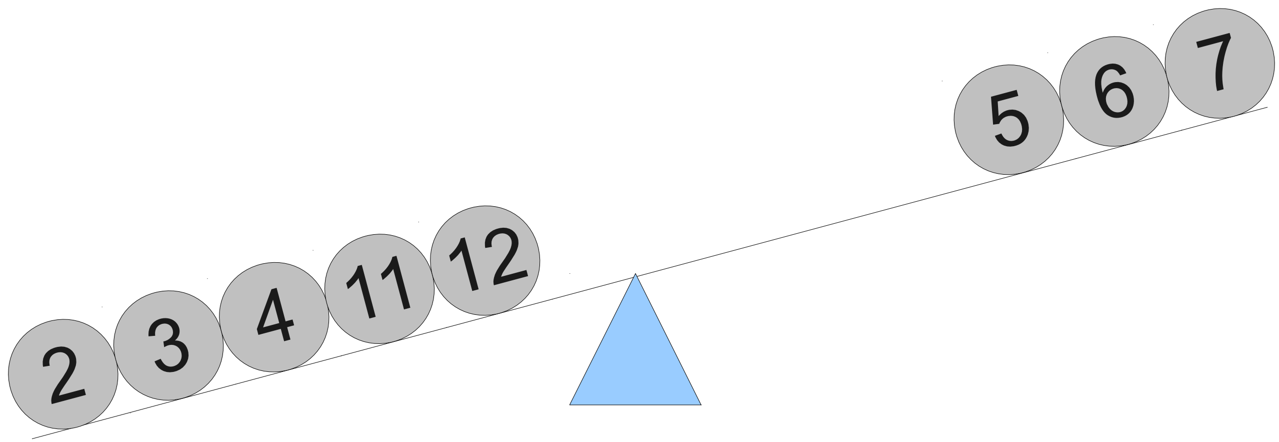
| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



| | |
|---|---|
| 1 | 00+ |
| 2 | 0+− |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +−− |
| 6 | +−0 |
| 7 | +−+ |
| 8 | +0− |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++− |
| 12 | ++0 |



| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| 8 | +0– |
| 9 | +00 |
| 10 | +0+ |
| 11 | ++– |
| 12 | ++0 |



# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| -1 | 00- | 1 | 00+ |
| -2 | 0-+ | 2 | 0+- |
| -3 | 0-0 | 3 | 0+0 |
| -4 | 0-- | 4 | 0++ |
| -5 | -++ | 5 | +-- |
| -6 | -+0 | 6 | +-0 |
| -7 | -+- | 7 | +-+ |
| -8 | -0+ | 8 | +0- |
| -9 | -00 | 9 | +00 |
| -10 | -0- | 10 | +0+ |
| -11 | --+ | 11 | ++- |
| -12 | --0 | 12 | ++0 |

# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| **−12** | **−−0** | **12** | **++0** |

# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| **12** | **++0** | **−12** | **−−0** |

# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| 12 | ++0 | −12 | −−0 |

# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| **−8** | **−0+** | **8** | **+0−** |
| **−9** | **−00** | **9** | **+00** |
| **−10** | **−0−** | **10** | **+0+** |
| −11 | −−+ | 11 | ++− |
| 12 | ++0 | −12 | −−0 |

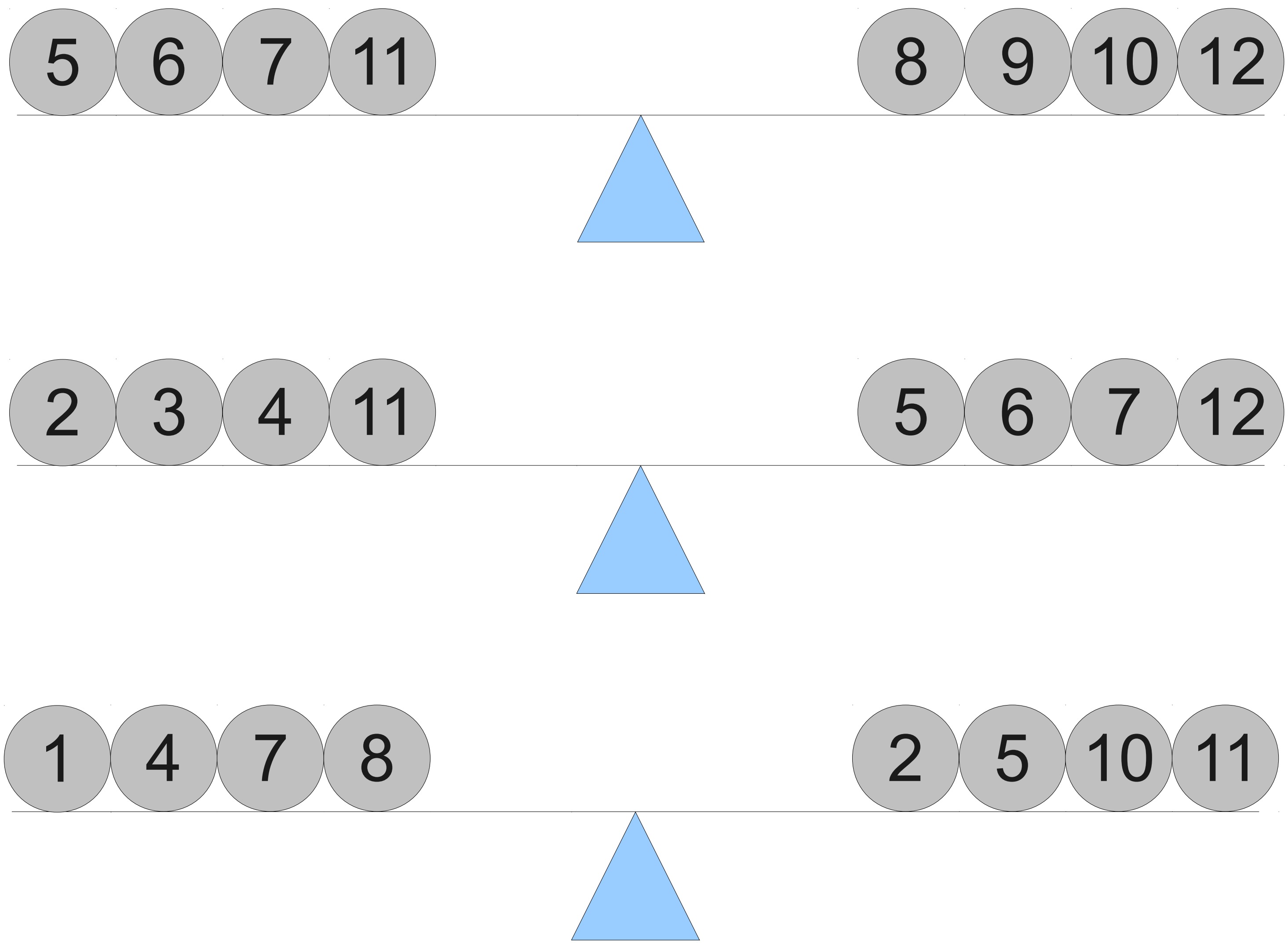
# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| **8** | **+0−** | **−8** | **−0+** |
| **9** | **+00** | **−9** | **−00** |
| **10** | **+0+** | **−10** | **−0−** |
| −11 | −−+ | 11 | ++− |
| 12 | ++0 | −12 | −−0 |

# Our Encoding Scheme

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| 8 | +0− | −8 | −0+ |
| 9 | +00 | −9 | −00 |
| 10 | +0+ | −10 | −0− |
| −11 | −−+ | 11 | ++− |
| 12 | ++0 | −12 | −−0 |

# The Solution




| | |
|---|---|
| 1 | 00+ |
| 2 | 0+– |
| 3 | 0+0 |
| 4 | 0++ |
| 5 | +–– |
| 6 | +–0 |
| 7 | +–+ |
| –8 | –0+ |
| –9 | –00 |
| –10 | –0– |
| 11 | ++– |
| –12 | ––0 |

# Generalizing the Result

- Why twelve balls?
  - With three trits, 27 possible combinations. 13 are positive, nine start with +.
  - Must discard one starting with + to ensure number of + and - in each column is the same, leaving 12 positive numbers.
- More generally:
  - With n trits, $3^n$ possible combinations. $(3^n – 1) / 2$ are positive.
  - $3^{n-1}$ numbers start with +. To balance + and -, we need to drop one starting with +, leaving **$(3^n – 3) / 2$** positive numbers.
- Can do 3, 12, 39, 120, 363, 1092, ...

# Example: Two Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 0− | 1 | 0+ |
| −2 | −+ | 2 | +− |
| −3 | −0 | 3 | +0 |
| −4 | −− | 4 | ++ |

# Example: Two Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 0− | 1 | 0+ |
| −2 | −+ | 2 | +− |
| −3 | −0 | 3 | +0 |
| **−4** | **−−** | **4** | **++** |

# Example: Two Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| -1 | 0- | 1 | 0+ |
| -2 | -+ | 2 | +- |
| -3 | -0 | 3 | +0 |

# Example: Two Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 0− | 1 | 0+ |
| −2 | −+ | 2 | +− |
| **−3** | **−0** | **3** | **+0** |

# Example: Two Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| –1 | 0– | 1 | 0+ |
| –2 | –+ | 2 | +– |
| **3** | **+0** | **–3** | **–0** |

# Example: Two Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 0− | 1 | 0+ |
| −2 | −+ | 2 | +− |
| 3 | +0 | −3 | −0 |

# Example: Three Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| −12 | −−0 | 12 | ++0 |
| −13 | −−− | 13 | +++ |

# Example: Three Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| −12 | −−0 | 12 | ++0 |
| **−13** | **−−−** | **13** | **+++** |

# Example: Three Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| −8 | −0+ | 8 | +0− |
| −9 | −00 | 9 | +00 |
| −10 | −0− | 10 | +0+ |
| −11 | −−+ | 11 | ++− |
| −12 | −−0 | 12 | ++0 |

# Example: Three Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| **−8** | **−0+** | **8** | **+0−** |
| **−9** | **−00** | **9** | **+00** |
| **−10** | **−0−** | **10** | **+0+** |
| −11 | −−+ | 11 | ++− |
| **−12** | **−−0** | **12** | **++0** |

# Example: Three Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| **8** | **+0−** | **−8** | **−0+** |
| **9** | **+00** | **−9** | **−00** |
| **10** | **+0+** | **−10** | **−0−** |
| −11 | −−+ | 11 | ++− |
| **12** | **++0** | **−12** | **−−0** |

# Example: Three Weighings

| Lighter | | Heavier | |
|---|---|---|---|
| −1 | 00− | 1 | 00+ |
| −2 | 0−+ | 2 | 0+− |
| −3 | 0−0 | 3 | 0+0 |
| −4 | 0−− | 4 | 0++ |
| −5 | −++ | 5 | +−− |
| −6 | −+0 | 6 | +−0 |
| −7 | −+− | 7 | +−+ |
| 8 | +0− | −8 | −0+ |
| 9 | +00 | −9 | −00 |
| 10 | +0+ | −10 | −0− |
| −11 | −−+ | 11 | ++− |
| 12 | ++0 | −12 | −−0 |

# Example: Four Weighings

| | | | | | |
|---|---|---|---|---|---|
| 1 | 000+ | 14 | +--- | 27 | +000 |
| 2 | 00+- | 15 | +--0 | 28 | +00+ |
| 3 | 00+0 | 16 | +--+ | 29 | +0+- |
| 4 | 00++ | 17 | +-0- | 30 | +0+0 |
| 5 | 0+-- | 18 | +-00 | 31 | +0++ |
| 6 | 0+-0 | 19 | +-0+ | 32 | ++-- |
| 7 | 0+-+ | 20 | +-+- | 33 | ++-0 |
| 8 | 0+0- | 21 | +-+0 | 34 | ++-+ |
| 9 | 0+00 | 22 | +-++ | 35 | ++0- |
| 10 | 0+0+ | 23 | +0-- | 36 | ++00 |
| 11 | 0++- | 24 | +0-0 | 37 | ++0+ |
| 12 | 0++0 | 25 | +0-+ | 38 | +++- |
| 13 | 0+++ | 26 | +00- | 39 | +++0 |
| | | | | 40 | ++++ |

# Example: Four Weighings

| | | | | | |
|---|---|---|---|---|---|
| 1 | 000+ | 14 | +--- | 27 | +000 |
| 2 | 00+- | 15 | +--0 | 28 | +00+ |
| 3 | 00+0 | 16 | +--+ | 29 | +0+- |
| 4 | 00++ | 17 | +-0- | 30 | +0+0 |
| 5 | 0+-- | 18 | +-00 | 31 | +0++ |
| 6 | 0+-0 | 19 | +-0+ | 32 | ++-- |
| 7 | 0+-+ | 20 | +-+- | 33 | ++-0 |
| 8 | 0+0- | 21 | +-+0 | 34 | ++-+ |
| 9 | 0+00 | 22 | +-++ | 35 | ++0- |
| 10 | 0+0+ | 23 | +0-- | 36 | ++00 |
| 11 | 0++- | 24 | +0-0 | 37 | ++0+ |
| 12 | 0++0 | 25 | +0-+ | 38 | +++- |
| 13 | 0+++ | 26 | +00- | 39 | +++0 |
| | | | | **40** | **++++** |

# Example: Four Weighings

| | |
|---|---|
| 1 | 000+ |
| 2 | 00+– |
| 3 | 00+0 |
| 4 | 00++ |
| 5 | 0+–– |
| 6 | 0+–0 |
| 7 | 0+–+ |
| 8 | 0+0– |
| 9 | 0+00 |
| 10 | 0+0+ |
| 11 | 0++– |
| 12 | 0++0 |
| 13 | 0+++ |
| 14 | +––– |
| 15 | +––0 |
| 16 | +––+ |
| 17 | +–0– |
| 18 | +–00 |
| 19 | +–0+ |
| 20 | +–+– |
| 21 | +–+0 |
| 22 | +–++ |
| 23 | +0–– |
| 24 | +0–0 |
| 25 | +0–+ |
| 26 | +00– |
| 27 | +000 |
| 28 | +00+ |
| 29 | +0+– |
| 30 | +0+0 |
| 31 | +0++ |
| 32 | ++–– |
| 33 | ++–0 |
| 34 | ++–+ |
| 35 | ++0– |
| 36 | ++00 |
| 37 | ++0+ |
| 38 | +++– |
| 39 | +++0 |

# Example: Four Weighings

| | | | | | |
|---|---|---|---|---|---|
| 1 | 000+ | 14 | +--- | **27** | **+000** |
| 2 | 00+- | 15 | +--0 | **28** | **+00+** |
| 3 | 00+0 | 16 | +--+ | **29** | **+0+-** |
| 4 | 00++ | 17 | +-0- | **30** | **+0+0** |
| 5 | 0+-- | 18 | +-00 | **31** | **+0++** |
| 6 | 0+-0 | 19 | +-0+ | 32 | ++-- |
| 7 | 0+-+ | 20 | +-+- | 33 | ++-0 |
| 8 | 0+0- | 21 | +-+0 | 34 | ++-+ |
| 9 | 0+00 | 22 | +-++ | **35** | **++0-** |
| 10 | 0+0+ | **23** | **+0--** | **36** | **++00** |
| 11 | 0++- | **24** | **+0-0** | **37** | **++0+** |
| 12 | 0++0 | **25** | **+0-+** | 38 | +++- |
| 13 | 0+++ | **26** | **+00-** | **39** | **+++0** |

# Example: Four Weighings

| | | | | | |
|---|---|---|---|---|---|
| 1 | 000+ | 14 | +--- | **-27** | **-000** |
| 2 | 00+- | 15 | +--0 | **-28** | **-00-** |
| 3 | 00+0 | 16 | +--+ | **-29** | **-0-+** |
| 4 | 00++ | 17 | +-0- | **-30** | **-0-0** |
| 5 | 0+-- | 18 | +-00 | **-31** | **-0--** |
| 6 | 0+-0 | 19 | +-0+ | 32 | ++-- |
| 7 | 0+-+ | 20 | +-+- | 33 | ++-0 |
| 8 | 0+0- | 21 | +-+0 | 34 | ++-+ |
| 9 | 0+00 | 22 | +-++ | **-35** | **--0+** |
| 10 | 0+0+ | **-23** | **-0++** | **-36** | **--00** |
| 11 | 0++- | **-24** | **-0+0** | **-37** | **--0-** |
| 12 | 0++0 | **-25** | **-0+-** | 38 | +++- |
| 13 | 0+++ | **-26** | **-00+** | **-39** | **---0** |

# Example: Four Weighings

| | | | | | |
|---|---|---|---|---|---|
| 1 | 000+ | 14 | +--- | -27 | -000 |
| 2 | 00+- | 15 | +--0 | -28 | -00- |
| 3 | 00+0 | 16 | +--+ | -29 | -0-+ |
| 4 | 00++ | 17 | +-0- | -30 | -0-0 |
| 5 | 0+-- | 18 | +-00 | -31 | -0-- |
| 6 | 0+-0 | 19 | +-0+ | 32 | ++-- |
| 7 | 0+-+ | 20 | +-+- | 33 | ++-0 |
| 8 | 0+0- | 21 | +-+0 | 34 | ++-+ |
| 9 | 0+00 | 22 | +-++ | -35 | --0+ |
| 10 | 0+0+ | -23 | -0++ | -36 | --00 |
| 11 | 0++- | -24 | -0+0 | -37 | --0- |
| 12 | 0++0 | -25 | -0+- | 38 | +++- |
| 13 | 0+++ | -26 | -00+ | -39 | ---0 |

# Some Insights

- What number did we drop?
  - With 2 trits, dropped ++.
  - With 3 trits, dropped +++.
  - With 4 trits, dropped ++++.
  - **Always drop ++…++**
- What numbers did we invert?
  - With two trits: +0
  - With three trits: ++0, +0-, +00, +0-
  - With four trits: +++0, ++0-, ++00, ++0+, +0--, +0-0, +0-+, +00-, +000, +00+, +0+-, +0+0, +0++
  - **Always invert numbers starting with ++…++0.**
- **This always works!**

# How Many Extra +s Per Column?

- Answer: The $3^j$ column has **$(3^j – 1) / 2$** extra +'s.

0+

+–

+0

# How Many Extra +s Per Column?

- Answer: The $3^j$ column has **$(3^j - 1) / 2$** extra +'s.

0 0 +

0 + –

0 + 0

# How Many Extra +s Per Column?

- Answer: The $3^j$ column has **$(3^j – 1) / 2$** extra +'s.

00+

0+–

0+0

0++

+––

+–0

+–+

+0–

+00

+0+

++–

++0

# How Many Extra +s Per Column?

- Answer: The $3^j$ column has $\mathbf{(3^j - 1) / 2}$ extra +'s.

| |
|---|
| 00+ |
| 0+– |
| 0+0 |
| 0++ |
| +–– |
| +–0 |
| +–+ |
| +0– |
| +00 |
| +0+ |
| ++– |
| ++0 |

# How Many +'s Get Flipped?

- Answer: The $3^j$ column has **$(3^j-1) / 2$** flipped +'s.

# How Many +'s Get Flipped?

- Answer: The $3^j$ column has $\mathbf{(3^j-1) / 2}$ flipped +'s.

0+

+−

+0

# How Many +'s Get Flipped?

- Answer: The $3^j$ column has **($3^j$-1) / 2** flipped +'s.

| | |
|---|---|
| 0 | + |
| + | – |
| + | 0 |

# How Many +'s Get Flipped?

- Answer: The $3^j$ column has **$(3^j-1) / 2$** flipped +'s.

00+
0+−
0+0
0++
+−−
+−0
+−+
+0−
+00
+0+
++−
++0

# How Many +'s Get Flipped?

- Answer: The $3^j$ column has $\mathbf{(3^j\text{-}1) / 2}$ flipped +'s.

0 0 +

0 + –

0 + 0

0 + +

+ – –

+ – 0

+ – +

+ 0 –

+ 0 0

+ 0 +

+ + –

+ + 0

# How Many +'s Get Flipped?

- Answer: The $3^j$ column has **($3^j$-1) / 2** flipped +'s.

00+
0+–
0+0
0++
+––
+–0
+–+
+0–
+00
+0+
++–
++0

$$\sum_{i=0}^{j-1} 3^i = \frac{3^j - 1}{2}$$

# Summary

- A three-way scale lends itself naturally to a **balanced ternary encoding** for each of the balls.
- Given an encoding with the same number of +'s and -'s in each column, we can use the scale to read off one trit of the answer at a time.
- Flipping numbers starting with +0, ++0, etc. guarantees an encoding with this property.

# Generating Permutations

“You are given a sorted string S of unique characters. Write a Java-style iterator that traverses all the permutations of S in lexicographical order.”

# Example

# Example

`abc`

# Example

```
abc
acb
bac
bca
cab
cba
```

# Example

| | |
|---|---|
| 0 | abc |
| 1 | acb |
| 2 | bac |
| 3 | bca |
| 4 | cab |
| 5 | cba |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | | | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | | | | |

# Lehmer Codes

| B | A | E | D | C |
|---|---|---|---|---|
| 1 | | | | |

# Lehmer Codes

| B | A | E | D | C |
|---|---|---|---|---|
| 1 | 0 | | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | | | |

# Lehmer Codes

| B | A | E | D | C |
|---|---|---|---|---|
| 1 | 0 | | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | | |

# Lehmer Codes

| B | A | E | D | C |
|---|---|---|---|---|
| 1 | 0 | 2 | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | | |

# Lehmer Codes

| B | A | E | D | C |
|---|---|---|---|---|
| 1 | 0 | 2 | | |

# Lehmer Codes

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

# Lehmer Codes

| B | A | E | D | C |
|---|---|---|---|---|
| 1 | 0 | 2 | 1 | 0 |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| C | D | A | B | E |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| C | D | A | B | E |
| 2 | | | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| C | D | A | B | E |
| 2 | 2 | | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| C | D | A | B | E |
| 2 | 2 | 0 | | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| C | D | A | B | E |
| 2 | 2 | 0 | 0 | |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| B | A | E | D | C |
| 1 | 0 | 2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| C | D | A | B | E |
| 2 | 2 | 0 | 0 | 0 |

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| A | B | C | D | E |

4 1 0 1 0

# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| A | B | C | D | E |

4 1 0 1 0

# Lehmer Codes

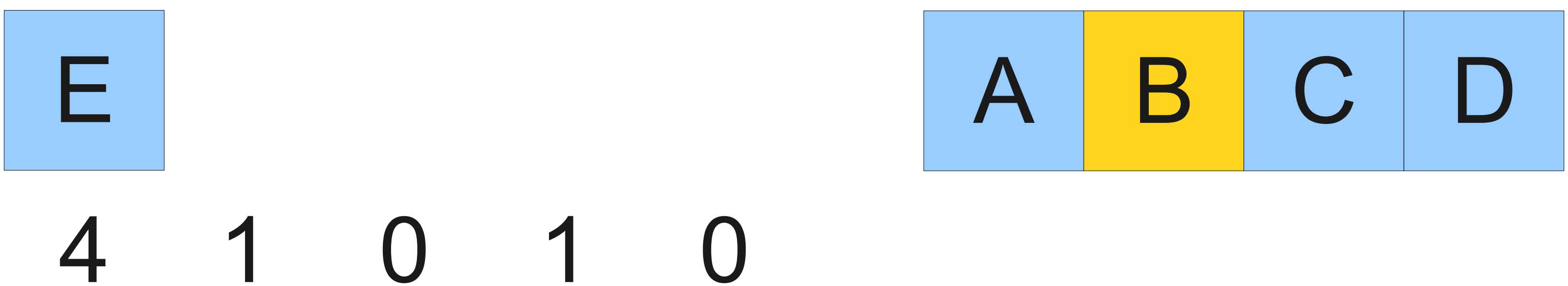
E

A B C D

4 1 0 1 0

# Lehmer Codes

# Lehmer Codes



# Lehmer Codes

# Lehmer Codes

# Lehmer Codes

E B　　　　　　A C D

4 1 0 1 0

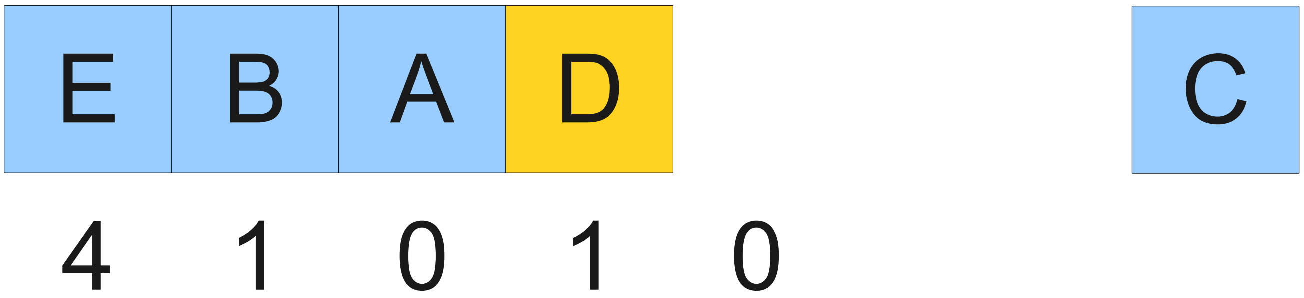
# Lehmer Codes

# Lehmer Codes

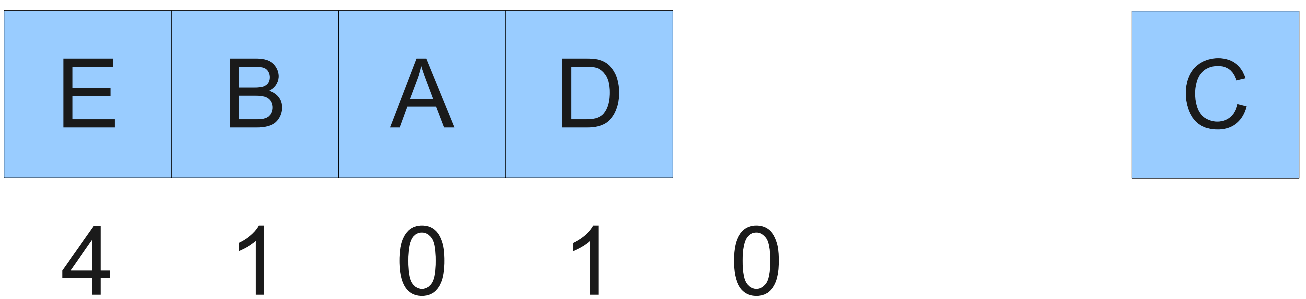
# Lehmer Codes

# Lehmer Codes

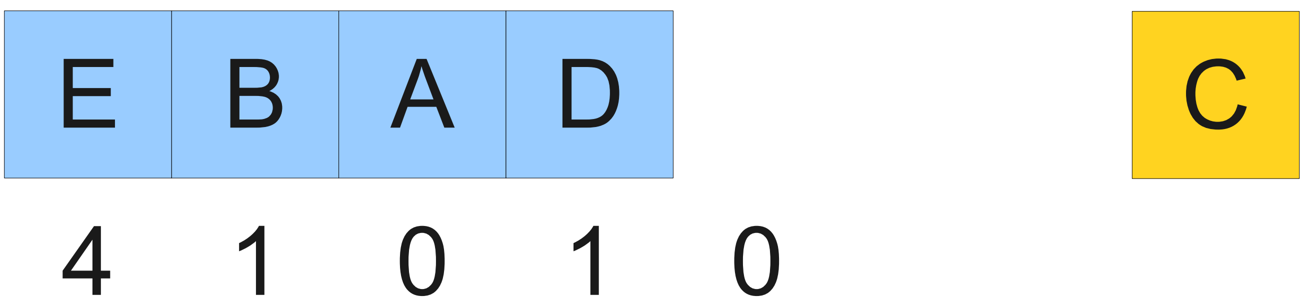
# Lehmer Codes



# Lehmer Codes



# Lehmer Codes



# Lehmer Codes

| | | | | |
|---|---|---|---|---|
| E | B | A | D | C |
| 4 | 1 | 0 | 1 | 0 |

# Lehmer Codes

| E | B | A | D | C |
|---|---|---|---|---|
| 4 | 1 | 0 | 1 | 0 |

# Listing Lehmer Codes

| | | |
|---|---|---|
| 0 | abcd | 0000 |
| 1 | abdc | 0010 |
| 2 | acbd | 0100 |
| 3 | acdb | 0110 |
| 4 | adbc | 0200 |
| 5 | adcb | 0210 |
| 6 | bacd | 1000 |
| 7 | badc | 1010 |
| 8 | bcad | 1100 |
| 9 | bcda | 1110 |
| 10 | bdac | 1200 |
| 11 | bdac | 1210 |
| 12 | cabd | 2000 |
| 13 | cadb | 2010 |
| 14 | cbad | 2100 |
| 15 | cbda | 2110 |
| 16 | cdab | 2200 |
| 17 | cdba | 2210 |
| 18 | dabc | 3000 |
| 19 | dacb | 3010 |
| 20 | dbac | 3100 |
| 21 | dbca | 3110 |
| 22 | dcab | 3200 |
| 23 | dcba | 3210 |

# Factoradic Numbers

- Mixed-radix number system.
- Nth digit in base n!.
- Nth digit can be 0, 1, 2, …, n
- Example: $3110_!$
  - 3 x 3! + 1 x 2! + 1 x 1! + 0 x 0! = 21
- Example: $1210_!$
  - 1 x 3! + 2 x 2! + 1 x 1! + 0 x 0! = 11

# Listing Lehmer Codes

| | | |
|---|---|---|
| 0 | abcd | 0000 |
| 1 | abdc | 0010 |
| 2 | acbd | 0100 |
| 3 | acdb | 0110 |
| 4 | adbc | 0200 |
| 5 | adcb | 0210 |
| 6 | bacd | 1000 |
| 7 | badc | 1010 |
| 8 | bcad | 1100 |
| 9 | bcda | 1110 |
| 10 | bdac | 1200 |
| 11 | bdac | 1210 |
| 12 | cabd | 2000 |
| 13 | cadb | 2010 |
| 14 | cbad | 2100 |
| 15 | cbda | 2110 |
| 16 | cdab | 2200 |
| 17 | cdba | 2210 |
| 18 | dabc | 3000 |
| 19 | dacb | 3010 |
| 20 | dbac | 3100 |
| 21 | dbca | 3110 |
| 22 | dcab | 3200 |
| 23 | dcba | 3210 |

# Listing Lehmer Codes

| | | | | | |
|---|---|---|---|---|---|
| 0 | abcd | 0000 | 12 | cabd | 2000 |
| 1 | abdc | 0010 | 13 | cadb | 2010 |
| 2 | acbd | 0100 | 14 | cbad | 2100 |
| 3 | acdb | 0110 | 15 | cbda | 2110 |
| 4 | adbc | 0200 | 16 | cdab | 2200 |
| 5 | adcb | 0210 | 17 | cdba | 2210 |
| 6 | bacd | 1000 | 18 | dabc | 3000 |
| 7 | badc | 1010 | 19 | dacb | 3010 |
| 8 | bcad | 1100 | 20 | dbac | 3100 |
| 9 | bcda | 1110 | **21** | **dbca** | **3110** |
| 10 | bdac | 1200 | 22 | dcab | 3200 |
| **11** | **bdac** | **1210** | 23 | dcba | 3210 |

Writing n in factoradic gives the nth Lehmer code.

# Converting to Factoradic

- Goal: convert k to factoradic.
  - Assume we know n, the number of elements to permute.
- To get the (n – 1)! place, divide k by (n – 1)!
  - Quotient is the (n – 1)! place.
- Repeat for the remaining digits using the remainder.
- Identical to converting to any other base, just using factorials instead of powers.

# Example: Convert 13 to Factoradic

# Example: Convert 13 to Factoradic

`13 = 2 x 3! + 1`

# Example: Convert 13 to Factoradic

```
13 = 2 x 3! + 1
 1 = 0 x 2! + 1
```

# Example: Convert 13 to Factoradic

```
13 = 2 x 3! + 1
 1 = 0 x 2! + 1
 1 = 1 x 1! + 0
```

# Example: Convert 13 to Factoradic

```
13 = 2 x 3! + 1
 1 = 0 x 2! + 1
 1 = 1 x 1! + 0
 0 = 0 x 0! + 0
```

# Example: Convert 13 to Factoradic

```
13 = 2 x 3! + 1
 1 = 0 x 2! + 1
 1 = 1 x 1! + 0
 0 = 0 x 0! + 0
```

# Example: Convert 13 to Factoradic

```
13 = 2 x 3! + 1

 1 = 0 x 2! + 1

 1 = 1 x 1! + 0

 0 = 0 x 0! + 0
```

**Answer:** $2010_!$

# Generating Permutations

```
public static String kthPermutation(String chars, int k)
{
    String result = "";

    for (int n = chars.length() - 1; n >= 0; --n) {
        int quotient  = k / factorial(n);
        int remainder = k % factorial(n);

        result += chars.charAt(quotient);

        chars = chars.substring(0, quotient) +
                chars.substring(quotient + 1);

        k = remainder;
    }

    return result;
}
```

# Analysis of Our Algorithm

- To get the kth permutation of n elements:
  - Converting k to factoradic takes O(n)
  - Building up the permutations takes $O(n^2)$
    - Elements stored in a list; O(n) to remove each.
  - Can reduce to O(n lg n) using **order statistic tree**.
- Easy to build an iterator from this.

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 1 |

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 1 |

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 1 |

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 1 |

# Incrementing Binary Numbers

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

# Incrementing Factoradic Numbers

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

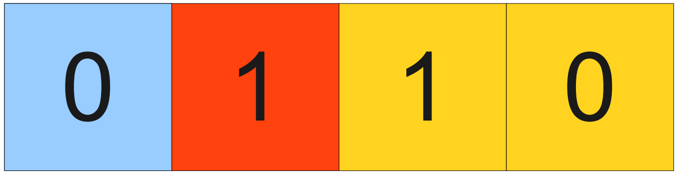
# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

# Incrementing Factoradic Numbers



# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 2 | 0 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 2 | 0 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 2 | 1 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 0 | 2 | 1 | 0 |

# Incrementing Factoradic Numbers

| | | | |
|---|---|---|---|
| 1 | 0 | 0 | 0 |

# Incrementing Factoradic Numbers

- Find the digit to increment.
    - Scan backwards from the end to find the first number not at its maximum.
- Increment that digit.
- Set the digits after that to zero.

# Incrementing Permutations

# Incrementing Permutations

| A | B | C | D |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

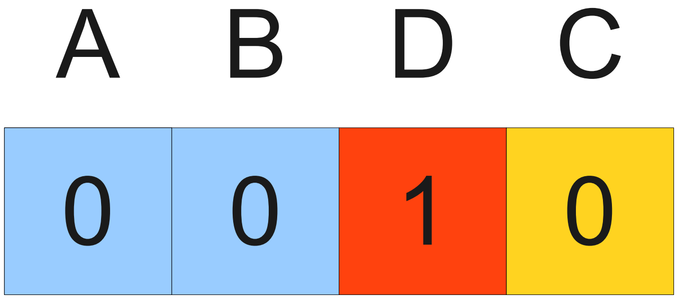
# Incrementing Permutations

| A | B | C | D |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

# Incrementing Permutations

| A | B | C | D |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

# Incrementing Permutations



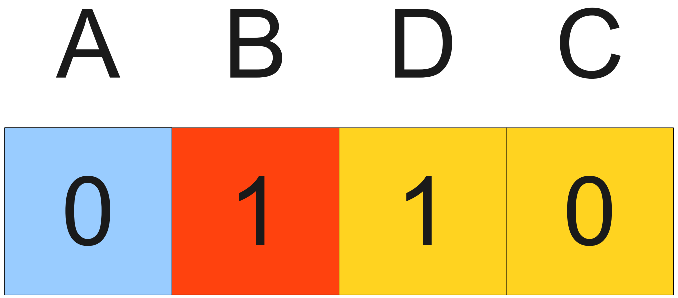
# Incrementing Permutations

| A | B | D | C |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

# Incrementing Permutations

| A | B | D | C |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

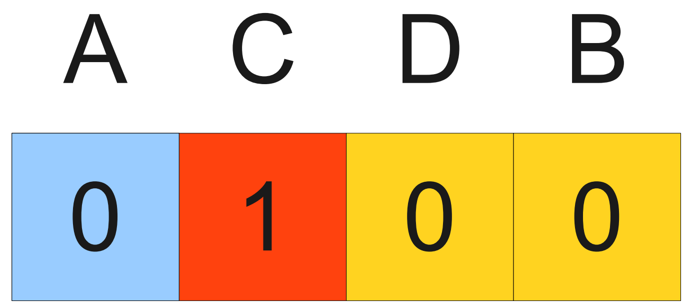
# Incrementing Permutations



# Incrementing Permutations

| A | C | D | B |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

# Incrementing Permutations



# Incrementing Permutations

| A | C | B | D |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

# Incrementing Permutations

| A | C | B | D |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

# Incrementing Permutations

| A | C | B | D |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

# Incrementing Permutations

| A | C | B | D |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

# Incrementing Permutations

| A | C | D | B |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

# Incrementing Permutations

| A | C | D | B |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

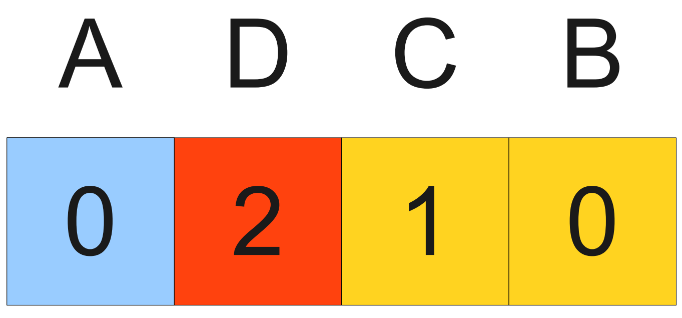
# Incrementing Permutations

| A | C | D | B |
|---|---|---|---|
| 0 | 1 | 1 | 0 |

# Incrementing Permutations

| A | C | D | B |
|---|---|---|---|
| 0 | 2 | 1 | 0 |

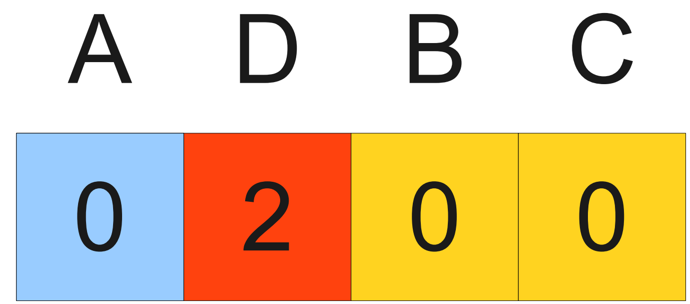
# Incrementing Permutations



# Incrementing Permutations

| A | D | C | B |
|---|---|---|---|
| 0 | 2 | 0 | 0 |

# Incrementing Permutations



# Incrementing Permutations

| A | D | B | C |
|---|---|---|---|
| 0 | 2 | 0 | 0 |

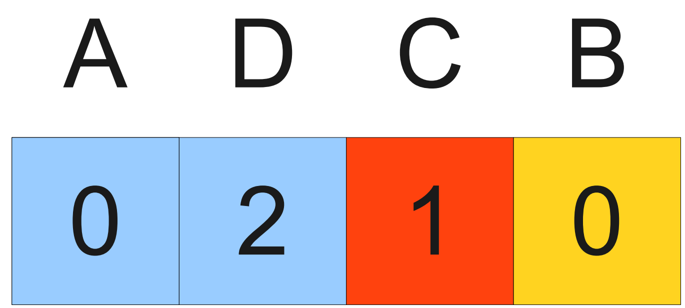
# Incrementing Permutations

| A | D | B | C |
|---|---|---|---|
| 0 | 2 | 0 | 0 |

# Incrementing Permutations

| A | D | B | C |
|---|---|---|---|
| 0 | 2 | 1 | 0 |

# Incrementing Permutations



# Incrementing Permutations

| A | D | C | B |
|---|---|---|---|
| 0 | 2 | 1 | 0 |

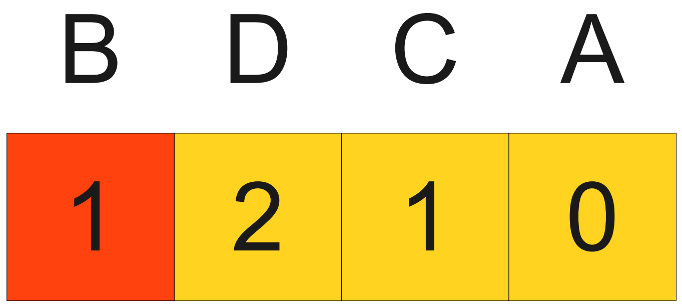
# Incrementing Permutations

| A | D | C | B |
|---|---|---|---|
| 0 | 2 | 1 | 0 |

# Incrementing Permutations

| A | D | C | B |
|---|---|---|---|
| 1 | 2 | 1 | 0 |

# Incrementing Permutations



# Incrementing Permutations

| B | D | C | A |
|---|---|---|---|
| 1 | 0 | 0 | 0 |

# Incrementing Permutations

| B | A | C | D |
|---|---|---|---|
| 1 | 0 | 0 | 0 |

# Incrementing Permutations

| B | A | C | D |
|---|---|---|---|
| 1 | 0 | 0 | 0 |

# Incrementing Permutations

- Find the digit to be incremented.
  - Scan backwards from the end of the sequence to find the longest increasing sequence.
- Increment that digit.
  - Find the smallest element bigger than the element right before that sequence.
  - Swap those two elements.
- Set the digits after that to zero.
  - Reverse the increasing sequence
- Runtime: **O(n) per permutation**.

# Summary

- Lehmer codes describe a permutation as a series of elements to choose in order.
- The factoradic number system maps directly onto Lehmer codes, and thus onto permutations.
- By simulating what **would** happen if we wrote out the Lehmer code, we can derive a fast algorithm for generating ordered permutations.

# Concluding Thoughts

Number systems are useful in **number recovery** by discovering one component of the number at a time.

Number systems are useful in **enumeration** by revealing structure hidden in the indices.

# Fun With Number Systems

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